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Effect of Couple Stresses on a Pulsatile Magnetohydrodynamic Viscoelastic Flow Through a Channel Bounded by Two Permeable Parallel Plates

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This paper is an analysis of an incompressible unsteady pulsatile laminar flow of electrically conducting non-Newtonian fluid through a channel bounded by two permeable plates taking into account the induced magnetic field and the effect of couple stresses on the flow. Solutions of the equations of motion are obtained by using *Mathematica* program and the effects of the various parameters entering the problem are discussed with the help of graphs. The main results show that the effect of couple stresses is to decrease the flow velocity and it is to increase (or decrease) the induced magnetic field near one of the boundaries and decrease (or increase) it near the other according to the time variation.

Keywords: magnetohydrodynamic (MHD), couple stresses, non-Newtonian, porous boundaries, viscoelastic.

1. Introduction

A transport processes through a channel is of great interest in diverse fields like ground-water hydrology and petroleum-reservoir specially the transport of non-Newtonian fluids.

When viscous non-Newtonian fluids are pumped through channels the flow may be steady or unsteady depending upon the type of pumping unit positive displacement pumps. Units in which pulsation is imposed upon the mean flow which varies with the design and speed of the pump produce unsteady flows like piston-cylinder peristaltic and stator-rotor units.

Previously, Edwards *et. al.*, discussed the problem for ordinary Newtonian fluid and the fluid of power-law type [1–4]. Eldabe [5] has studied a theoretical study of pulsatile non-Newtonian conducting fluid obeying Rivlin-Ericksen type [6] flows through a channel bounded by permeable parallel walls and the fluid was stressed by a uniform transverse magnetic field. Also, Eldabe and Hassan [7] have studied non-Newtonian fluid flow between two parallel walls one of them moving with a uniform velocity under the action of a transverse magnetic field.

This paper treats computationally and graphically the flow of pulsatile elastico-viscous, incompressible and electrically conducting fluid taking into account the induced magnetic field and the effect of couple stresses. The idea of this problem is very important to some of the applications concerning with such the flow of the oil under ground where there is a natural magnetic field and the earth is considered as a porous boundary. Also, the motion of the blood through the arteries where the boundaries are porous.

Couple stresses are consequence of assuming that the mechanical action of one part of a body on another, across a surface, is equivalent to a force and a moment distribution; the state of stress is measured by a stress tensor (T_{ij}) and a couple stress tensor (M_{ij}). Eldabe [8] has studied analytically the effect of couple stresses on a flow of pulsating hydromagnetic, incompressible electrically conducting fluid flowing between two parallel fixed plates.

The purpose of this paper is to investigate the effect of couple stresses on the flow by changing the couple stresses parameter besides the effects of other different parameters. The results are discussed computationally by using *Mathematica* program and the effects of various parameters on both the velocity and the induced magnetic field are discussed graphically. The field equations are [9]:

- the continuity equation

$$\dot{\rho} + \rho V_{i,i} = 0$$

- Cauchy's first law of motion

$$\rho a_i = T_{ji,j} + \rho b_i$$

- Cauchy's second law of motion

$$M_{ji,j} + \rho \ell_i + e_{ijk} T_{jk} = 0$$

where, ρ is the density of the fluid, V_i is the velocity vector, a_i is the acceleration, T_{ji} is the stress tensor, b_i is the body force and ℓ_i is the body moment per unit mass.

The non-polar theory of fluid is characterized by the conditions $m_i = 0$ (a couple stresses vector), $\ell_i = 0$, only the effect of stress tensor T_{ji} is considered. In the polar theory m_i is not identically zero. Stokes [9] obtained the constitutive equations for a polar fluid by replacing the strains E_{ij} in Mindlin and Tiersten [10] equations for a linear perfectly elastic solid. The constitutive equations are:

$$\begin{aligned} T^s_{ij} &= -P\delta_{ij} + \alpha(D_{rr})\delta_{ij} + 2\mu D_{ij}, \\ M^D_{ij} &= 4\eta K_{ij} + 4\eta' K_{ji}\,, \end{aligned} \tag{1}$$

where, P is the pressure, D_{ij} is the rate of deformation tensor, δ_{ij} is the Kronecker delta, α and μ are the material constants of viscosity, K_{ij} is the curvature-twist tensor, η and η' are the material constants of momentum and M_{ij} is the couple stress tensor.

