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The Change in the Hyperbolic Orbital Elements Due to Application of a Small Impulse

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The differential variations in the hyperbolic orbital classical elements due to a small impulse in the direction of the velocity vector are computed. We applied the method of Gauss for secular perturbations using the Lagrangian form of planetary equations.

Keywords: rocket dynamics, Gauss method for secular perturbations, Lagrange planetary equations, orbital motion.

1. Introduction

In this paper the influence of applying a small impulse on the hyperbolic orbital elements is considered. The change in the hyperbolic elements depend solely on the change in magnitude and direction of the velocity vector since the radius vector remains unaltered during the operation. Small intermediate impulse is applied in the case of bi-elliptic transfer and always involves going to infinity and coming back again.

A hyperbola - hyperbola transfer can always be performed with six infinitesimal impulses given at either infinity or origin. If the two hyperbolas intersect, the transfer can be done by a single impulse. Also ellipse-hyperbola transfer may be achieved. In this case-five impulses are required [3]. The effect of small impulses on the orbital elements is required to be determined"in transfer and rendezvous problems, for the purpose of infinitesimal or small a corrections of the orbital elements of the trajectory [4].

2. Method and results

2.1. *Application of Gauss Method for Hyperbolic Orbits*

According to Gauss' procedure for elliptic orbits and from the basic formulae of hyperbolic motion, namely [5,6]

$$r = \frac{a(e^2-1)}{1+e\cos f}\,, \tag{1}$$

$$e\sinh F - F = \nu(t-\tau)\,, \tag{2}$$

$$r = a(e\cosh F - 1) = M\,, \tag{3}$$

$$\frac{\partial r}{\partial \sigma} = \frac{\partial a}{\partial \sigma}(e\cosh F - 1) + a\left(\frac{\partial F}{\partial \sigma}e\sinh F + \frac{\partial e}{\partial \sigma}\cosh F\right) \tag{4}$$

$$\frac{\partial F}{\partial \sigma}\frac{r}{a} + \frac{\partial e}{\partial \sigma}\sinh F = \frac{\partial(\epsilon - \varpi)}{\partial \sigma}\,, \tag{5}$$

where σ is either one of the six hyperbolic orbital elements a, e, i, ϖ, Ω, ϵ; ν is a quantity defined by $\nu^2 a^3 = \mu$; F is analogous to the elliptic eccentric anomaly and is defined by Eq. (2); f is the true anomaly.

$$r\cos f = ae - a\cosh F\,, \tag{6}$$

$$a\cosh F = ae - r\cos f\,, \tag{7}$$

After some reductions we acquire

$$\frac{\partial r}{\partial \sigma} = \frac{\partial a}{\partial \sigma}\frac{r}{a} + \frac{a^2 e\sinh F}{r}\frac{\partial(\epsilon-\varpi)}{\partial \sigma} + \frac{\partial e}{\partial \sigma}\left(-r\cos f + ae - \frac{a^2 e\sinh^2 F}{r}\right)\,. \tag{8}$$

We have

$$\tan\frac{f}{2} = \left(\frac{e+1}{e-1}\right)^{\frac{1}{2}}\tanh\frac{F}{2}\,. \tag{9}$$

After differentiating Eq.(9) with respect to σ and after some algebraic and trigonometric reductions we get

$$\frac{\partial f}{\partial \sigma}\frac{1}{\sin f} = \frac{\partial e}{\partial \sigma}\frac{-1}{e^2-1} + \frac{\partial F}{\partial \sigma}\frac{1}{\sinh F}\,. \tag{10}$$

From Eqs. (5) and (10), after some calculations, we can write

$$\frac{\partial f}{\partial \sigma} = \sin f\left[\frac{\partial e}{\partial \sigma}\left(\frac{-1}{e^2-1} - \frac{a}{r}\right) + \frac{\partial(\epsilon-\varpi)}{\partial \sigma}\frac{ab}{r^2}\right]\,, \tag{11}$$

where $b = a(e^2-1)^{\frac{1}{2}}$. Let

$$u = \omega + f = \varpi - \Omega + f\,.$$

Whence from Eq. (10) we get

$$\frac{\partial u}{\partial \sigma} = \frac{\partial(\varpi - \Omega)}{\partial \sigma} - \sin f \left(\frac{a}{r} + \frac{1}{e^2 - 1} \right) \frac{\partial e}{\partial \sigma} + \frac{ab}{r^2} \frac{\partial(\epsilon - \varpi)}{\partial \sigma} . \tag{12}$$

