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Generalized Thermoelasticity Plane Waves in Rotating Media with Thermal Relaxation under the Temperature Dependent Properties

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A two-dimensional coupled problem in generalized thermoelasticity for rotating media under the temperature dependent properties is studied. The problem is in the context of the Lord-Shulman's theory with one relaxation time. The normal mode analysis is used to obtain the expressions for the temperature distribution, displacement components and thermal stresses. The resulting formulation is applied to two different problems. The first concerns the case of a heat punch moving across the surface of a semi-infinite thermoelastic half-space subjected to appropriate boundary conditions. The second deals with a thick plate subject to a time-dependent heat source on each face. Numerical results are illustrated graphically for each problem considered. Comparisons are made with the results obtained predicted by the two theories in case of absence of rotation.

Keywords: Generalized thermoelasticity, rotating media, thermal relaxation, temperature dependent properties.

1. Introduction

In recent years due to the progress in various fields in science and technology the necessity of taking into consideration the real behavior of the material characteristics become actual. We have seen a rapid development of thermoelasticity stimulated by various engineering sciences [1]. Most of investigations were done under the assumption of the temperature-independent material properties, which limit the applicability of the solutions obtained to certain ranges of temperature. At high temperature the material characteristics such as the modulus of elasticity, the Poisson's ratio, the coefficient of thermal expansion and the thermal conductivity are no longer constants. In some investigations they were taken as functions of coordinates [2] and [3].

In this work we consider the modulus of elasticity is the only temperature-dependent material parameter. The experimental data [4] show that the changes of Poisson's ratio and the coefficient of linear thermal expansion due to the high temperature can be neglected.

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of parabolic type predicting infinite speeds of propagation for heat waves.

Biot [5] introduced the theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming since heat equation for the coupled theory is also parabolic.

Two generalizations to the coupled theory are introduced. The first is due to Lord and Shulman [6] who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier's law. This new law contains the heat flux vector as well as its time derivative. It contains also a new constant that acts as a relaxation time. Since the heat equation of this theory is of the wave-type, it automatically ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motions and constitutive relations remain the same as those for the coupled and the uncoupled theories. Following the Lord-Shulman's theory, several authors including Puri [7], Ezzat et al. [8] studied the plane thermoelastic wave propagations in a medium of perfect conductivity. This theory was extended by Dhaliwal and Sherief [9] to general anisotropic media in the presence of heat sources. Sherief and Dhaliwal [10] solved a thermal shock problem. Nayfeh and Nasser [11] used Lord-Shulman theory to study plane thermo-elastic surface waves in a half-space. Ezzat et al. [12] studied the effect of reference temperature on thermal distribution for one-dimensional problem. Recently, Othman [13] established the model of equations of generalized thermo-elasticity based on the Lord-Shulman theory in an isotropic elastic medium under the dependence of the modulus of elasticity on reference temperature.

The second generalization to the coupled theory of the thermoelasticity is what known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate-dependent thermoelasticity. Müller [14] in a review of thermodynamic of thermoelastic solids has proposed an entropy production inequality, with the help of which, he considered restrictions on a class of constitutive equations.

A generalization of this inequality was proposed by Green and Laws [15]. Green and Lindsay have obtained an explicit version of the constitutive equations in [16]. These equations were also obtained independently by Erbay and Şuhubi [17]. This theory contains two constants that act as relaxation times and modifies all the equations of the coupled theory not the heat equation only. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Using this theory, Agarwal [18], [19] considered, respectively, thermoelastic and magneto-thermoelastic plane wave propagation in an infinite elastic medium. Ezzat and Othman [20] studied the electromagneto-thermoelasticity plane wave with two relaxation times. Othman [21] applied the Green-Lindsay's theory to study the effect of rotation and relaxation time on plane waves in generalized thermoelasticity. Ignaczak [22] studied a strong discontinuity wave and obtained a decomposition theorem for this theory [23].

In the present paper we shall use the normal mode method to problems of generalized thermoelasticity with thermal relaxation in an isotropic medium when the modulus of elasticity is taken as a linear function of reference temperature under the effect of rotation. The resulting formulation is applied in case of two dimensional to two concrete problems. The exact expressions for temperature, displacement components and thermal stress are obtained for the two problems considered.

2. Formulation of the problem

We shall consider an infinite isotropic, homogeneous, thermally conducting elastic medium. The medium is rotating uniformly with an angular velocity $\mathbf{\Omega} = \Omega\mathbf{n}$, where $\mathbf{n}$ is a unit vector representing the direction of the axis of rotation. The displacement equation of motion in the rotating frame of reference has two additional terms [24]; centripetal acceleration $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u})$ due to the time-varying motion only and the Coriolis acceleration $2\mathbf{\Omega} \times \dot{\mathbf{u}}$ where $\mathbf{u}$ is the dynamic displacement vector measured from a steady state deformed position and supposed to be small. These two terms do not appear in the equations for non-rotating media.

The constitutive law for the theory of generalized thermo-elasticity with one relaxation time is

$$\sigma_{ij} = \lambda \mathrm{e}\delta_{ij} + 2\mu\varepsilon_{ij} - \gamma (T - T_0)\,\delta_{ij}\,, \tag{1}$$

The equation of heat conduction

$$k\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)(\rho C_E T + \gamma T_0 \mathrm{e})\,, \tag{2}$$

The strain-displacement constitutive relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

and

$$\varepsilon_{ii} = \mathrm{e} = u_{i,i} \tag{3}$$

We assume that

$$E = E_0 f(T), \quad \lambda = E_0\lambda_0 f(T), \quad \mu = E_0\mu_0 f(T) \text{ and } \gamma = E_0\gamma_0 f(T)\,. \tag{4}$$

where $f(T)$ is a given non-dimensional function of temperature, in case of temperature-independent modulus of elasticity $f(T) \equiv 1$, and $E = E_0$.

The equations of motion, in the absence of body forces, are

$$\sigma_{ij,j} = \rho\left[\ddot{u}_i + \{\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u})\}_i + (2\mathbf{\Omega} \times \dot{\mathbf{u}})_i\right]\,. \tag{5}$$

where all the terms have the same significance as in [16].