The equation of motion as obtained by Stokes is:

$$\rho a_i = -P_{,i} + (\alpha + \mu)V_{r,ri} + \mu V_{i,rr} - \eta V_{i,ttrr} + \eta V_{r,ttri} + \frac{1}{2}e_{irs}(\rho \ell_s)_{,r} + \rho f_i \,. \quad (2)$$

The boundary conditions, which are needed to solve equation (2), are six. Three conditions are provided by assuming that the velocity of the fluid relative to the surface is zero at all solid boundaries. The remaining three conditions are provided by assuming either the couple stress is identically zero at the boundary, or that the vorticity at the boundary equal the rate of rotation of the boundary.

2. Mathematical formulation

Consider the flow of an incompressible, non-Newtonian unsteady electrically conducting and viscoelastic fluid in a channel bounded by two permeable parallel plates situated at distant $y=\pm h$ under the effect of couple stresses. The coordinate system used is given in Fig. 1. We assume that a uniform magnetic field H_0 acting along y-axis, and the fluid is being injected into the channel through the wall $y = -h$, and is being sucked through the wall $y = +h$ with uniform velocity V_0.The momentum, Maxwell's and continuity equations are:

$$\frac{\partial \underline{V}}{\partial t} + (\underline{V}.\nabla)\underline{V} = \frac{-1}{\rho}\nabla P + \frac{1}{\rho}\nabla.\tau + \frac{\mu_e}{\rho}\underline{J} \wedge \underline{H} + \frac{\eta}{\rho}\nabla^4 \underline{V}\,, \quad (3)$$

$$\frac{\partial \underline{H}}{\partial t} - \nabla \wedge (\nabla \wedge \underline{H}) = -\nabla \wedge (\lambda \nabla \wedge \underline{H})\,, \quad (4)$$

$$\nabla.\underline{H} = 0\,, \quad \nabla \wedge \underline{E} = -\mu_e \frac{\partial \underline{H}}{\partial t}\,, \quad \underline{J} = \sigma\{\underline{E} + \mu_e \underline{V} \wedge \underline{H}) \quad (5)$$

$$\nabla.\underline{V} = 0\,, \quad (6)$$

where τ is the stress tensor, $\underline{V}$ is the velocity vector, P is the fluid pressure, J is the current density, $\underline{H}$ is the magnetic field vector, μ_e is the magnetic permeability, $\underline{J}$ is the current density vector, $\underline{E}$ is the electric field, η is the coefficient of couple stresses and $\lambda=1/\mu_e\sigma$ is the magnetic diffusivity, where σ is the electrical conductivity. The following assumptions are considered:

(a) the physical properties ρ, μ, μ_2 andμ_3 are constants.

(b) the fluid is injected into the channel through the wall $y = -h$, and is being sucked through the wall $y = +h$ with uniform velocity V_0.

(c) the induced magnetic field is taking into account.

(d) The external electric field is zero and the electric field due to polarization of charges is negligible.

For two dimensional flow, let $\underline{V}= (u, v,0)$ and $\underline{H}= (H_x, H_y,0)$ since the two walls are infinite in extent so, all quantities are functions of y and t only. From equations (5) and (6) we can conclude that H_y=constant$\equiv H_0$, and $\underline{H} \equiv H_x$ also, $v = V_0$ and $\underline{V}= u$. Then equations (3) and (4) are reduced to:

$$\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P^m}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}+\frac{\mu_2}{\rho}\left(\frac{\partial^3 u}{\partial y^2\partial t}+V_0\frac{\partial^3 u}{\partial y^3}\right)+\frac{\mu_e}{\rho}H_0\frac{\partial H_x}{\partial y}+\frac{\eta}{\rho}\frac{\partial^4 u}{\partial y^4} \quad (7)$$