By the hyperbolic relationships Eqs. (1),(2) and (3) we can prove that the squared bracket on the R.H.S. of Eq. (8) is equal to $(+a \cos f)$, whence

$$\frac{\partial r}{\partial \sigma} = \frac{\partial a}{\partial \sigma} \frac{r}{a} + \frac{a^2 e \sinh F}{r} \frac{\partial(\epsilon - \varpi)}{\partial \sigma} + a \cos f \frac{\partial e}{\partial \sigma} . \tag{13}$$

From Gauss treatment of secular inequalities we have:

$$\frac{1}{k} \frac{\partial R}{\partial \sigma} = \frac{\partial r}{\partial \sigma} .S + r \left[n_3 \frac{\partial \Omega}{\partial \sigma} + \frac{\partial u}{\partial \sigma} \right] .T + r \left[-n_2 \frac{\partial \Omega}{\partial \sigma} + \frac{\partial i}{\partial \sigma} \sin u \right] .W , \tag{14}$$

here $n_2 = \cos u . \sin i$ and $n_3 = \cos i$. R is the planetary disturbing function, S, T, W are the orthogonal components of the attraction between the disturbed and disturbing planets, $k = Gm_1$. Whence from Eqs. (14), (13), (12) we can find

$$\begin{aligned} \frac{1}{k} \frac{\partial R}{\partial \sigma} \quad = \quad & \left[\frac{\partial a}{\partial \sigma} \frac{r}{a} + \frac{a^2 e \sinh F}{r} \left(\frac{\partial \epsilon}{\partial \sigma} - \frac{\partial \varpi}{\partial \sigma} \right) + a \cos f \frac{\partial e}{\partial \sigma} \right] .S + \\ & r \left[\cos i \frac{\partial \Omega}{\partial \sigma} - \left(\frac{1}{e^2 - 1} + \frac{a}{r} \right) \frac{\partial e}{\partial \sigma} \sin f + \frac{\partial(\varpi - \Omega)}{\partial \sigma} + \frac{\partial(\epsilon - \varpi)}{\partial \sigma} \frac{ab}{r^2} \right] .T + \\ & r \left[- \cos u . \sin i \frac{\partial \Omega}{\partial \sigma} + \sin u . \frac{\partial i}{\partial \sigma} \right] .W . \end{aligned} \tag{15}$$

From Eq. (15) we can immediately deduce the following

$$\begin{aligned} \frac{1}{k} \frac{\partial R}{\partial a} &= \left(\frac{r}{a} \right) .S \\ \frac{1}{k} \frac{\partial R}{\partial e} &= a \cos f . S - r \left(\frac{1}{e^2 - 1} + \frac{a}{r} \right) \sin f . T \\ \frac{1}{k} \frac{\partial R}{\partial i} &= r \sin u . W \\ \frac{1}{k} \frac{\partial R}{\partial \Omega} &= -r(1 - \cos i) . T - r \sin i \cos u . W \\ \frac{1}{k} \frac{\partial R}{\partial \epsilon} &= \left(\frac{a^2 e \sinh F}{r} \right) .S + \left(\frac{ab}{r} \right) .T \\ \frac{1}{k} \frac{\partial R}{\partial \varpi} &= \left(\frac{a^2 e \sinh F}{r} \right) .S + \left(1 - \frac{ab}{r^2} \right) .T . \end{aligned} \tag{16}$$

3. Effect of a small impulse on the hyperbolic orbital elements

For hyperbolic trajectories

$$r = \frac{a(e^2 - 1)}{1 + e \cos f} , \text{ i.e., } \cos f = \frac{a(e^2 - 1) - r}{er} ,$$

$$e \sinh F - F = M = \nu(t - \tau)\,,$$

$$r = a(e \cosh F - 1)\,.$$

From the above equations, we find:

$$e\left[F + \frac{F^3}{6} + \ldots\right] - F = \nu(t - \tau) = M\,,$$

i.e. we get up to F^3, the equation:

$$F^3 + F\frac{6(e-1)}{e} - \frac{6M}{e} = 0\,. \tag{17}$$

Whence

$$F = \frac{\nu(t-\tau)}{e-1} + A(t-\tau)^3\,, \tag{18}$$

$$\sinh F = \frac{\nu(t-\tau)}{e-1} + B(t-\tau)^3\,, \tag{19}$$

$$\cosh F = 1 + \frac{1}{2}\left(\frac{\nu(t-\tau)}{e-1}\right)^2\,. \tag{20}$$

The proof for the Eqs. (18), (19) and (20) is the following: firstly, by using Mathematica software, to solve Eq. (17) with respect to F, we get one real and two complex roots, the real one is:

$$F = \frac{-2e(e-1) + \left\{3e^2 M + \sqrt{e^3\left[8(e-1)^3 + 9eM^2\right]}\right\}^{\frac{2}{3}}}{e\left\{3e^2 M + \sqrt{e^3\left[8(e-1)^3 + 9eM^2\right]}\right\}^{\frac{1}{3}}}$$

Expanding as a power series in M up to M^3, we get

$$F = \frac{1}{e-1}M - \frac{e}{6(e-1)^4}M^3\,,$$

i.e.

$$F = \frac{\nu(t-\tau)}{e-1} + A(t-\tau)^3\,,$$

where

$$A = \frac{-e\nu^3}{6(e-1)^4}\,.$$

Secondly, we have

$$\sinh F = \frac{M+F}{e} = \frac{1}{e}\left[M + \frac{M}{e-1} - \frac{eM^3}{6(e-1)^4}\right]\,,$$

$$\sinh F = \frac{M}{e-1} - \frac{M^3}{6(e-1)^4}$$

Therefore,

$$\sinh F = \frac{\nu(t-\tau)}{e-1} + B(t-\tau)^3\,,$$

where

$$B = -\frac{1}{6(e-1)^4}\,.$$

Thirdly, from

$$\frac{\partial M}{\partial F} = e\cosh F - 1$$

and by differentiating with respect to F, we obtain:

$$\cosh F = \frac{\partial}{\partial M}\left(\frac{M}{e-1} - \frac{M^3}{6(e-1)^4}\right)\frac{\partial M}{\partial F}\,,$$

$$\cosh F = \left(\frac{1}{e-1} - \frac{M^2}{2(e-1)^4}\right)(e\cosh F - 1)\,,$$

after some algebraic reductions, we get

$$\cosh F = 1 + \frac{M^2}{2(e-1)^2}\,,$$

i.e.

$$\cosh F = 1 + \frac{\nu^2(t-\tau)^2}{2(e-1)^2}\,.$$

Now, let

$$\zeta = r\cos f\,, \quad \eta = r\sin f\,,$$

whence

$$\zeta = a(e - \cosh F)\,. \tag{21}$$

By substitution for $\cos f$

$$\eta = r\left(1-\cos^2 f\right)^{\frac{1}{2}} = r\left\{1 - \left[\frac{a(e^2-1)-r}{er}\right]^2\right\}^{\frac{1}{2}} \tag{22}$$

$$\eta = \frac{1}{e}\left\{(e^2-1)\left[r(r+2a) - a^2(e^2-1)\right]\right\}^{\frac{1}{2}} \tag{23}$$

$$\eta = \frac{1}{e}\left\{(e^2-1)\left[a(e\cosh F-1)(ae\cosh F - a + 2a) - a^2(e^2-1)\right]\right\}^{\frac{1}{2}} \tag{24}$$

After some reductions, we have

$$\eta = a\sqrt{e^2-1}\sinh F\,, \tag{25}$$

From Eqs. (20), (21), we have

$$\zeta = a(e-1) - \frac{a\nu^2}{2(e-1)^2}(t-\tau)^2\,, \tag{26}$$

$$\zeta = -\frac{a\nu^2}{(e-1)^2}(t-\tau)\,. \tag{27}$$

From Eqs. (19) and (25), we get

$$\eta = \frac{a\nu(e+1)^{\frac{1}{2}}}{(e-1)^{\frac{1}{2}}}(t-\tau) + C(t-\tau)^3\,, \tag{28}$$

$$\eta = \frac{a\nu(e+1)^{\frac{1}{2}}}{(e-1)^{\frac{1}{2}}} + D(t-\tau)^2\,, \tag{29}$$

Evaluation of A, B, C, D is not needed.

