Mechanics and Mechanical Engineering
Vol. 7, No. 2 (2004) 77–87

The Uniqueness and Reciprocity Theorems for Generalized Thermo-Viscoelasticity with Thermal Relaxation Times

Mohamed I.A. OTHMAN
Faculty of Education, Department of Mathematics
Salalah-211, P.O. Box 2801, Sultanate of Oman
e-mail: m_i_othman@yahoo.com

Received (22 September 2003)
Revised (27 October 2003)
Accepted (15 December 2003)

The equations of generalized thermo-viscoelasticity based on Lord-Shulman (L-S), Green and Lindsay (G-L) and Classical dynamical coupled (CD) theories are given. Using Laplace transforms, a uniqueness theorem for these equations is proved. Also, a reciprocity theorem is obtained.

Keywords: uniqueness, receprocity, thermo-viscoelasticity, relaxation times.

1. Introduction

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of parabolic type predicting infinite speeds of propagation for heat waves [1].

Lord and Shulmann [2] introduced the theory of generalized thermoelasticity with one relaxation time by postulating a new law of heat conduction to replace the classical Fourier law. This new law contains the heat flux vector as well as its time derivative. It contains also a new constant that acts as a relaxation time. The heat equation of this theory is of the wave-type, ensuring finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motions and constitutive relations remain the same as those for the coupled and the uncoupled theories. Dhaliwal and Sherief [3] extended this theory to general anisotropic media in the presence of heat sources. In two-dimensional generalized thermo-viscoelasticity this theory was used recently by Ezzat *et al.* [4].

Müller [5] was the first introduced the theory of generalized thermoelasticity with two relaxation times. A more explicit version was then introduced by Green and Laws [6], Green and Lindsay [7] and independently by Suhubi [8]. In this theory the temperature rates are considered among the constitutive variables. This theory also predicts finite speeds of propagation as in Lord-Shulmann's theory. Ignaczak [9] studied a strong discontinuity wave and obtained a decomposition theorem for this theory [10]. Ezzat *et al.* [11] established the model of one-dimensional equations of generalized thermo-viscoelasticity with two relaxation times. Ezzat *et al.* [12] studied the model of two-dimensional equations of generalized thermo-viscoelasticity with two relaxation times. The state space formulation for these investigations is introduced. Ezzat and Othman [13] have studied some problems in magneto-thermoelasticity with thermal relaxation in a medium of perfect conductivity.

Gross [14], Staverman and Schwarzl [15], Alfery and Gurnee [16] and Ferry [17] investigated the mechanical-model representation of linear viscoelastic behaviour results. A reciprocity theorem for the theory of viscoelasticity was derived by Fung [18] and Pobedria [19] derived the reciprocity theorem for the coupled thermo-viscoelasticity.

2. Formulation of the fundamental equations

In the isotropic thermo-viscoelastic medium, [20,21] give the constitutive equation

$$S_{ij} = \int_0^t R(t-\tau)\frac{\partial e_{ij}(x,\tau)}{\partial \tau}\, d\tau = \hat{R}(\mathrm{e}_{ij})\,. \tag{1}$$

where $R(t)$ is the relaxation function such that $R(0) = 2\mu$, $R(\infty) > 0$,

$$S_{ij} = \sigma_{ij} - \sigma\delta_{ij}\,, \quad \mathrm{e}_{ij} = \varepsilon_{ij} - \frac{\mathrm{e}}{3}\delta_{ij}\,, \quad \mathrm{e} = \varepsilon_{kk}\,, \quad \sigma = \frac{\sigma_{kk}}{3}\,, \quad \sigma_{ij} = \sigma_{ji}\,, \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\,, \tag{3}$$

with the assumption [20]

$$\sigma_{ij}(x,t) = 0\,, \quad \varepsilon_{ij}(x,t) = 0\,, \quad -\infty < t < 0\,.$$

Assuming that the relaxation effects of the volume properties of the material are ignored [18], one can write for the generalized theory of thermo-viscoelasticity with thermal relaxation times

$$\sigma = Ke - \gamma(T - T_0 + \nu\dot{T})\,. \tag{4}$$

The equation of motion

$$\sigma_{ij,j} + F_i = \rho\ddot{u}_i\,. \tag{5}$$

The generalized heat conduction equation

$$kT_{,ii} = \rho C_E(\dot{T} + \tau_0\ddot{T}) + \gamma T_0(\dot{e} + \tau_0\delta\ddot{e}) - Q \tag{6}$$

where k, C_E, μ, γ, K, T_0, ρ, τ_0, ν are positive constants, together with the previous equations constitutes a complete system of generalized thermo-viscoelasticity with thermal relaxation times for isotropic medium.