and,

$$\frac{\partial H_x}{\partial t}+V_0\frac{\partial H_x}{\partial y}-H_0\frac{\partial u}{\partial y}=\lambda\frac{\partial^2 H_x}{\partial y^2}\,, \quad (8)$$

where, the modified pressure is given by

$$P^m = P-(2\mu_2+\mu_3)\left(\frac{\partial u}{\partial y}\right)^2+\frac{\mu_e}{2}H_x^2,$$

and

$$\nu=\frac{\mu_1}{\rho},$$

where, μ_1, μ_2 and μ_3 are the viscosity, viscoelastic and cross-viscosity coefficients respectively.

Suppose pulsation pressure gradient of the form

$$\frac{\partial P^m}{\partial x}=\left(\frac{\partial P^m}{\partial x}\right)_s+\left(\frac{\partial P^m}{\partial x}\right)_0 e^{i\omega t} \quad (9)$$

and, the appropriate boundary conditions are:

$$\left.\begin{array}{lll} u=0\,, & u''=0\,, & H=0 \text{ at } y=-h \\ u=0\,, & u''=0\,, & H=0 \text{ at } y=+h \end{array}\right\}. \quad (10)$$

Let us introduce the non-dimensional quantities as follows:

$$\left.\begin{array}{l} x=\frac{\nu}{V_o}x^*\;,\; y=\frac{\nu}{V_o}y^*\;,\; t=\frac{\nu}{V_0^2}t^*\;,\; \omega=\frac{V_0^2}{\nu}\omega^*, \\ u=V_0u^*\;,\; H_x=H_0H^*\;,\; P^m=\rho V_0^2P^*\;,\; a^2=\frac{h^2}{L^2}=\frac{\mu\nu^2}{\eta V_0^2} \\ \text{where, } L^2=\frac{\eta}{\mu}\,. \end{array}\right\} \quad (11)$$

Equations (7) and (8) after substituting from equations (9) and (11) may be rewritten in dimensionless form after dropping the star mark as:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial y}=P_s+P_0e^{i\omega t}+\frac{\partial^2 u}{\partial y^2}+S\left\{\frac{\partial^3 u}{\partial y^2\partial t}+\frac{\partial^3 u}{\partial y^3}\right\}+M\frac{\partial H}{\partial y}+\frac{1}{a^2}\frac{\partial^4 u}{\partial y^4} \quad (12)$$

and,

$$\frac{\partial H}{\partial t}+\frac{\partial H}{\partial y}-\frac{\partial u}{\partial y}=\frac{1}{R_m}\frac{\partial^2 H}{\partial y^2} \quad (13)$$

subjected to the boundary conditions

$$\left.\begin{array}{lll} u=0\,, & u''=0\,, & H=0 \text{ at } y=-1 \\ u=0\,, & u''=0\,, & H=0 \text{ at } y=+1 \end{array}\right\}. \quad (14)$$

where, $S=\frac{\mu_2 V_0^2}{\rho\nu^2}$ is the elasticity parameter, $M=\frac{\mu_e H_0^2}{\rho V_0^2}$ is the magnetic pressure number, $\frac{1}{a^2}=\frac{\eta V_0^2}{\rho\nu^3}$ is the couple stresses parameter, $R_m=\frac{\nu}{\lambda}$ is the magnetic Prandtl number.

Here, we point out that S=0 leads to the pulsating flow of ordinary viscous conducting fluid through a channel, while in the absence of pressure gradient and couple stresses this problem have been studied by Eldabe *et. al.* [5] and Eldabe [7]. Also in the absence of couple stresses only the problem will reduced to that discussed by Eldabe [11].