The displacement equation of motion in the rotating frame of reference as

$$\begin{aligned}\rho\left[\ddot{u}_i + \{\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u})\}_i + (2\mathbf{\Omega} \times \dot{\mathbf{u}})_i\right] &= E_0 f[(\lambda_0 + \mu_0)e_{,i} + \mu_0\nabla^2 u_i - \gamma_0 T_{,i}] \qquad (6)\\ &\quad + E_0 f_{,j}[\lambda_0 e\delta_{ij} + 2\mu_0\varepsilon_{ij} - \gamma_0(T - T_0)\delta_{ij}]\,.\end{aligned}$$

Now we introduce the following non-dimensional variables

$$\begin{array}{llll} x' = c_0\eta_0 x\,, & y' = c_0\eta_0 y\,, & u' = c_0\eta_0 u\,, & v' = c_0\eta_0 v\,, \\ t' = c_0^2\eta_0 t\,, & \tau_0' = c_0^2\eta_0\tau_0\,, & \theta = \frac{\gamma_0 E_0}{\rho c_0^2}(T - T_0)\,, & \sigma_{ij}' = \frac{\sigma_{ij}}{\rho c_0^2}\,. \end{array} \tag{7}$$

Omitting the dashes for convenience, we have

$$\sigma_{ij} = [(2\beta - 1)e\delta_{ij} + (1-\beta)(u_{i,j} + u_{j,i}) - \theta\delta_{ij}]\, f(\theta)\,, \tag{8}$$

$$\begin{aligned} &[\ddot{u}_i + \{\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u})\}_i + (2\mathbf{\Omega} \times \dot{\mathbf{u}})_i] \\ &= \quad [\beta e_{,i} + (1-\beta)\nabla^2 u_i - \theta_{,i}] f(\theta) \\ &\quad + [(2\beta - 1)e f_{,i} + (1-\beta)(u_{i,j} + u_{j,i}) f_{,j} - \theta f_{,i}]\,. \end{aligned} \tag{9}$$

The heat conduction Eq. (2) can be rewritten by using Eq. (3) as:

$$\nabla^2\theta = (\dot{\theta} + \tau_0\ddot{\theta}) + \varepsilon_1 f(\theta)\delta_0(\dot{e} + \tau_0\ddot{e})\,. \tag{10}$$

We consider a special case when $|T - T_0| \ll 1$, $0 \le \delta_0 \le 1$ and $f(\theta) = (1 - \alpha^* T_0)$. The equations of motion in two-dimension take the form

$$\alpha\left[\frac{\partial^2 u}{\partial t^2} - \Omega^2 u - 2\Omega\dot{v}\right] = \nabla^2 u + \beta\frac{\partial^2 v}{\partial x\partial y} - \beta\frac{\partial^2 u}{\partial y^2} - \frac{\partial\theta}{\partial x}\,, \tag{11}$$

$$\alpha\left[\frac{\partial^2 v}{\partial t^2} - \Omega^2 v - 2\Omega\dot{u}\right] = \nabla^2 v + \beta\frac{\partial^2 u}{\partial x\partial y} - \beta\frac{\partial^2 v}{\partial x^2} - \frac{\partial\theta}{\partial y}\,. \tag{12}$$

The equation of heat conduction

$$\nabla^2\theta = (\dot{\theta} + \tau_0\ddot{\theta}) + \varepsilon(\dot{e} + \tau_0\ddot{e})\,. \tag{13}$$

$$\alpha\sigma_{xx} = \frac{\partial u}{\partial x} + (2\beta - 1)\frac{\partial v}{\partial y} - \theta\,, \tag{14}$$

$$\alpha\sigma_{yy} = \frac{\partial v}{\partial y} + (2\beta - 1)\frac{\partial u}{\partial x} - \theta\,, \tag{15}$$

$$\alpha\sigma_{xy} = (1-\beta)(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\,, \tag{16}$$

$$\alpha\sigma_{zz} = (2\beta - 1)e - \theta\,. \tag{17}$$

where

$$\alpha = \frac{1}{1 - \alpha^* T_0}\,, \quad \varepsilon = \varepsilon_1\delta_0(1 - \alpha^* T_0)\,. \tag{18}$$

Differentiating Eq. (11) with respect to x, and Eq. (12) with respect to y, then adding, we arrive at

$$\left[\nabla^2 - \alpha\left(\frac{\partial^2}{\partial t^2} - \Omega^2\right)\right] e = \nabla^2\theta + 2\Omega\frac{\partial\zeta}{\partial t}\,. \tag{19}$$

Differentiating Eq. (11) with respect to y, and Eq. (12) with respect to x, then subtracting, we arrive at

$$\left[(1-\beta)\nabla^2 - \alpha\left(\frac{\partial^2}{\partial t^2} - \Omega^2\right)\right]\zeta = -2\alpha\Omega\frac{\partial e}{\partial t}\,. \tag{20}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is Laplace's operator in a two-dimensional space and

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\,. \tag{21}$$

3. Normal mode analysis

The solution of the considered physical variables can be decomposed in terms of normal mode as the following form

$$[u, v, e, \zeta, \theta, \sigma_{ij}]\,(x, y, t) = \left[u^*(y), v^*(y), e^*(y), \zeta^*(y), \theta^*(y), \sigma^*_{ij}(y)\right]\exp(\omega t + iax)\,. \tag{22}$$

where ω is the (complex) time constant, i is imaginary unit, a is the wave number in the x-direction and $u^*(y)$, $v^*(y)$, $e^*(y)$, $\zeta^*(y)$, $\theta^*(y)$ and $\sigma^*_{ij}(y)$ are the amplitudes of the functions.