In the above equations a dot denotes differentiation with respect to time, while a comma denotes material derivatives. The summation notation is used.

Moreover, the use of the relaxation times τ_0, ν and a parameter δ makes the aforementioned fundamental equations possible for the three different theories:

1. Classical Dynamical Coupled theory [1]: $\tau_0 = 0$, $\nu = 0$, $\delta = 0$.

2. Lord-Shulman's theory [2]: $\nu = 0$, $\tau_0 > 0$, $\delta = 1$.

3. Green-Lindsay's theory [7]: $\nu \geq 0$, $\tau_0 > 0$, $\delta = 0$.

3. Uniqueness theorem

Assuming that a linear isotropic thermo-viscoelastic material occupies a regular region D [22] with boundary surface B in the three-dimensional space, there is only one system of functions: $u_i(x,t)$, $T(x,t)$ of class $C^{(2)}$ and $\sigma_{ij}(x,t)$, $\varepsilon_{ij}(x,t)$ of class $C^{(1)}$, in the point $P \in (D+B)$ having coordinates $x = (x_1, x_2, x_3)$ at $t \geq 0$, which satisfy Eqs. (1), (3) and (4) for $x \in (D+B)$, $t \geq 0$ and (5) and (6) for $x \in D$, $t > 0$, with the boundary conditions

$$T = \Phi^{(1)}(x_B, t)\,, \quad u_i = G_i^{(1)}(x_B, t)\,, \quad x_B \in B\,, \quad t > 0\,, \tag{7}$$

and the initial conditions

$$T = \Phi^{(2)}(x, 0)\,, \quad u_i = G_i^{(2)}(x, 0)\,, \quad \dot{u}_i = G_i^{(3)}(x, 0)\,, \quad x \in D\,, \quad t = 0\,, \tag{8}$$

Let $u_i^{(1)}$, $T^{(1)}$, ... be two solution sets of Eqs. (1)–(6) with the same body forces, the same relaxation function, the same boundary conditions (7) and the same initial conditions (8). Consider the difference functions

$$u_i^* = u_i^{(1)} - u_i^{(2)}\,, \quad T^* = T^{(1)} - T^{(2)}\,, \quad \varepsilon_{ij}^* = \varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(2)}\,, \ldots \tag{9}$$

which satisfy Eqs. (1)–(3), thus Eqs. (4)–(6) for the difference functions become

$$\sigma_{ij,j}^* = \rho \ddot{u}_i^*\,, \quad \sigma^* = K e^* - \gamma \left(T^* + \nu \dot{T}^* \right)\,. \tag{10}$$

$$k T_{,ii}^* = \rho C_E \left(\dot{T}^* + \tau_0 \ddot{T}^* \right) + \gamma T_0 \left(\dot{e}^* + \tau_0 \delta \ddot{e}^* \right) \tag{11}$$

The difference functions (9) satisfy the homogeneous boundary conditions, thus

$$T^*(x_B, t) = 0\,, \quad u_i^*(x_B, t) = 0\,, \quad x_B \in B\,, \quad t > 0\,, \tag{12}$$

$$T^*(x, 0) = 0\,, \quad u_i^*(x, 0) = 0\,, \quad \dot{u}_i^*(x, 0) = 0\,, \quad x \in D\,, \quad t = 0\,, \tag{13}$$

The Laplace transform of the difference functions (9) with real s is defined as

$$\bar{f}^*(x, s) = \int_0^\infty e^{-st} f^*(x, t)\, dt\,, \quad s > 0\,. \tag{14}$$

Applying (14) to the system of equations obtained for the difference functions and omitting the asterisks and bars since the following analysis concerns only the difference functions we get

$$\sigma_{ij,j} = s^2\rho, u_i\,, \tag{15}$$

$$kT_{,ii} = \rho C_E s(1+\tau_0 s)T + \gamma T_0 s(1+\tau_0\delta s)e\,, \tag{16}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})\,, \tag{17}$$

$$S_{ij} = sRe_{ij}\,, \quad S_{ij} = \sigma_{ij} - \sigma\delta_{ij}\,, \quad e_{ij} = \varepsilon_{ij} - \frac{e}{3}\delta_{ij}\,, \quad \sigma = Ke - \gamma(1+\nu s)T\,, \tag{18}$$

$$T(x_B, s) = 0\,, \quad u_i(x_B, s) = 0\,, \quad x_B \in B\,. \tag{19}$$

$R = R(s) > 0$ the Laplace transform of the relaxation function.