In order to solve Equations (12) and (13) subjected to the boundary conditions (14), we use the perturbation method as follows:

$$\left.\begin{aligned} u &= u_0(y)+u_1(y)e^{i\omega t} \\ H &= H_0(y)+H_1(y)e^{\mathrm{i}\omega \mathrm{t}} \end{aligned}\right\} \tag{15}$$

Then equations (12), (13) are reduced to the following system of equations after equating the terms of zero and first order of $\mathrm{e}^{i\omega t}$.

$$\frac{d^4u_0}{dy^4}+Sa^2\frac{d^3u_0}{dy^3}+a^2\frac{d^2u_0}{dy^2}-a^2\frac{du_0}{dy}+Ma^2\frac{dH_0}{dy}=-P_sa^2 \tag{16}$$

$$\frac{d^4u_1}{dy^4}+Sa^2\frac{d^3u_1}{dy^3}+(1+Si\omega)a^2\frac{d^2u_1}{dy^2}-a^2\frac{du_1}{dy}-i\omega a^2u_1+Ma^2\frac{dH_1}{dy}=-P_0a^2 \tag{17}$$

$$\frac{1}{R_m}\frac{d^2H_0}{dy^2}-\frac{dH_0}{dy}+\frac{du_0}{dy}=0 \tag{18}$$

$$\frac{1}{R_m}\frac{d^2H_1}{dy^2}-\frac{dH_1}{dy}+\frac{du_1}{dy}-i\omega H_1=0 \tag{19}$$

with the boundary conditions

$$\left.\begin{aligned} u_0=u_1=0\,,\quad u_0''=u_1''=0\,,\quad H_0=H_1=0 \text{ at } y=-1 \\ u_0=u_1=0\,,\quad u_0''=u_1''=0\,,\quad H_0=H_1=0 \text{ at } y=+1 \end{aligned}\right\}. \tag{20}$$

To have the solutions of the equations (16)–(19) under the boundary conditions (20) we use a *Mathematica* program of version (4) and the effects of various parameters entering the problem are discussed with the help of graphs. Actually, due to the big size of the detailed solutions we'll only show the graphical representations of these solutions here.

3. Results and discussion

The effect of the properties of the parameters entering the problem on the velocity and the induced magnetic field is illustrated graphically through Figs. 1–11. The velocity and the magnetic field distributions depend periodically on time and it is found that the phase of the velocity and the induced magnetic field decrease with increasing position y, see Figs. 1(a), (b).

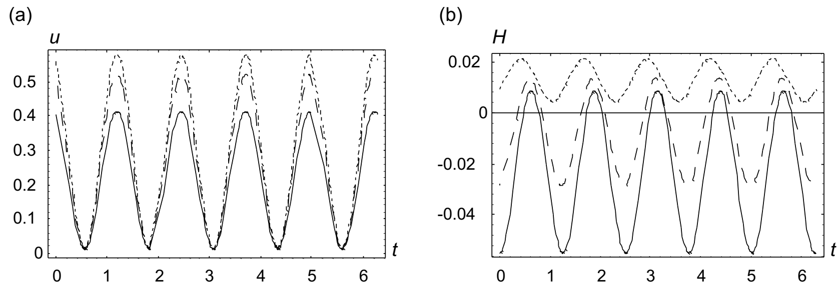

Figure 1 (a) Velocity distribution plotted against time in the case: a=0.5, S=0.2, M=5, P_0=5, P_s=5, ω=5; and (b) Magnetic field distribution plotted against time in the case: a=0.5, S=0.2, M=5, P_0=5, P_s=5, ω=5. Lines shown for y values of: dotted – 0.1, dashed – 0.3, and continuous – 0.5.

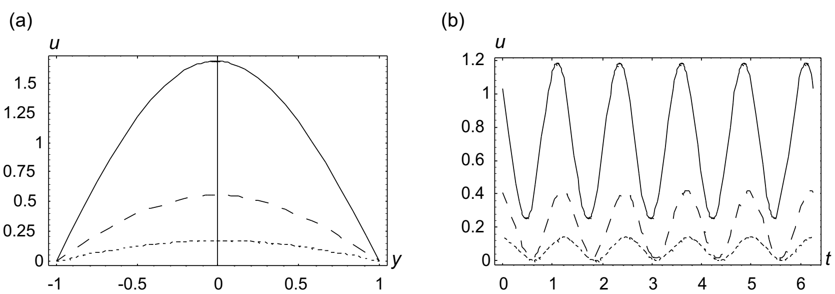

Figure 2 (a) Velocity distribution plotted against position in the case: t=2π/3, S=0.2, M=5, P_0=5, P_s=5, ω=5, and (b) against time in the case: t=2π/3, S=0.2, M=5, P_0=5, P_s=5, ω=5. Lines shown for a values of: dotted – 0.3, dashed – 0.5, and continuous – 0.8.