Using Eq. (22), we can obtain the following equations from Eqs. (13), (19) and (20) respectively

$$\left[D^2 - a^2 - \omega(1+\tau_0\omega)\right]\theta^*(y) = \varepsilon\omega(1+\tau_0\omega)e^*(y)\,, \tag{23}$$

$$\left[D^2 - a^2 - \alpha(\omega^2 - \Omega^2)\right]e^*(y) = (D^2 - a^2)\theta^*(y) + 2\Omega\omega\zeta^*\,, \tag{24}$$

$$\left[D^2 - a^2 - \beta_1(\omega^2 - \Omega^2)\right]\zeta^*(y) = -2\beta_1\omega\Omega e^*\,. \tag{25}$$

where

$$D = \frac{\partial}{\partial y}\,, \quad \beta_1 = \frac{\alpha}{1-\beta}\,. \tag{26}$$

Eliminating $\theta^*(y)$ and $\zeta^*(y)$ between Eqs. (23)–(25), we get the following sixth-order partial differential equation satisfied by $e^*(y)$

$$(D^6 - a_1 D^4 + a_2 D^2 - a_3)e^*(y) = 0\,, \tag{27}$$

where,

$$a_1 = 3a^2 + b_1\,, \tag{28}$$

$$a_2 = 3a^4 + 2a^2 b_1 + b_2\,, \tag{29}$$

$$a_3 = a^6 + a^4 b_1 + a^2 b_2 + b_3\,, \tag{30}$$

$$b_1 = (\alpha + \beta_1)\omega_2 + (\varepsilon + 1)\omega_1\,, \tag{31}$$

$$b_2 = \omega_2[\omega_1(\beta_1 + \alpha + \varepsilon\beta_1) + \alpha\beta_1\omega_2] + 4\omega^2\Omega^2\beta_1\,, \tag{32}$$

$$b_3 = \beta_1\omega_1[\alpha\omega_2^2 + 4\omega^2\Omega^2]\,, \tag{33}$$

$$\omega_1 = \omega(1+\tau_0\omega)\,, \quad \omega_2 = \omega^2 - \Omega^2\,. \tag{34}$$

Equation (27) can be factorized as

$$(D^2 - k_1^2)(D^2 - k_2^2)(D^2 - k_3^2)e^*(y) = 0\,. \tag{35}$$

where, k_j, $j = 1, 2, 3$ are the roots of the following characteristic equation

$$k^6 - a_1 k^4 + a_2 k^2 - a_3 = 0\,. \tag{36}$$

The solution of Eq. (35) is given by:

$$e^*(y) = \sum_{j=1}^{3} e_j^*(y)\,. \tag{37}$$

where $e_j^*(y)$ is the solution of the equation

$$(D^2 - k_j^2)e_j^*(y) = 0\,, \quad j = 1, 2, 3\,. \tag{38}$$

The solution of Eq. (38), which is bounded as $y \to \infty$, is given by

$$e_j^*(y) = R_j(a, \omega)e^{-k_j y}\,. \tag{39}$$

Substituting from Eq.(39) into the Eq. (37), we obtain:

$$e^*(y) = \sum_{j=1}^{3} B_j(a, \omega)e^{-k_j y}\,. \tag{40}$$

In a similar manner, we get

$$\theta^*(y) = \sum_{j=1}^{3} B'_j(a, \omega)e^{-k_j y}\,, \tag{41}$$

$$\zeta^*(y) = \sum_{j=1}^{3} B''_j(a, \omega)e^{-k_j y}\,. \tag{42}$$

where $B_j(a, \omega)$, $B'_j(a, \omega)$ and $B''_j(a, \omega)$ are parameters depending on a, ω.
Substituting from Eqs. (40)–(42) into Eqs. (23) and (25), we obtain

$$B'_j(a, \omega) = \frac{\varepsilon \omega_1}{k_j^2 - a^2 - \omega_1} B_j(a, \omega)\,, \quad j = 1, 2, 3\,, \tag{43}$$

$$B''_j(a, \omega) = \frac{-2\omega \Omega \beta_1}{k_j^2 - a^2 - \beta_1 \omega_2} B_j(a, \omega)\,, \quad j = 1, 2, 3\,. \tag{44}$$

Substituting from Eqs. (43) and (44) into Eqs. (41) and (42), respectively, we obtain

$$\theta^*(y) = \sum_{j=1}^{3} \frac{\varepsilon \omega_1}{k_j^2 - a^2 - \omega_1} B_j(a, \omega)e^{-k_j y}\,, \tag{45}$$

$$\zeta^*(y) = \sum_{j=1}^{3} \frac{-2\omega\Omega\beta_1}{k_j^2 - a^2 - \beta_1\omega_2} B_j(a,\omega)e^{-k_j y} \,. \tag{46}$$

Since,

$$e^* = iau^* + Dv^* \,, \tag{47}$$

$$\zeta^* = Du^* - iav^* \,. \tag{48}$$

In order to obtain the amplitude of displacement components u^* and v^*, which are bounded as $y \to \infty$, in terms of Eq. (22) from Eqs. (40), (46), (47) and (48) we can obtain

$$u^*(y) = Ce^{a_y} + \sum_{j=1}^{3} \frac{1}{k_j^2 - a^2} \left[ia + \frac{2\omega\Omega\beta_1 k_j}{k_j^2 - a^2 - \beta_1\omega_2} \right] B_j(a,\omega)e^{-k_j y} \,, \tag{49}$$

$$v^*(y) = -iCe^{a_y} - \sum_{j=1}^{3} \frac{1}{k_j^2 - a^2} \left[k_j - \frac{2ia\omega\Omega\beta_1}{k_j^2 - a^2 - \beta_1\omega_2} \right] B_j(a,\omega)e^{-k_j y} \,. \tag{50}$$

where $C = 0$ to make Eqs. (49) and (50) are bounded as $y \to \infty$.

In terms of Eq. (22), substituting from Eqs. (40), (45), (46), (49) and (50) into Eqs. (14)–(17), respectively, we obtain the stress components in the form

$$\begin{aligned} \sigma_{xx}^*(y) \quad = \quad & \sum_{j=1}^{3} \left\{ \left[\frac{(2\beta - 1)k_j^2 - a^2}{k_j^2 - a^2} - \frac{\varepsilon\omega_1}{k_j^2 - a^2 - \omega_1} \right] \right. \\ & \left. - \frac{4ia(\beta - 1)\omega\Omega\beta_1 k_j}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta_1\omega_2]} \right\} B_j e^{-k_j y} \,, \end{aligned} \tag{51}$$

$$\begin{aligned} \sigma_{yy}^*(y) \quad = \quad & \sum_{j=1}^{3} \left\{ \left[\frac{k_j^2 - (2\beta - 1)a^2}{k_j^2 - a^2} - \frac{\varepsilon\omega_1}{k_j^2 - a^2 - \omega_1} \right] \right. \\ & \left. - \frac{4ia(\beta - 1)\omega\Omega\beta_1 k_j}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta_1\omega_2]} \right\} B_j e^{-k_j y} \,, \end{aligned} \tag{52}$$

$$\sigma_{xy}^*(y) = (1-\beta) \sum_{j=1}^{3} \left\{ \frac{2\omega\Omega\beta_1(k_j^2 + a^2)}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta_1\omega_2]} + \frac{2iak_j}{k_j^2 - a^2} \right\} B_j e^{-k_j y} \,, \tag{53}$$

$$\sigma_{zz}^*(y) = \sum_{j=1}^{3} \left\{ 2\beta - 1 - \frac{\varepsilon\omega_1}{k_j^2 - a^2 - \omega_1} \right\} B_j e^{-k_j y} \,. \tag{54}$$

The normal mode analysis is, in fact, to look for the solution in Fourier transformed domain. This assumes that all the field quantities are sufficiently smooth on the real line such that the normal mode analysis of these functions exist.