Consider the integral

$$\int_D \sigma_{ij}\varepsilon_{ij}\,dV = \int_D \sigma_{ij}u_{i,j}\,dV = \int_D (\sigma_{ij}u_i)_{,j}\,dV - \int_D \sigma_{ij,j}u_i\,dV\,. \tag{20}$$

Using the divergence theorem and taking into considering Eq. (19) one obtains

$$\int_D (\sigma_{ij}u_i)_{,j}\,dV = \int_B u_i\sigma_{ij}n_j\,dA = 0\,, \tag{21}$$

where, n_j the outward unit normal to the surface B. Thus Eq. (20) takes the form

$$\int_D \sigma_{ij}\varepsilon_{ij}\,dV + \int_D \sigma_{ij,j}u_i\,dV = 0\,. \tag{22}$$

From Eq. (18) we obtain

$$\sigma_{ij}\varepsilon_{ij} = \sigma e + S_{ij}e_{ij} = Ke^2 + sRe_{ij}e_{ij} - \gamma(1+\nu s)Te\,. \tag{23}$$

Substituting from Eqs. (15) and (23) into Eq. (22) we obtain

$$\int_D \left[Ke^2 + sRe_{ij}e_{ij} + \rho s^2 u_i^2 - \gamma(1+\nu s)Te\right]\,dV = 0\,. \tag{24}$$

Since

$$TT_{,ii} = \left(TT_{,i}\right)_{,i} - T_{,i}T_{,i} \text{ and } \int_D \left(TT_{,i}\right)_{,i}\,dV = \int_B TT_{,i}n_i\,dA = 0\,, \tag{25}$$

we have

$$\int_D T\left(kT_{,ii}\right)_{,i}\,dV = -k\int_D T_{,i}T_{,i}\,dV\,. \tag{26}$$

Substituting from Eq. (16) into Eq. (26) we obtain

$$\int_D \gamma Te\,dV = -\frac{\rho C_E(1+\tau_0 s)}{T_0(1+\tau_0\delta s)}\int_D T^2\,dV - \frac{k}{sT_0(1+\tau_0\delta s)}\int_D T_{,i}T_{,i}\,dV\,. \tag{27}$$

Eq. (24) with Eq. (27) gives

$$\int_D \left[Ke^2 + sRe_{ij}e_{ij} + \rho s^2 u_i u_i + \left(\frac{m_1}{T_0(1+\tau_0\delta s)}\right) T^2 + \left(\frac{m_2}{T_0(1+\tau_0\delta s)}\right) T_{,i}T_{,i}\right] dV = 0\,, \tag{28}$$

where

$$m_1 = \rho C_E(1+\tau_0 s)(1+\nu s)\,, \quad m_2 = \frac{k(1+\nu s)}{s}\,.$$

The integrand function in Eq. (28), is the sum of squares, thus we conclude that

$$u_i = 0\,, \quad T = 0\,, \quad e = 0\,, \quad e_{ij} = 0\,,$$

i.e., the Laplace transforms of all the difference functions (9) are zeros and according to Learch's theorem [23] the inverse Laplace transform of each is unique, consequently

$$u_i^* = 0\,, \quad T^* = 0\,, \quad \varepsilon^* = 0\,, \quad \sigma_{ij}^* = 0\,.$$

This proves the uniqueness of the solution of the system of Eqs. (1)–(8) for the Classical Dynamical Coupled theory when $\tau_0 = 0$, $\nu = 0$, $\delta = 0$, Lord-Shulman' theory when $\nu = 0$, $\tau_0 > 0$, $\delta = 1$ and Green and Lindsay' theory when $\nu \geq 0$, $\tau_0 > 0$, $\delta = 0$.