From Eqs. (26)–(29) we get at $t = \tau$

$$\frac{\partial\zeta}{\partial a} = e - 1\,; \quad \frac{\partial\zeta}{\partial e} = a\,;$$

$$\frac{\partial\zeta}{\partial\tau} = \frac{\partial\zeta}{\partial a} = \frac{\partial\zeta}{\partial e} = \frac{\partial\eta}{\partial a} = \frac{\partial\eta}{\partial e} = \frac{\partial\eta}{\partial\tau} = 0\,;$$

$$\frac{\partial\zeta}{\partial\tau} = \frac{a\nu^2}{(e-1)^2}\,;$$

$$\frac{\partial\eta}{\partial\tau} = -\frac{a\nu(e+1)^{\frac{1}{2}}}{(e-1)^{\frac{1}{2}}}\,;$$

$$\frac{\partial\eta}{\partial a} = -\frac{\nu(e+1)^{\frac{1}{2}}}{2(e-1)^{\frac{1}{2}}}\,;$$

$$\frac{\partial\eta}{\partial e} = -\frac{a\nu}{(e+1)^{\frac{1}{2}}(e-1)^{\frac{3}{2}}}\,.$$

From the definition of the Lagrange's brackets,

$$[\alpha_r, \alpha_s] = \frac{\partial\zeta}{\partial\alpha_r}\frac{\partial\dot{\zeta}}{\partial\alpha_s} - \frac{\partial\zeta}{\partial\alpha_s}\frac{\partial\dot{\zeta}}{\partial\alpha_r} + \frac{\partial\eta}{\partial\alpha_r}\frac{\partial\dot{\eta}}{\partial\alpha_s} - \frac{\partial\eta}{\partial\alpha_s}\frac{\partial\dot{\eta}}{\partial\alpha_r}\,,$$

where α is any one of the elements a, e, χ. We get

$$[a, e] = [e, \tau] = 0\,, \tag{30}$$

$$[a, \tau] = -\frac{a\nu^2}{2}\,. \tag{31}$$

If we put $\chi = -\nu t$, then

$$[a, e] = [e, \chi] = 0\,, \tag{32}$$

$$[a, \chi] = \frac{a\nu}{2}\,. \tag{33}$$

Now,

$$h = a^2\nu(e^2-1)^{\frac{1}{2}}\,.$$

From
$$[\beta_r, \beta_s] = -h \sin i \frac{\partial(\Omega, i)}{\partial(\beta_r, \beta_s)} \,,$$
where β is either of the elements Ω, ω, i. We have:
$$[\omega, \Omega] = [\omega, i] = 0 \,, \tag{34}$$
$$[\Omega, i] = -a^2 \nu (e^2 - 1)^{\frac{1}{2}} \sin i \tag{35}$$
and from
$$[\alpha, \beta] = -\left[\cos i \frac{\partial \Omega}{\partial \beta} + \frac{\partial \omega}{\partial \beta}\right] \frac{\partial h}{\partial \alpha}$$
we have
$$[\chi, \Omega] = [\chi, \omega] = [\chi, i] = [a, i] = [e, i] = 0 \,, \tag{36}$$
$$[a, \Omega] = -\cos i \frac{\partial h}{\partial a} \,, \tag{37}$$
i.e.
$$[a, \Omega] = -\frac{a\nu\sqrt{e^2 - 1}\cos i}{2} \,, \tag{38}$$
$$[e, \Omega] = -\cos i \frac{\partial h}{\partial e} \,, \tag{39}$$
i.e.
$$[e, \Omega] = -\frac{a^2 \nu e \cos i}{\sqrt{e^2 - 1}} \,, \tag{40}$$
$$[a, \omega] = -\frac{\partial h}{\partial a} \,, \tag{41}$$
i.e.
$$[a, \omega] = -\frac{a\nu\sqrt{e^2 - 1}}{2} \,, \tag{42}$$
$$[e, \omega] = -\frac{\partial h}{\partial e} \,, \tag{43}$$
i.e.
$$[e, \omega] = -\frac{a^2 \nu e}{\sqrt{e^2 - 1}} \,. \tag{44}$$
From, the two Eqs.
$$\sum_{i=1}^{3} [(\alpha_r, \alpha_i)\alpha_i + (\alpha_r, \beta_i)\beta_i] = \frac{\partial R}{\partial \alpha_r} \,,$$
$$\sum_{i=1}^{3} [(\beta_r, \alpha_i)\alpha_i + (\beta_r, \beta_i)\beta_i] = \frac{\partial R}{\partial \beta_r} \,,$$
Whence,
$$\frac{\partial R}{\partial a} = \frac{a\nu}{2} \left[\chi - \sqrt{e^2 - 1}(\cos i . \Omega + \omega)\right] \tag{45}$$