4. Applications

4.1. *Problem I*

A time-dependent heat punch across the surface of semi-infinite thermo-elastic half space [1].

We consider a homogeneous isotropic thermoelastic solid occupying the region G given by $G = \{(x, y, z)| \, x, z \in \mathbf{R}|, \; y \geq 0\}$.

In the physical problem, we should suppress the positive exponentials that are unbounded at infinity. The constants B_j, $j = 1, 2, 3$ have to be chosen such that the boundary conditions on the surface $y = 0$ take the form

$$\theta(x, y, t) = n(x, t) \qquad \text{on } y = 0\,, \tag{55}$$

$$\sigma_{yy}(x, y, t) = P(x, t) \qquad \text{on } y = 0\,, \tag{56}$$

$$\sigma_{xy}(x, y, t) = 0 \qquad \text{on } y = 0\,, \tag{57}$$

where n, P are given functions of x and t.

Eqs. (55)–(57) in the normal mode form together with Eqs. (45), (52) and (53) respectively, give

$$L_1 B_1 + L_2 B_2 + L_3 B_3 = n^*(a, \omega)\,, \tag{58}$$

$$M_1 B_1 + M_2 B_2 + M_3 B_3 = P^*(a, \omega)\,, \tag{59}$$

$$N_1 B_1 + N_2 B_2 + N_3 B_3 = 0\,. \tag{60}$$

Eqs. (58)–(60) can be solved for the three unknowns B_1, B_2 and B_3.

The solution of these equations can be written as

$$B_1 = \frac{1}{\Delta_1^2 + \Delta_2^2}\left[(\mu_1 \Delta_1 + \mu_2 \Delta_2) + i(\mu_2 \Delta_1 - \mu_1 \Delta_2)\right], \tag{61}$$

$$B_2 = \frac{1}{\Delta_1^2 + \Delta_2^2}\left[(\mu_3 \Delta_1 + \mu_4 \Delta_2) + i(\mu_4 \Delta_1 - \mu_3 \Delta_2)\right], \tag{62}$$

$$B_3 = \frac{1}{\Delta_1^2 + \Delta_2^2}\left[(\mu_5 \Delta_1 + \mu_6 \Delta_2) + i(\mu_6 \Delta_1 - \mu_5 \Delta_2)\right], \tag{63}$$

where

$$L_j = \frac{\varepsilon \omega_1}{k_j^2 - a^2 - \omega_1} \qquad j = 1, 2, 3\,, \tag{64}$$

$$M_j = (\alpha_{1j} + i\beta_{1j})\,, \quad j = 1, 2, 3, \tag{65}$$

$$N_j = (\alpha_{2j} + i\beta_{2j})\,, \quad j = 1, 2, 3, \tag{66}$$

$$\alpha_{1j} = \left[\frac{k_j^2 - (2\beta - 1)a^2}{k_j^2 - a^2} - \frac{\varepsilon \omega_1}{k_j^2 - a^2 - \omega_1}\right], \quad j = 1, 2, 3\,, \tag{67}$$

$$\alpha_{2j} = \frac{2\omega \Omega \beta_1 (k_j^2 + a^2)}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta_1 \omega_2]}\,, \quad j = 1, 2, 3\,, \tag{68}$$

$$\beta_{1j} = \frac{4a(\beta - 1)\omega \Omega \beta_1 k_j}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta_1 \omega_2]}\,, \quad j = 1, 2, 3. \tag{69}$$

$$\beta_{2j} = \frac{2ak_j}{k_j^2 - a^2}\,, \quad j = 1, 2, 3\,, \tag{70}$$

$$\lambda_1 = \alpha_{12}\alpha_{23} - \beta_{12}\beta_{23} - \alpha_{13}\alpha_{22} + \beta_{13}\beta_{22}\,, \tag{71}$$

$$\lambda_2 = \alpha_{12}\beta_{23} + \alpha_{23}\beta_{12} - \alpha_{13}\beta_{22} - \alpha_{22}\beta_{13}\,, \tag{72}$$

$$\lambda_3 = \alpha_{11}\alpha_{23} - \beta_{11}\beta_{23} - \alpha_{13}\alpha_{21} + \beta_{13}\beta_{21}\,, \tag{73}$$

$$\lambda_4 = \alpha_{11}\beta_{23} + \alpha_{23}\beta_{11} - \alpha_{13}\beta_{21} - \alpha_{21}\beta_{13}\,, \tag{74}$$

$$\lambda_5 = \alpha_{11}\alpha_{22} - \beta_{11}\beta_{22} - \alpha_{12}\alpha_{21} + \beta_{12}\beta_{21}\,, \tag{75}$$

$$\lambda_6 = \alpha_{11}\beta_{22} + \alpha_{22}\beta_{11} - \alpha_{12}\beta_{21} - \alpha_{21}\beta_{12}\,, \tag{76}$$

$$\mu_1 = n^*\lambda_1 - P^*(L_2\alpha_{23} - L_3\alpha_{22})\,, \tag{77}$$

$$\mu_2 = n^*\lambda_2 - P^*(L_2\beta_{23} - L_3\beta_{22})\,, \tag{78}$$

$$\mu_3 = -n^*\lambda_3 + P^*(L_1\alpha_{23} - L_3\alpha_{21})\,, \tag{79}$$

$$\mu_4 = -n^*\lambda_4 + P^*(L_1\beta_{23} - L_3\beta_{21})\,, \tag{80}$$

$$\mu_5 = n^*\lambda_5 - P^*(L_1\alpha_{22} - L_2\alpha_{21})\,, \tag{81}$$

$$\mu_6 = n^*\lambda_6 + P^*(L_1\beta_{22} - L_2\beta_{21})\,, \tag{82}$$

$$\Delta_1 = L_1\lambda_1 - L_2\lambda_3 + L_3\lambda_5\,, \tag{83}$$

$$\Delta_2 = L_1\lambda_2 - L_2\lambda_4 + L_3\lambda_6\,. \tag{84}$$

4.2. *Problem II*

A plate subjected to time-dependent heat sources on both sides [25].