4. Reciprocity theorem

We derive the dynamic reciprocity relationship for a generalized thermo-visco-elastic bounded body subjected to the action of a given body force $F_i(x,t)$, a given heat source $Q(x,t)$, surface traction T_i over a part of the surface B_σ, while over the other part B_u it is assigned the displacement g_i and heating of the surface $B = B_\sigma + B_u$ to a given temperature Φ, under zero initial conditions, the mentioned actions start at $t > 0$. Therefore, we assume the system of Eqs (1)–(6) to be given with the following boundary and initial conditions:

$$T_i^n = \sigma_{ij}n_j = f_i(x_B,t)\,, \quad x_B \in B_\sigma\,, \quad t > 0\,, \tag{29}$$

where n_j is the outward-pointing unit normal vector to B_σ,

$$u_i = g_i(x_B,t)\,, \quad x_B \in B_u\,, \quad t > 0\,, \tag{30}$$

$$\Theta = \Phi\,(x_B,t)\,, \quad x_B \in B\,, \quad t > 0\,, \tag{31}$$

$$u_i(x,t) = \frac{\partial u_i(x,t)}{\partial t}\,, \quad x \in D\,, \quad t \leq 0\,, \tag{32}$$

$$\Theta(x,t) = \frac{\partial \Theta(x,t)}{\partial t}\,, \quad x \in D\,, \quad t \leq 0\,, \tag{33}$$

$$F_i = Q(x,t) = 0\,, \quad x \in B\,, \quad t \leq 0\,, \tag{34}$$

$$\Phi(x_B,t) = 0 \quad x_B \in D\,, \quad t \leq 0\,, \tag{35}$$

$$f_i(x_B,t) = 0\,, \quad x_B \in D\,, \quad t \leq 0\,, \tag{36}$$

$$g_i(x_B, t) = 0 \quad x_B \in D\,, \quad t \leq 0\,, \tag{37}$$

where $\Theta = T - T_0$, f_i, g_i and Φ are the given functions.

Substituting from Eq. (4) into Eq. (1) and taking Eq. (2) into consideration we obtain

$$\sigma_{ij} = \hat{R}\left(\varepsilon_{ij} - \frac{e}{3}\delta_{ij}\right) + Ke\delta_{ij} - \gamma\left(\Theta + \nu\dot{\Theta}\right)\delta_{ij}\,. \tag{38}$$

Performing the Laplace transform (14) over the system of equations (1)–(6), (29)–(31) and (38) in view of Eqs. (32), (33) and omitting the bars, we get the following system in Laplace transform domain:

$$S_{ij} = sRe_{ij}\,, \tag{39}$$

$$S_{ij} = \sigma_{ij} - \sigma\delta_{ij}\,, \quad e_{ij} = \varepsilon_{ij} - \frac{e}{3}\delta_{ij}\,, \tag{40}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\,, \tag{41}$$

$$\sigma = Ke - \gamma(1 + \nu s)\Theta\,, \tag{42}$$

$$\sigma_{ij,j} + F_i = \rho s^2 u_i\,, \tag{43}$$

$$k\Theta_{,ii} = \rho C_E s(1 + \tau_0 s)T + \gamma T_0 s(1 + \tau\delta s)e - Q\,, \tag{44}$$

$$\sigma_{ij} = sR(\varepsilon_{ij} - \frac{e}{3}\delta_{ij}) + Ke\delta_{ij} - \gamma(1 + \nu s)\Theta\delta_{ij}\ ,, \tag{45}$$

$$u_i = g_i \text{ on } B_u\,, \quad \sigma_{ij}n_j = f_i \text{ on } B_\sigma\,, \quad \Theta = \Phi \text{ on } B\,. \tag{46}$$

Now consider two problems where applied body forces, heat sources, surface tractions, assigned surface displacements and surface temperature are specified differently. Let the variables involved in these two problems is distinguished by superscripts in parentheses. Thus, we have $u_i^{(1)}$, $\varepsilon_{ij}^{(1)}$, $e^{(1)}$, $\sigma_{ij}^{(1)}$, $\Theta^{(1)}, \ldots$ for the first problem and $u_i^{(2)}$, $\varepsilon_{ij}^{(2)}$, $e^{(2)}$, $\sigma_{ij}^{(2)}$, $\Theta^{(2)}, \ldots$ for the second problem. Each set of variables satisfies the system of Equations (39)–(46).