$$\frac{\partial R}{\partial e} = -\frac{a^2\nu e}{\sqrt{e^2-1}}(\cos i.\Omega + \omega) \tag{46}$$

$$\frac{\partial R}{\partial \chi} = -\frac{a\nu a}{2}\,. \tag{47}$$

$$\frac{\partial R}{\partial \Omega} = \frac{a\nu\sqrt{e^2-1}\cos i}{2}\left[a + \frac{2ae.e}{e^2-1} - 2a\tan i\frac{\partial i}{\partial t}\right] \tag{48}$$

$$\frac{\partial R}{\partial \omega} = \frac{a\nu\sqrt{e^2-1}}{2}\left[a + \frac{2ae.e}{e^2-1}\right] \tag{49}$$

$$\frac{\partial R}{\partial i} = a^2\nu\sqrt{e^2-1}\sin i.\Omega\,. \tag{50}$$

From equations (45)–(50), we obtain

$$a = -\frac{2}{a\nu}\frac{\partial R}{\partial \chi}\,. \tag{51}$$

$$e = \frac{1}{a^2\nu e}\left[\sqrt{e^2-1}\frac{\partial R}{\partial \omega} + (e^2-1)\frac{\partial R}{\partial \chi}\right]\,. \tag{52}$$

$$\chi = -\frac{e^2-1}{a^2\nu e}\frac{\partial R}{\partial e} + \frac{2}{a\nu}\frac{\partial R}{\partial a}\,. \tag{53}$$

$$\Omega = \frac{1}{a^2\nu\sqrt{e^2-1}\sin i}\frac{\partial R}{\partial i} \tag{54}$$

$$\omega = -\frac{\sqrt{e^2-1}}{a^2\nu e}\frac{\partial R}{\partial e} - \frac{\cot i}{a^2\nu\sqrt{e^2-1}}\frac{\partial R}{\partial i} \tag{55}$$

$$\frac{di}{dt} = \frac{1}{a^2\nu\sqrt{e^2-1}}\left[\cot i\frac{\partial R}{\partial \omega} - \csc i\frac{\partial R}{\partial \Omega} - \right] \tag{56}$$

Let R' denotes the disturbing function, expressed in terms of a, e, i, Ω, ϖ, ϵ where $\varpi = \omega + \Omega$ and $\epsilon = \omega + \Omega + \chi$. Then

$$\begin{aligned}
\frac{\partial R}{\partial \Omega} &= \frac{\partial R'}{\partial \Omega} + \frac{\partial R'}{\partial \varpi} + \frac{\partial R}{\partial \epsilon} \\
\frac{\partial R}{\partial \omega} &= \frac{\partial R'}{\partial \varpi} + \frac{\partial R'}{\partial \epsilon} \\
\frac{\partial R}{\partial \chi} &= \frac{\partial R'}{\partial \epsilon}
\end{aligned}$$

After substitution in the above equations, we obtained the following equations for hyperbolic motion

$$a = -\frac{2}{a\nu}\frac{\partial R}{\partial \epsilon}\,. \tag{57}$$

but

$$\epsilon = \omega + \Omega + \chi\,.$$

Whence

$$\epsilon = \frac{2}{a\nu}\frac{\partial R}{\partial a} - \frac{\sqrt{e^2-1}(\sqrt{e^2-1}+1)}{a^2\nu e}\frac{\partial R}{\partial e} + \frac{\tan\frac{i}{2}}{a^2\nu\sqrt{e^2-1}}\frac{\partial R}{\partial i} \tag{58}$$

$$e = \frac{\sqrt{e^2-1}}{a^2\nu e}\left[\frac{\partial R}{\partial \varpi} + \left(1+\sqrt{e^2-1}\right)\frac{\partial R}{\partial \epsilon}\right] \tag{59}$$

$$\Omega = \frac{1}{a^2\nu\sqrt{e^2-1}\sin i}\frac{\partial R}{\partial i} \tag{60}$$

and

$$\varpi = \omega + \Omega\,.$$

then

$$\varpi = -\frac{\sqrt{e^2-1}}{a^2\nu e}\frac{\partial R}{\partial e} + \frac{\tan\frac{i}{2}}{a^2\nu\sqrt{e^2-1}}\frac{\partial R}{\partial i} \tag{61}$$

$$\frac{di}{dt} = -\frac{1}{a^2\nu\sqrt{e^2-1}}\left[\tan\frac{i}{2}\left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \epsilon}\right) + \csc i\frac{\partial R}{\partial \Omega}\right]. \tag{62}$$