We shall consider a homogeneous isotropic thermo-elastic infinite conductivity thick flat plate of a finite thickness $2L$ occupying the region G^* given by:

$$G^* = \{(x, y, z)\,|x, z \in \mathbf{R}\,, \quad -L \le y \le L\}$$

with the middle surface of the plate coinciding with the plane $y = 0$.

The boundary conditions of the problem are taken as:

(i) The normal and tangential stress components are zero on both surfaces of the plate; thus,

$$\sigma_{yy} = 0 \quad \text{on} \quad y = \pm L\,, \tag{85}$$

$$\sigma_{xy} = 0 \quad \text{on} \quad y = \pm L\,. \tag{86}$$

(ii) The thermal boundary condition

$$q_n + h_0\theta_0 = r(x, t) \quad \text{on} y = \pm L\,. \tag{87}$$

where q_n denotes the normal component of the heat flux vector, h_0 is Biot's number and $r(x, t)$ represents the intensity of the applied heat sources.

We now make use of the generalized Fourier's law of heat conduction in the non-dimensional form, namely,

$$q_n + \tau_0 \frac{\partial q_n}{\partial t} = -\frac{\partial \theta}{\partial y}\,. \tag{88}$$

Eq. (88) in the normal mode form

$$q_n^* = -\frac{1}{(1+\tau_0\omega)} \frac{\partial \theta^*}{\partial y}\,. \tag{89}$$

Combining Eqs. (45), (87) and (89) we arrive at

$$H_1 B_1 \cosh(k_1 L) + H_2 B_2 \cosh(k_2 L) + H_3 B_3 \cosh(k_3 L) = (1+\tau_0\omega) r^*\,. \tag{90}$$

Equations (85) and (86) in the normal mode form together with Eqs. (52) and (53) respectively give:

$$M_1 B_1 \cosh(k_1 L) + M_2 B_2 \cosh(k_2 L) + M_3 B_3 \cosh(k_3 L) = 0\,, \tag{91}$$

$$N_1 B_1 \sinh(k_1 L) + N_2 B_2 \sinh(k_2 L) + N_3 B_3 \sinh(k_3 L) = 0\,. \tag{92}$$

Equation (90), (91) and (92) can be solved for the three unknowns B_1, B_2 and B_3.

$$B_1 = \frac{(1+\tau_0\omega) r^*}{(\Delta_3^2+\Delta_4^2)\cosh(k_1 L)} \left[(\lambda_7\Delta_3 + \lambda_8\Delta_4) + i(\lambda_7\Delta_4 - \lambda_8\Delta_3)\right]\,, \tag{93}$$

$$B_2 = \frac{-(1+\tau_0\omega) r^*}{(\Delta_3^2+\Delta_4^2)\cosh(k_2 L)} \left[(\lambda_9\Delta_3 + \lambda_{10}\Delta_4) + i(\lambda_9\Delta_4 - \lambda_{10}\Delta_3)\right]\,, \tag{94}$$

$$B_3 = \frac{(1+\tau_0\omega) r^*}{(\Delta_3^2+\Delta_4^2)\cosh(k_3 L)} \left[(\lambda_{11}\Delta_3 + \lambda_{12}\Delta_4) + i(\lambda_{11}\Delta_4 - \lambda_{12}\Delta_3)\right]\,, \tag{95}$$

where

$$H_j = \frac{\varepsilon\omega_1}{k_j^2 - a^2 - \omega_1} \left[-k_j \tanh(k_j L) + h_0(1+\tau_0\omega)\right]\,, \quad j=1,2,3\,, \tag{96}$$

$$\lambda_7 = (\alpha_{21}\alpha_{23} - \beta_{12}\beta_{23})\tanh(k_3 L) - (\alpha_{13}\alpha_{22} - \beta_{13}\beta_{22})\tanh(k_2 L)\,, \tag{97}$$

$$\lambda_8 = (\alpha_{12}\beta_{23} + \alpha_{23}\beta_{12})\tanh(k_3 L) - (\alpha_{13}\beta_{22} + \alpha_{22}\beta_{13})\tanh(k_2 L)\,, \tag{98}$$

$$\lambda_9 = (\alpha_{11}\alpha_{23} - \beta_{12}\beta_{23})\tanh(k_3 L) - (\alpha_{13}\alpha_{21} - \beta_{13}\beta_{21})\tanh(k_1 L)\,, \tag{99}$$

$$\lambda_{10} = (\alpha_{11}\beta_{23} + \alpha_{23}\beta_{11})\tanh(k_3 L) - (\alpha_{13}\beta_{21} + \alpha_{21}\beta_{13})\tanh(k_1 L)\,, \tag{100}$$

$$\lambda_{11} = (\alpha_{11}\alpha_{22} - \beta_{11}\beta_{22})\tanh(k_2 L) - (\alpha_{12}\alpha_{21} - \beta_{12}\beta_{21})\tanh(k_1 L)\,, \tag{101}$$

$$\lambda_{12} = (\alpha_{11}\beta_{22} + \alpha_{22}\beta_{11})\tanh(k_2 L) - (\alpha_{12}\beta_{21} + \alpha_{21}\beta_{12})\tanh(k_1 L)\,, \tag{102}$$

$$\Delta_3 = H_1\lambda_7 - H_2\lambda_9 + H_3\lambda_{11}\,, \tag{103}$$

$$\Delta_4 = H_1\lambda_8 - H_2\lambda_{10} + H_3\lambda_{12}\,. \tag{104}$$

5. Numerical results

The copper material was chosen for the purpose of numerical evaluations. Since we have $\omega = \omega_0 + i\zeta$, $e^{\omega t} = e^{\omega_0 t}(\cos\zeta t + i\sin\zeta t)$ and for small values of time, we can take $\omega = \omega_0$ (real). The numerical constants of the problems were takes as: $\varepsilon = 0.003$; $\delta_0 = 0.0199$; $\rho = 8954$; $\tau_0 = 0.03$; $n^* = 500$; $P^* = 100$; $h_0 = 100$; $r^* = 1000$; $a = 1.2$; $\omega = 1$; $\alpha = 1.8$; $\alpha^* = 0.001517$; $(1/K)$ and $T_0 = 293$ K. The computations were carried out for a value of time $t = 0.0001$. The numerical technique, outlined above, was used for the real part of the thermal temperature θ distribution, the displacement components u, v and the stress components σ_{xx}, σ_{yy} and σ_{xy} for each problem, for Problem I on the plane $y = 5$ and for Problem II on surface at $y = 2$ and on the middle plane $y = 0$, where $L = 4$ for two different values of $\Omega = 0$ and $\Omega = 0.01$. The results are shown in Figs. 1–16.