Using Eq. (41) the assumption $\sigma_{ij} = \sigma_{ji}$ and the divergence theorem we get

$$\begin{aligned}\int_D \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}\,dV &= \int_D \sigma_{ij}^{(1)}u_{i,j}^{(2)}\,dV = \int_D (\sigma_{ij}^{(1)}u_i^{(2)})_{,j}\,dV - \int_D \sigma_{ij,j}^{(1)}u_i^{(2)}\,dV \\ &= \int_B \sigma_{ij}^{(1)}n_j u_i^{(2)}\,dA - \int_D \sigma_{ij,j}^{(1)}u_i^{(2)}\,dV\end{aligned} \tag{47}$$

Substituting from Eqs. (43) and (46) into Eq. (47) we obtain

$$\begin{aligned}\int_D \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}\,dV \int_{B\sigma} f_i^{(1)}u_i^{(2)}\,dA \quad + \\ \int_{Bu} \sigma_{ij}^{(1)}n_j g_i^{(2)}\,dA - \int_D \rho s^2 u_i^{(1)}u_i^{(2)}\,dV + \int_D F_i^{(1)}u_i^{(2)}\,dV\,.\end{aligned} \tag{48}$$

A similar expression is obtained for the integral $\int_D \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}\,dV$, from which together with Eq. (48) it follows that

$$\int_D(\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}-\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)})\,dV=\int_{B\sigma}(f_i^{(1)}u_i^{(2)}-f_i^{(2)}u_i^{(1)})\,dA \quad +$$
$$\int_{Bu}(\sigma_{ij}^{(1)}g_i^{(2)}-\sigma_{ij}^{(2)}g_i^{(1)})n_j\,dA+\int_D(F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)})\,dV\,. \tag{49}$$

Now multiplying $\varepsilon_{ij}^{(2)}$ by the corresponding Eq. (45) for the first problem, $\varepsilon_{ij}^{(1)}$ by the analogous equation for the second problem, subtracting and integrating over the region D we obtain

$$\int_D(\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}-\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)})\,dV=\int_D sR\left[(\varepsilon_{ij}^{(1)}-\frac{e^{(1)}}{3}\delta_{ij})\varepsilon_{ij}^{(2)}-(\varepsilon_{ij}^{(2)}-\frac{e^{(2)}}{3}\delta_{ij})\varepsilon_{ij}^{(1)}\right]\,dV+$$
$$K\int_D(e^{(1)}\varepsilon_{ij}^{(2)}\delta_{ij}-e^{(2)}\varepsilon_{ij}^{(1)}\delta_{ij})\,dV-\gamma(1+\nu s)\int_D(\Theta^{(1)}\varepsilon_{ij}^{(2)}\delta_{ij}-\Theta^{(2)}\varepsilon_{ij}^{(1)}\delta_{ij})\,dV \tag{50}$$

Since $\varepsilon_{ij}\delta_{ij}=e$, we obtain

$$\int_D(\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}-\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)})\,dV=\gamma(1+\nu s)\int_D(\Theta^{(2)}e^{(1)}-\Theta^{(1)}e^{(2)})\,dV\,. \tag{51}$$

From Eqs. (50) and (51) we get the first part of the reciprocity theorem:

$$\int_{B\sigma}(f_i^{(1)}u_i^{(2)}-f_i^{(2)}u_i^{(1)})\,dA+\int_{Bu}(\sigma_{ij}^{(1)}n_jg_i^{(2)}-\sigma_{ij}^{(2)}n_jg_i^{(1)})\,dA+\,,$$
$$\int_D(F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)})\,dV+\gamma(1+\nu s)\int_D(\Theta^{(1)}e^{(2)}-\Theta^{(2)}e^{(1)})\,dV=0\,. \tag{52}$$

which contains the mechanical causes of motion F_i, f_i and the prescribed surface displacements g_i.

To derive the second part we multiply $\Theta^{(2)}$ by the corresponding equation (44) for the first problem, $\Theta^{(1)}$ by the analogous equation for the second problem, subtracting and integrating over D, we obtain

$$k\int_D(\Theta^{(2)}\Theta_{,ii}^{(1)}-\Theta^{(1)}\Theta_{,ii}^{(2)})\,dV=\gamma T_0s(1+\tau_0s\delta)\int_D(\Theta^{(2)}e^{(1)}-\Theta^{(1)}e^{(2)})\,dV-$$
$$\int_D(\Theta^{(2)}Q^{(1)}-\Theta^{(1)}Q^{(2)})\,dV\,, \tag{53}$$

since,

$$\Theta^{(2)}\Theta_{,ii}^{(1)}=(\Theta^{(2)}\Theta_{,ii}^{(1)})_{,i}-\Theta_{,i}^{(1)}\Theta_{,i}^{(2)} \text{ and } \Theta^{(1)}\Theta_{,ii}^{(2)}=(\Theta^{(1)}\Theta_{,ii}^{(2)})_{,i}-\Theta_{,i}^{(2)}\Theta_{,i}^{(1)} \tag{54}$$