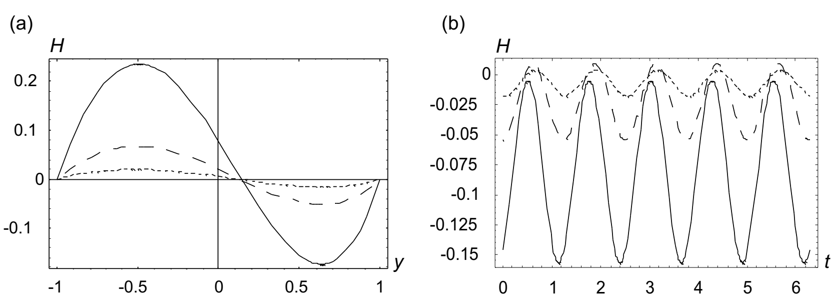

Figure 3 (a) Magnetic field distribution plotted against position in the case: $t = 2\pi/3$, $S = 0.2$, $M = 5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$, and (b) against time in the case: $t = 2\pi/3$, $S = 0.2$, $M = 5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$. Lines shown for a values of: dotted – 0.3, dashed – 0.5, and continuous – 0.8.

From Figs. 2 and 3 we can see that the velocity profile increases with the increase of the parameter a_2 which is the inverse of the coefficient of couple stresses i.e. the velocity decreases with the increase of couple stresses, whereas the induced magnetic field decreases (or increases) near one of the boundaries and increases (or decreases) near the other one with the increase of couple stresses parameter, according to the time variation.

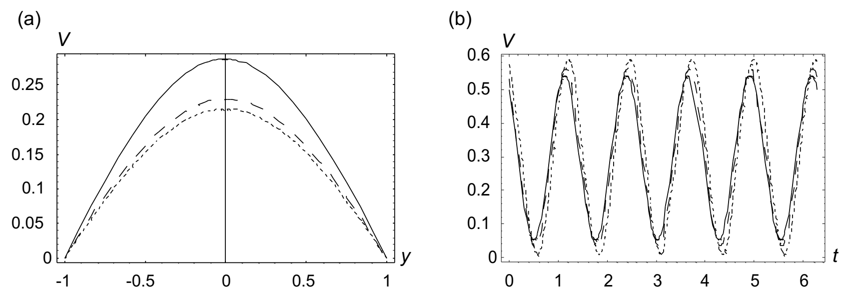

Figure 4 (a) Velocity distribution plotted against position in the case: $t = 2\pi/3$, $a = 0.5$, $M = 5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$, and (b) against time in the case: $t = 2\pi/3$, $a = 0.5$, $M = 5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$. Lines shown for S values of: dotted – 0.01, dashed – 0.1, and continuous – 0.5.

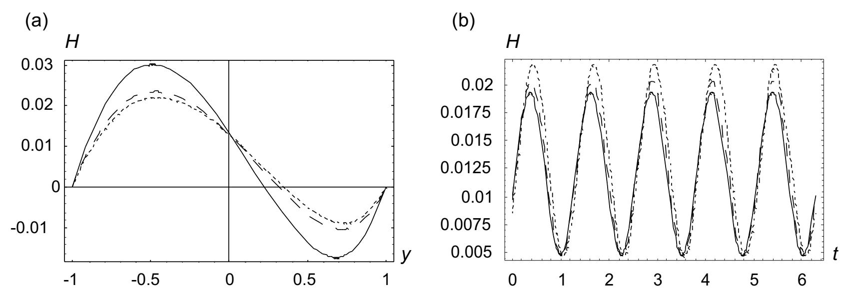

Figure 5 (a) Magnetic field distribution plotted against position in the case: $t = 2\pi/3$, $a = 0.5$, $M = 5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$, and (b) against time in the case: $t = 2\pi/3$, $a = 0.5$, $M = 5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$. Lines shown for S values of: dotted – 0.01, dashed – 0.1, and continuous – 0.5.