Brouwer, Boccaletti and Murray derived Eqs (57)–(62) for the elliptic motion [1,2,5].

After substitution from Eq (16) in Eqs (57)–(62) we find the Gaussian form for a, ϵ, e, i, Ω, ϖ for a hyperbolic orbit as follows:

$$a = -\frac{2}{a\nu}\left(\frac{a^2 e\sinh F}{r}\,.\,S + \frac{ab}{r}\,.\,T\right) \tag{63}$$

$$\begin{aligned}\epsilon \;=\; & \frac{2}{a\nu}\left[\frac{r}{a} - \frac{\sqrt{e^2-1}(\sqrt{e^2-1}+1)}{2e}\cos f\right].\,S + \\ & \frac{\sqrt{e^2-1}(\sqrt{e^2-1}+1)}{a^2\nu e} r\sin f\left(\frac{a}{r} + \frac{1}{e^2-1}\right).\,T + \frac{r\sin u\tan\frac{i}{2}}{a^2\nu\sqrt{e^2-1}}\,.\,W\,,\end{aligned} \tag{64}$$

$$e = \frac{\sqrt{e^2-1}}{a^2\nu e}\left[\frac{a^2 e\sqrt{e^2-1}}{r}\sinh F\,.\,S + \left(r + \frac{a^2(e^2-1)}{r}\right).\,T\right] \tag{65}$$

$$\Omega = \frac{r\sin u}{a^2\nu\sqrt{e^2-1}\sin i}\,.\,W \tag{66}$$

$$\varpi = -\frac{\sqrt{e^2-1}}{a^2\nu e}\left[a\cos f\,.\,S - r\sin f\left(\frac{a}{r} + \frac{1}{e^2-1}\right).\,T\right] + \frac{r\sin u\tan\frac{i}{2}}{a^2\nu\sqrt{e^2-1}}\,.\,W \tag{67}$$

$$\frac{di}{dt} = \frac{r\cos u}{a^2\nu\sqrt{e^2-1}}\,.\,W \tag{68}$$

The mean longitude l is given by

$$l = \rho + \epsilon = \int \nu\, dt + \epsilon\,. \tag{69}$$

For hyperbolic motion ρ is defined by

$$\rho = \int v \, dt \, . \tag{70}$$

We have

$$\mu = \nu^2 a^3$$

i.e.

$$\nu = \nu^{\frac{1}{2}} a^{-\frac{3}{2}} \, ,$$

whence

$$\rho = \mu^{\frac{1}{2}} \int a^{-\frac{3}{2}} \, dt \, .$$

By differentiation

$$\rho = \mu^{\frac{1}{2}} a^{-\frac{3}{2}} = \nu \, .$$

Evidently ν is a function of a only. Then

$$\frac{d\nu}{da} = -\frac{3}{2} \mu^{\frac{1}{2}} a^{-\frac{5}{2}} = -\frac{3\nu}{2a} \, ,$$

and we have

$$\begin{aligned} \rho &= \frac{d\nu}{dt} = \frac{d\nu}{da}\frac{da}{dt} \\ \rho &= \frac{2}{a\nu}\frac{d\nu}{da}\frac{\partial R}{\partial \epsilon} \end{aligned}$$

From the above, we get

$$\ddot{\rho} = -\frac{3}{a^2}\frac{\partial R}{\partial \epsilon} \, . \tag{71}$$

Substituting for $\frac{\partial R}{\partial \epsilon}$ from Eq. (16) we may write

$$\ddot{\rho} = -\frac{3}{a^2}\left(\frac{a^2 e \sinh F}{r}\right) . S + \frac{ab}{r} . T \tag{72}$$

assuming k to be absorbed in S, T.