Eq. (53) can be written, using the divergence theorem and Eq. (46), in the form

$$k\int_B(\Phi^{(2)}\Theta_{,N}^{(1)}-\Phi^{(1)}\Theta_{,N}^{(2)})\,dA-\gamma T_0s(1+\tau_0s\delta)\int_D(\Theta^{(2)}e^{(1)}-\Theta^{(1)}e^{(2)})\,dV+$$

$$\int_D (\Theta^{(2)}Q^{(1)} - \Theta^{(1)}Q^{(2)})\, dV = 0\,, \tag{55}$$

where $\Theta_{,N} = \Theta_{,i}N_i$, the derivative of Θ in the direction of the normal to the surface B, N_i is the outward-pointing unit normal to the surface B. Eq. (55) constitutes the second part of reciprocity theorem which contains the thermal causes of motion Q, Φ.

Combining (52) and (55) we obtain the general reciprocity theorem in Laplace transform domain:

$$T_0 s(1+\tau_0 s\delta)\left\{\int_{B\sigma}(f_i^{(1)}u_i^{(2)} - f_i^{(2)}u_i^{(1)})\, dA + \int_{Bu}(\sigma_{ij}^{(1)}n_j g_i^{(2)} - \sigma_{ij}^{(2)}n_j g_i^{(1)})\, dA + \int_D (F_i^{(1)}u_i^{(2)} - F_i^{(2)}u_i^{(1)})\, dV\right\} - k(1+\nu s)\int_D(\Theta_{,N}^{(1)}\Phi^{(2)} - \Theta_{,N}^{(2)}\Phi^{(1)})\, dA - (1+\nu s)\int_D (Q^{(1)}\Theta^{(2)} - Q^{(2)}\Theta^{(1)})\, dV = 0 \tag{56}$$

Using the convolution theorem [23]:

$$\pounds^{-1}\{F(s)G(s)\} = \int_0^t f(x,t-\tau)g(x,\tau)\, d\tau = \int_0^t g(x,t-\tau)f(x,\tau)\, d\tau\,, \tag{57}$$

using the symbolic notation

$$L(f) = f(x,\tau) + \nu\frac{\partial f(x,\tau)}{\partial\tau} \quad , \tag{58}$$

and inverting Eq. (52) we obtain the first part of reciprocity theorem in the final form

$$\int_D\int_0^t F_i^{(1)}(x,t-\tau)u_i^{(2)}(x,\tau)\, d\tau\, dV + \int_{B\sigma}\int_0^t f_i^{(1)}(x,t-\tau)u_i^{(2)}(x,\tau)\, d\tau\, dA + \int_{Bu}\int_0^t \sigma_{ij}^{(1)}(x,t-\tau)n_j g_i^{(2)}(x,\tau)\, d\tau\, dA + \gamma\int_D\int_0^t \Theta^{(1)}(x,t-\tau)L(e^{(2)})\, d\tau\, dV =$$
$$\int_D\int_0^t F_i^{(2)}(x,t-\tau)u_i^{(1)}(x,\tau)\, d\tau\, dV + \int_{B\sigma}\int_0^t f_i^{(2)}(x,t-\tau)u_i^{(1)}(x,\tau)\, d\tau\, dA + \int_{Bu}\int_0^t \sigma_{ij}^{(2)}(x,t-\tau)n_j g_i^{(1)}(x,\tau)\, d\tau\, dA + \gamma\int_D\int_0^t \Theta^{(2)}(x,t-\tau)L(e^{(1)})\, d\tau\, dV\,, \tag{59}$$

and inverting Eq.(55) we obtain the second part of reciprocity theorem in the final form

$$\gamma T_0(1+\tau_0\delta s)\int_D\int_0^t \Theta^{(1)}(x,t-\tau)\frac{\partial e^{(2)}(x,\tau)}{\partial\tau}\, d\tau\, dV + \int_D\int_0^t Q^{(1)}(x,t-\tau)\Theta^{(2)}(x,\tau)\, d\tau\, dV - k\int_B\int_0^t \Phi^{(1)}(x,t-\tau)\Theta_{,N}^{(2)}(x,\tau)\, d\tau\, dV =$$
$$\gamma T_0(1+\tau_0\delta s)\int_D\int_0^t \Theta^{(2)}(x,t-\tau)\frac{\partial e^{(1)}(x,\tau)}{\partial\tau}\, d\tau\, dV +$$

$$\int_D \int_0^t Q^{(2)}(x, t-\tau)\Theta^{(1)}(x,\tau)\, d\tau\, dV - k\int_B \int_0^t \Phi^{(2)}(x,t-\tau)\Theta^{(1)}_{,N}(x,\tau)\, d\tau\, dV\,. \quad (60)$$