The graphs shows the four curves predicted by different theories of thermoelasticity. In these figures, the solid lines represent the solution corresponding to using the Coupled theory of heat conduction ($\tau_0 = 0$), the dashed lines represent the solution for the Lord-Shulman's theory ($\tau_0 = 0.03$). It can be seen from these figures that the rotation acts to decrease the magnitude of the real part of the temperature, displacement and the stress components.

We notice also that the results for the temperature, displacement and stress components when the relaxation time is appeared in the heat equation are distinctly different from those the relaxation time is not mentioned in the heat equation. This is due to the fact that thermal waves in the Fourier theory of heat equation travel with an infinite speed of propagation as opposed to finite speed in the non-Fourier case. This demonstrates clearly the difference between the coupled and the generalized theory of thermoelasticity.

Figs. 1–6 demonstrate the effect of rotation for Problem I on the plane $y = 5$. Figs. 7-12 represent the effect of rotation on the surface of Problem II at $y = 2$ and Figs. 13–16 represent the effect of rotation on the middle plane at $y = 0$, where $L = 4$.

6. Conclusion

We can obtain the following conclusions according to the analysis above and the illustrated figures

1. It is clear that the rotation has decreasing effect with the modulus of elasticity being dependent on the reference temperature.

2. The horizontal component of displacement is vanishes on the surface and middle plane of problem II when the rotation equal to zero.

3. We note that since the vertical component of displacement v and the stress component σ_{xy} are odd function of y there values on the middle plane of Problem II are always zero.

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Nomenclature

Symbol	Description
$\lambda,\ \mu$	Lamé's constants
ρ	density
C_E	specific heat at constant strain
$E(T)$	Temperature dependent modulus of elasticity
α^*	empirical material constant $\left[\frac{1}{K}\right]$
η_0	$=\frac{\rho C_E}{k}$
c_0^2	$=\frac{(\lambda_0+2\mu_0)E_0}{\rho}$
ε_1	$=\frac{\gamma_0 E_0}{\rho C_E}$
E_0	const. modulus of elasticity at $\alpha^*=0$
ν	Poisson's ratio
t	time
T	absolute temperature
α_T	coefficient of linear thermal expansion
δ_0	non-dimensional constant
μ_0	$=\frac{1}{2(1+\nu)}$
$f(\theta)$	is a given nondimensional function of temperature
ε	$=\varepsilon_1\delta_0 f(\theta)$
β	$=\frac{E_0(\lambda_0+\mu_0)}{\rho c_0^2}=\frac{1}{2(1-\nu)}$
λ_0	$=\frac{\nu}{(1+\nu)(1-2\nu)}$
γ_0	$=\frac{\alpha_T}{1-2\nu}$
T_0	$=\frac{\delta_0\rho c_0^2}{\gamma_0 E_0}=\frac{\delta_0}{\alpha_T}\left(\frac{1-\nu}{1+\nu}\right)$ reference temperature
σ_{ij}	components of stress tensor
ε_{ij}	components of strain tensor
$e=\varepsilon_{ii}$	dilatation
u_i	components of displacement vector
k	thermal conductivity
τ_0	one relaxation time