The reason for the introduction of ρ is to avoid the appearance of mixed terms in the trigonometric series encountered through the analysis. We have

$$S = \frac{\Delta \nu_S}{\Delta t} \, ; \quad T = \frac{\Delta \nu_T}{\Delta t} \, ; \quad W = \frac{\Delta \nu_W}{\Delta t} \, . \tag{73}$$

Thus from the Gaussian form, we obtain the variations in the hyperbolic orbital elements due to a small impulse as the following:

$$\Delta a = -\frac{2}{\nu}\left(\frac{ae \sinh F}{r}\Delta \nu_S + \frac{\sin f}{\sinh F}\Delta \nu_T\right) \tag{74}$$

$$\begin{aligned} \Delta \epsilon &= \frac{1}{a\nu}\left\{\left[\frac{2r}{a} - \frac{\sqrt{e^2-1}(\sqrt{e^2-1}+1)}{e}\cos f\right]\right\}\Delta v_S + \\ &\quad \frac{\sqrt{e^2-1}(\sqrt{e^2-1}+1)}{ae}\left(\frac{a}{r} + \frac{1}{e^2-1}\right) r \sin f \, . \, \Delta v_T + \frac{r \sin u \tan \frac{i}{2}}{a\sqrt{e^2-1}} \, . \, \Delta v_W \end{aligned} \tag{75}$$

$$\Delta e = \frac{\sqrt{e^2-1}}{\nu}\left[\frac{\sqrt{e^2-1}}{r}\sinh F\,.\,\Delta v_S + \left(\frac{r}{a^2 e} + \frac{e^2-1}{er}\right).\,\Delta v_T\right] \tag{76}$$

$$\Delta\Omega = \frac{r\sin u}{a^2\nu\sqrt{e^2-1}\sin i}\,.\,\Delta v_W \tag{77}$$

$$\Delta\varpi = -\frac{\sqrt{e^2-1}}{a\nu e}\left[\cos f\,.\,\Delta v_S - \left(\frac{1}{r} + \frac{1}{a(e^2-1)}\right) r\sin f\,.\,\Delta v_T + \frac{er\tan\frac{i}{2}\sin u}{a(e^2-1)}\,.\,\Delta v_W\right] \tag{78}$$

$$\Delta i = \frac{r\cos u}{a^2\nu\sqrt{e^2-1}}\,.\,\Delta v_W\,. \tag{79}$$

4. Discussion

The subject of the present paper is the change in the orbital elements of a rocket moving in a hyperbolic trajectory due to a small impulse. We calculated the Lagrange's brackets that appear in the partial derivatives of the perturbation function with respect to the orbital elements. Then we computed the expressions for a, e, χ, Ω, ω, $\frac{di}{dt}$ which are the Lagrange's equations for the hyperbolic trajectory.

Moreover we derived the modified hyperbolic equations when a, e, χ, Ω, ω, i are replaced by a, e, ϵ, Ω, ϖ, i. Also, for the hyperbolic motion, we should acquire the Gaussian form, for the partial derivatives of the disturbing function with respect to a, e, i, ϵ, Ω, ϖ. We computed the Gauss' equation of the first form for hyperbolic orbits, from which, we finally obtained the differential changes in the hyperbolic orbital elements, due to a small impulse applied to the rocket's hyperbolic trajectory by the substitutions $S = \frac{\Delta v_S}{\Delta t}$, $T = \frac{\Delta v_T}{\Delta t}$, $W = \frac{\Delta v_W}{\Delta t}$.

The impulse change in velocity $\Delta\mathbf{v}$ splits into three components as shown in the analysis of the problem, and we have $\Delta\mathbf{v} = \Delta\mathbf{v}_S + \Delta\mathbf{v}_T + \Delta\mathbf{v}_W$, where $\Delta\mathbf{v}_W$, is perpendicular to the orbital plane and $\Delta\mathbf{v}_S$, $\Delta\mathbf{v}_T$ are along and at right angles to the radius vector and are lying in the orbital plane. Evidently, if $\Delta\mathbf{v}_W = 0$, the impulse does not alter the inclination or the longitude of the ascending node. The above treatment is a completely rigorous mathematical one. No approximations or non-closed forms are dealt with, through the analysis. This research work is the first publication about the importance of applying a small impulse in transfer and rendezvous problems for hyperbolic trajectories. Most of the involved equations are new, since we are concerned with the hyperbolic motion.

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