Finally, inverting (56) we obtain the general reciprocity theorem in the form

$$\begin{aligned}
&\int_D \int_0^t F_i^{(1)}(x,t-\tau)\frac{\partial u_i^{(2)}(x,\tau)}{\partial \tau}\, d\tau\, dV + \int_{B\sigma}\int_0^t f_i^{(1)}(x,t-\tau)\frac{\partial u_i^{(2)}(x,\tau)}{\partial \tau}\, d\tau\, dA + \\
&\int_{Bu}\int_0^t \sigma_{ij}^{(1)}(x,t-\tau)n_j \frac{\partial g_i^{(2)}(x,\tau)}{\partial\tau}\, d\tau\, dA + \\
&\frac{k}{T_0(1+\tau_0\delta s)}\int_B \int_0^t \Phi^{(1)}(x,t-\tau)L(\Theta^{(2)}_{,N})\, d\tau\, dA - \\
&\frac{1}{T_0(1+\tau_0\delta s)}\int_D\int_0^t Q^{(1)}(x,t-\tau)L(\Theta^{(2)})\, d\tau\, dV = \\
&\int_D \int_0^t F_i^{(2)}(x,t-\tau)\frac{\partial u_i^{(1)}(x,\tau)}{\partial \tau}\, d\tau\, dV + \int_{B\sigma}\int_0^t f_i^{(2)}(x,t-\tau)\frac{\partial u_i^{(1)}(x,\tau)}{\partial \tau}\, d\tau\, dA + \\
&\int_{Bu}\int_0^t \sigma_{ij}^{(2)}(x,t-\tau)n_j \frac{\partial g_i^{(1)}(x,\tau)}{\partial\tau}\, d\tau\, dA + \\
&\frac{k}{T_0(1+\tau_0\delta s)}\int_B \int_0^t \Phi^{(2)}(x,t-\tau)L(\Theta^{(1)}_{,N})\, d\tau\, dA - \\
&\frac{1}{T_0(1+\tau_0\delta s)}\int_D\int_0^t Q^{(2)}(x,t-\tau)L(\Theta^{(1)})\, d\tau\, dV\,. \qquad (61)
\end{aligned}$$

In the particular case of an infinite thermo-viscoelasticity medium, assuming that the body forces and the heat sources act only in a bounded region, the surface integrals are absent and Eq. (56) takes the form

$$T_0 s(1+\tau_0 s\delta)\int_D \left(F_i^{(1)}u_i^{(2)} - F_i^{(2)}u_i^{(1)}\right) dV = (1+\nu s)\int_D \left(Q^{(1)}\,\Theta^{(2)} - Q^{(2)}\Theta^{(1)}\right) dV\,. \quad (62)$$

Inverting Eq. (62) we get

$$\begin{aligned}
&\int_D \int_0^t F_i^{(1)}(x,t-\tau)\frac{\partial u_i^{(2)}(x,\tau)}{\partial\tau}\, d\tau\, dV - \\
&\frac{1}{T_0(1+\tau_0 s\delta)}\int_D\int_0^t Q^{(1)}(x,t-\tau)L(\Theta^{(2)})\, d\tau\, dV = \\
&\int_D \int_0^t F_i^{(2)}(x,t-\tau)\frac{\partial u_i^{(1)}(x,\tau)}{\partial\tau}\, d\tau\, dV - \\
&\frac{1}{T_0(1+\tau_0 s\delta)}\int_D\int_0^t Q^{(2)}(x,t-\tau)L(\Theta^{(1)})\, d\tau\, dV\,. \qquad (63)
\end{aligned}$$

This proves the reciprocity theorem of the solution of the system of Eqs. (39)–(45) for the Classical Dynamical Coupled theory when $\tau_0 = 0$, $\nu = 0$, $\delta = 0$, Lord-Shulman' theory when $\nu = 0$, $\tau_0 > 0$, $\delta = 1$ and Green and Lindsay' theory when $\nu \geq 0$, $\tau_0 > 0$, $\delta = 0$.

References

[1] **Biot, M**: Thermoelasticity and irreversible thermodynamics, *J. Appl. Phys.*, (1956), **27**, 240.