Also it is found that the increasing of the elasticity parameter S will increase the velocity distribution and decrease (or increase) the induced magnetic field near the upper plate while the inverse will occur near the lower plate as we can see through Figs. 4 and 5. We can see from Figs. 6 and 7 that the velocity increases with the increase of the magnetic parameter M and the induced magnetic field increases (or decreases) with the increase of M near one of the two plates and the inverse will occur near the other one according to the time variation. Finally, it has been observed that the effect of the amplitude of the pulsation P_0 implies increase or decrease of these distributions according to the time variation as illustrated in Figs. 8–11.

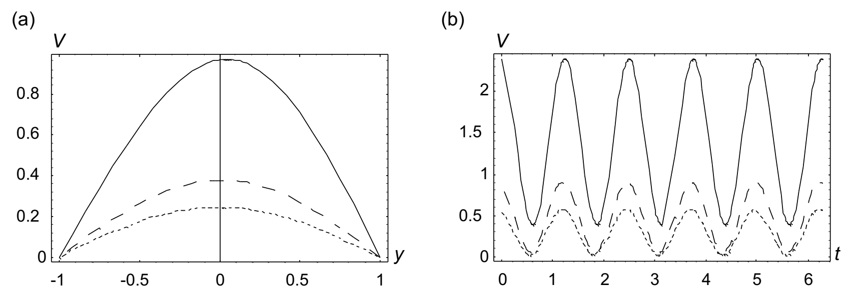

Figure 6 (a) Velocity distribution plotted against position in the case $t = 2\pi/3$, $S = 0.2$, $a = 0.5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$, and (b) against time in the case: $t = 2\pi/3$, $S = 0.2$, $a = 0.5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$. Lines shown for M values of: dotted – 5, dashed – 50, and continuous – 100.

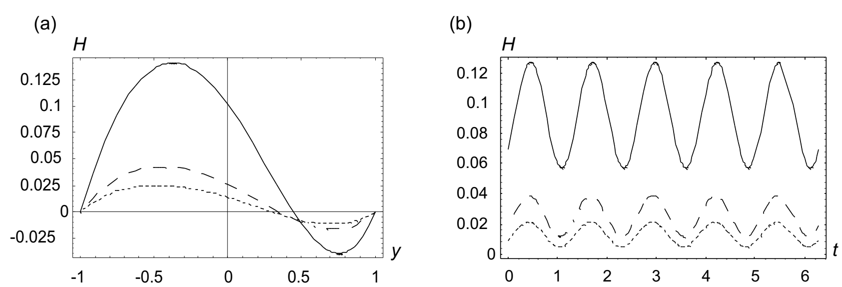

Figure 7 Magnetic field distribution plotted against position in the case: $t = 2\pi/3$, $S = 0.2$, $a = 0.5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$, and (b) against time in the case: $t = 2\pi/3$, $S = 0.2$, $a = 0.5$, $P_0 = 5$, $P_s = 5$, $\omega = 5$. Lines shown for M values of : dotted – 5, dashed – 50, and continuous – 100.