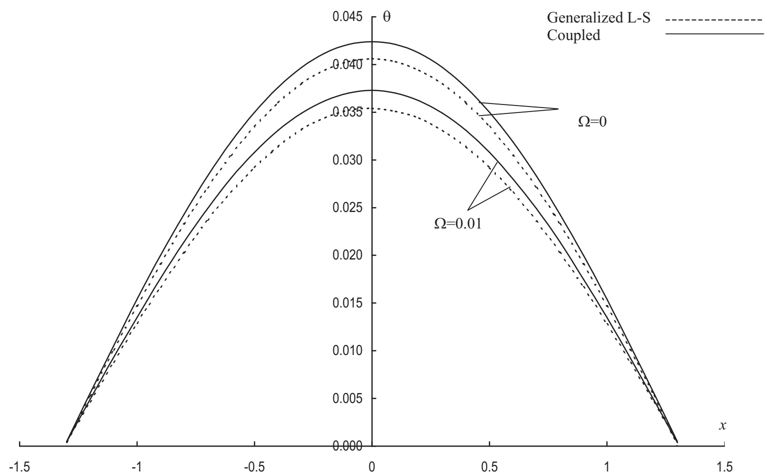

Figure 1 Temperature distribution θ for Problem I at $\alpha = 1.8$

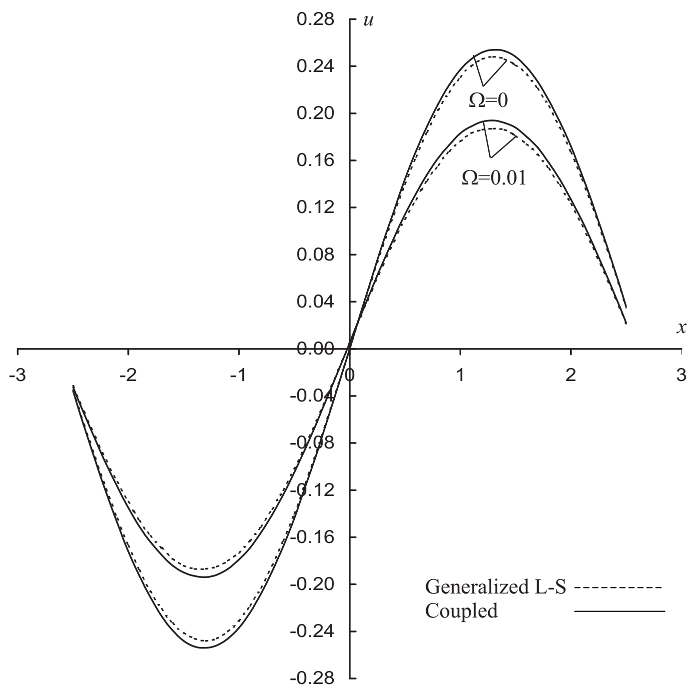

Figure 2 Horizontal displacement distribution u for Problem I at $\alpha = 1.8$

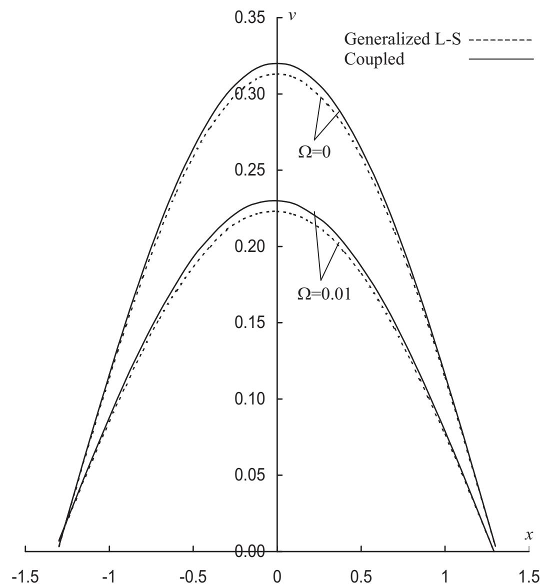

Figure 3 Vertical displacement distribution v for Problem I at $\alpha = 1.8$

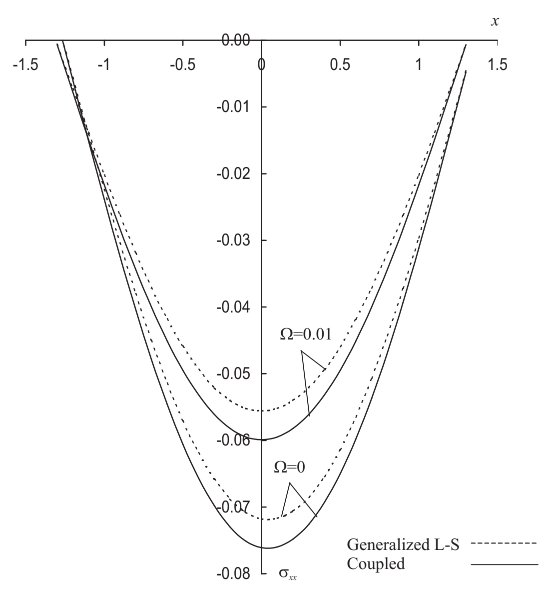

Figure 4 The distribution of stress component σ_{xx} for Problem I at $\alpha = 1.8$

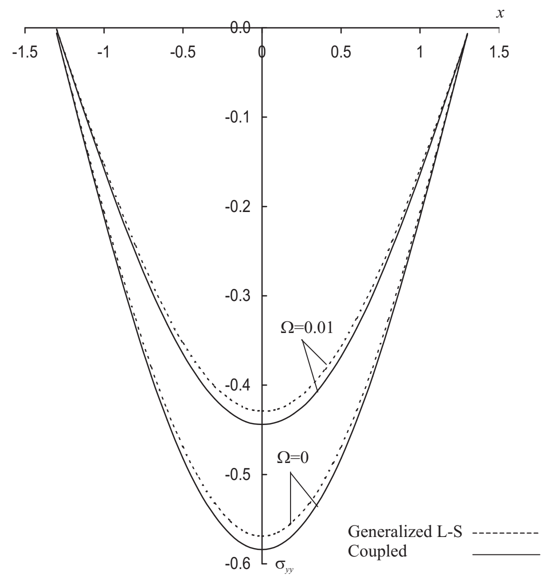

Figure 5 The distribution of stress component σ_{yy} for Problem I at $\alpha = 1.8$

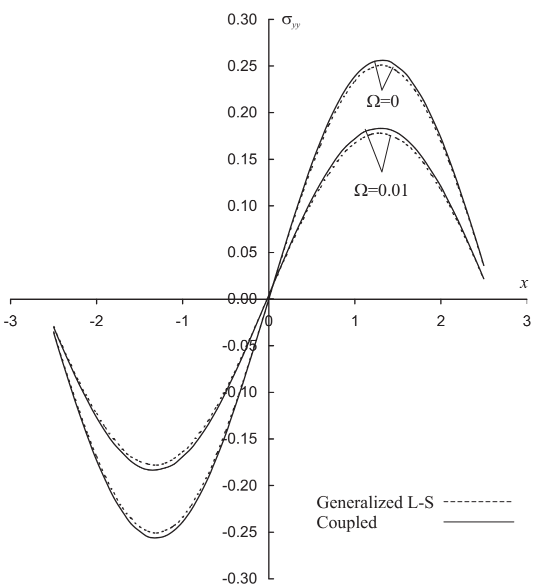

Figure 6 The distribution of stress component σ_{xy} for Problem I at $\alpha = 1.8$