[2] **Lord, H**, **Shulman, Y**: A Generalized dynamical theory of thermo-elasticity, *J. Mech. Phys. Solid*, (1976), **15**, 299-309.

[3] **Dhaliwal, R**, **Sherief, H**: Generalized thermoelasticity for an isotropic media, *Quart. Appl. Math.*, (1980), **33**, 1-8.

[4] **Ezzat, MA**, **Othman, MIA**, **El Karamany, AS**: State space approach to two-dimensional generalized thermo-viscoelasticity with one relaxation time, *J. of Thermal Stresses*, (2002), **25**, 295-316.

[5] **Müller, I**: The Coldness, A Universal function in thermo-elastic Solids, *Arch. Rat. Mech. Anal.*, (1971), **41**, 319.

[6] **Green, AE**, **Laws, N**: On the entropy production inequality, *Arch. Rat. Mech. Anal.*, (1972), **45**, 47.

[7] **Green, AE**, **Lindsay, KA**: Thermoelasticity, *Journal of Elasticity*, (1972), **2**, 1.

[8] **Suhubi, ES**: Thermoelastic solids, in: *Continuum Physics II*, Ch.2, **Eringen, AC**, Ed.,, (1975), Academic, Press, New York.

[9] **Ignaczak, J**: A strong discontinuity wave in thermoelastic with relaxation times, *J. Thermal Stresses*, (1985), **8**, 25-40.

[10] **Ignaczak, J**: Decomposition theorem for thermoelasticity with finite wave speeds, *J. Thermal Stresses*, (1985), **1**, 41.

[11] **Ezzat, MA**, **Othman, MIA**, **El Karamany, AS**: State space approach to generalized thermo-viscoelasticity with two relaxation times, (2002), *Int. J. Engng. Sci.*, **40**, 283-302.

[12] **Ezzat, MA**, **Othman, MIA**, **El Karamany, AS**: State space approach to two-dimensional generalized thermo-viscoelasticity with two relaxation times, *J. Engng. Sci.*, (2002), **40**, 1251-1274.

[13] **Ezzat, MA**, **Othman, MIA**: State space approach to generalized magneto- thermoelasticity with thermal relaxation in a medium of perfect conductivity, *J. Thermal Stresses*, (2002), **25**, 409-429.

[14] **Gross, B**: *Mathematical structure of the theories of viscoelasticity*, (1953), Hermann, Paris.

[15] **Staverman, AJ**, **Schwarzl, F**: *Die physik der hochpolymeren*, **Stuart, HA**, Ed., (1956), **4**, Springer Verlag, Chapter 1.

[16] **Alfrey, T**, **Gurnee, EF**: *Rheology theory and applications*, **Eirich, FR**, Ed., **1**, (1956), Academic Press, New York.

[17] **Ferry, JD**: *Viscoelastic properties of polymers*, (1970), John Wiley & Sons, New York.

[18] **Fung, YC**: *Foundation of solid mechanics*, (1980), Prentice-Hall, Englewood Cliffs, NJ.

[19] **Pobedria, BE**: Coupled problems in continuum mechanics, *J. Durabilility and Plasticity*, Moscow State University, Moscow.

[20] **Sternberg, E**: On the analysis of thermal stresses in viscoelastic solids, *Brown Univ. Dir, J. Appl. Math. TR.*, (1963), **9**, 213-219.

[21] **Ilioushin, AA**, **Pobederia, BE**: *Mathematical theory of thermal visco-elastic*, (1970), Nauka, Moscow.

[22] **Boley, B**, **Weiner, JH**: *Theory of thermal stresses*, (1960), John Wiley & Sons, Inc. New York.

[23] **Churchill, RV**: *Operational mathematics*, third ed., (1972), McGraw-Hill, NewYork.

Nomenclature

λ, μ	Lamé's constants,
$K = \lambda + \frac{2}{3}\mu$	bulk modulus,
ρ	density,
C_E	specific heat at constant strain,
t	time,
T	absolute temperature,
T_0	reference temperature chosen so that $\left\lvert \frac{T-T_0}{T_0} \right\rvert \ll 1$,
u_i	components of displacement vector,
ε_{ij}	components of strain tensor,
σ_{ij}	components of stress tensor,
S_{ij}	components of stress deviator tensor,
e_{ij}	components of strain deviator tensor,
k	thermal conductivity,
τ_0, ν	two relaxation times,
α_T	coefficient of linear thermal expansion,
γ	$3K\alpha_T$,
Q	the strength of the applied heat source per unit mass.