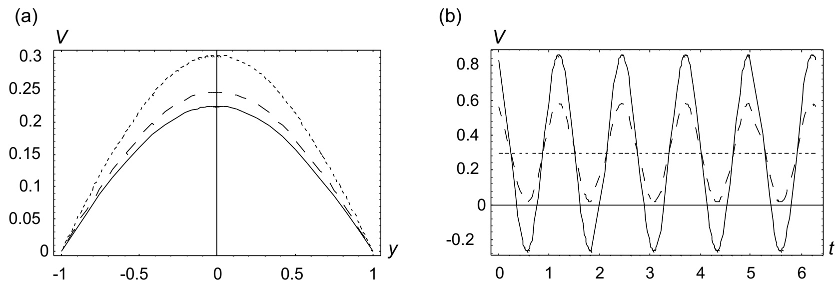

Figure 8 Velocity distribution plotted against position in the case $t = 2\pi/3$, $S = 0.2$, $M = 5$, $a = 0.5$, $P_s = 5$, $\omega = 5$, and (b) against time in the case $t = 2\pi/3$, $S = 0.2$, $M = 5$, $a = 0.5$, $P_s = 5$, $\omega = 5$. Lines shown for P_0 values of: dotted – 0, dashed – 5, and continuous – 10.

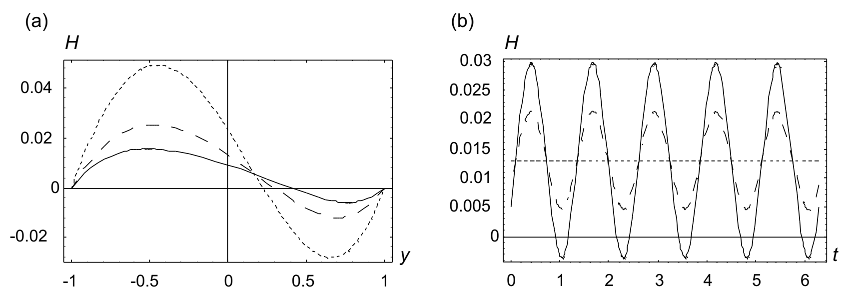

Figure 9 (a) Magnetic field distribution plotted against position in the case $t = 2\pi/3$, $S = 0.2$, $M = 5$, $a = 0.5$, $P_s = 5$, $\omega = 5$, and (b) against time in the case: $t = 2\pi/3$, $S = 0.2$, $M = 5$, $a = 0.5$, $P_s = 5$, $\omega = 5$. Lines shown for P_0 values of: dotted – 0, dashed – 5, and continuous – 10.

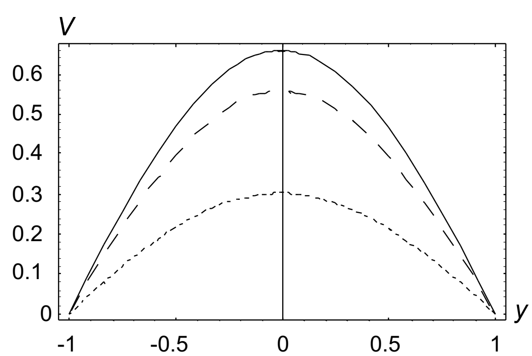

Figure 10 Velocity distribution plotted against position in the case $t = 3\pi/4$, $S = 0.2$, $M = 5$, $a = 0.5$, $P_s = 5$, $\omega = 5$. Lines shown for P_0 values of: dotted – 0, dashed – 5, and continuous – 10.

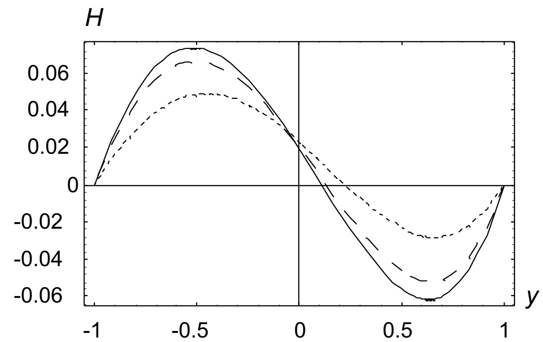

Figure 11 Magnetic field distribution plotted against position in the case $t = 3\pi/4$, $S = 0.2$, $M = 5$, $a = 0.5$, $P_s = 5$, $\omega = 5$. Lines shown for P_0 values of: dotted – 0, dashed – 5, and continuous – 10.

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