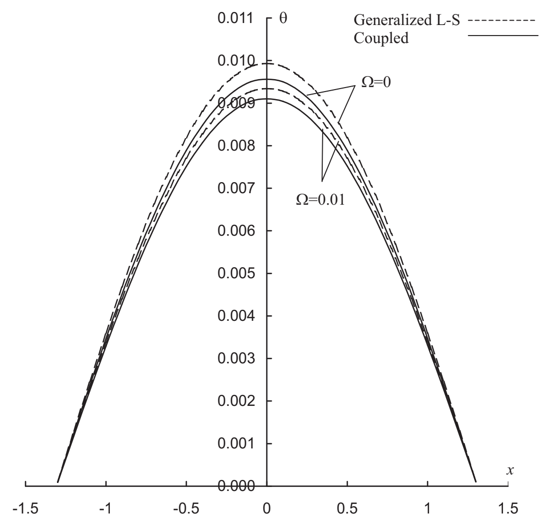

Figure 7 Temperature distribution θ for Problem II at $\alpha = 1.8$

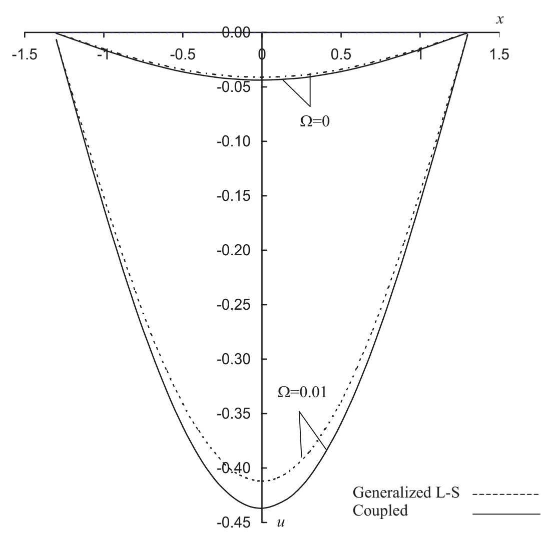

Figure 8 Horizontal displacement distribution u for Problem II on the surface at $\alpha = 1.8$

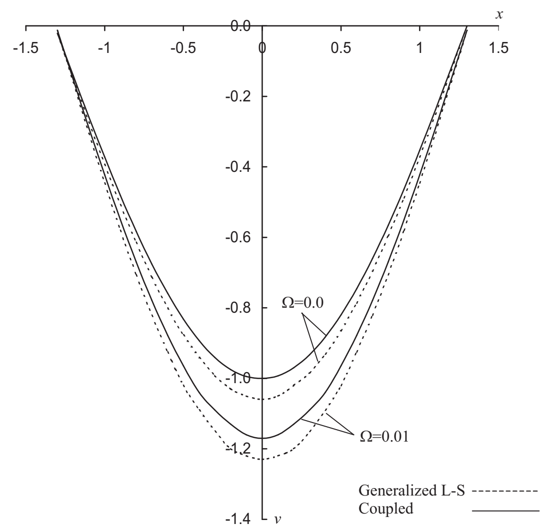

Figure 9 Vertical displacement distribution v for Problem II on the surface at $\alpha = 1.8$

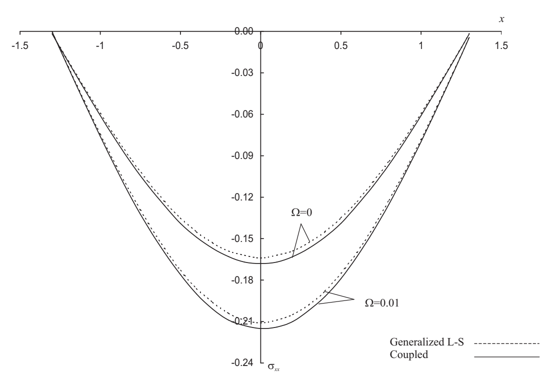

Figure 10 The distribution of stress component σ_{xx} for Problem II on the surface at $\alpha = 1.8$

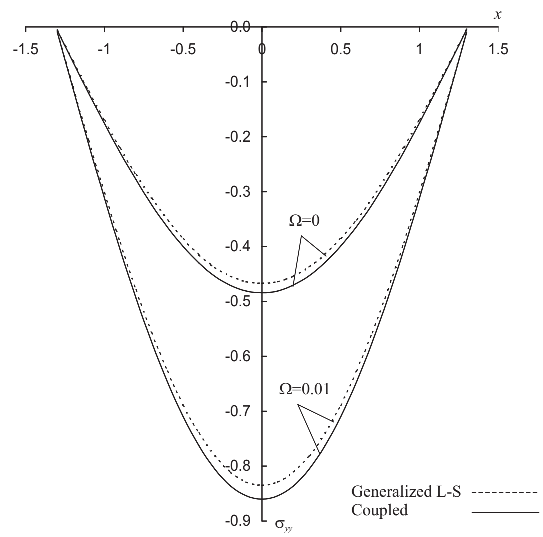

Figure 11 The distribution of stress component σ_{yy} for Problem II on the surface at $\alpha = 1.8$

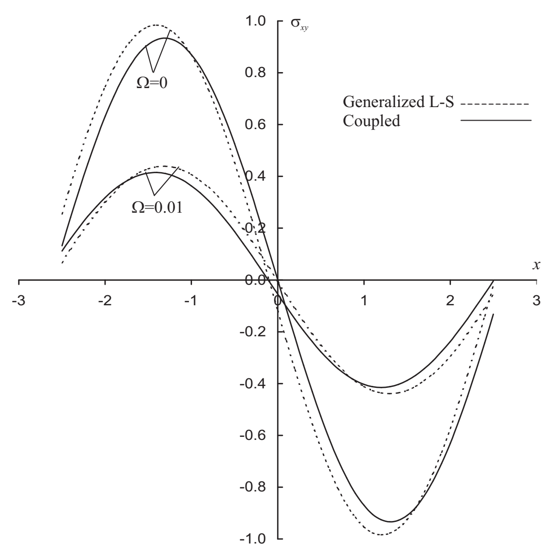

Figure 12 The distribution of stress component σ_{xy} for Problem II on the surface at $\alpha = 1.8$

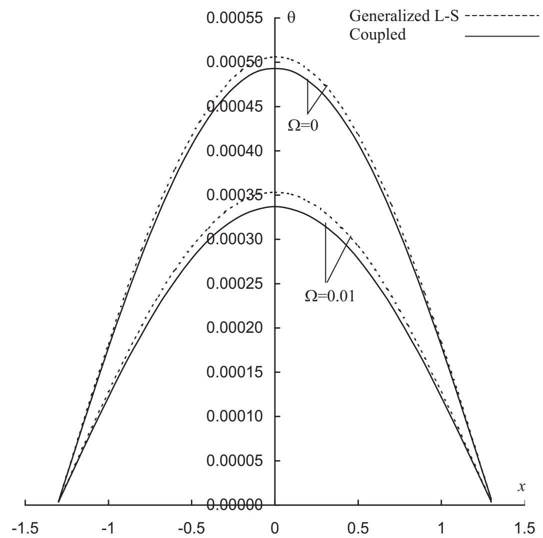

Figure 13 Temperature distribution θ for Problem II on the middle plane at $\alpha = 1.8$

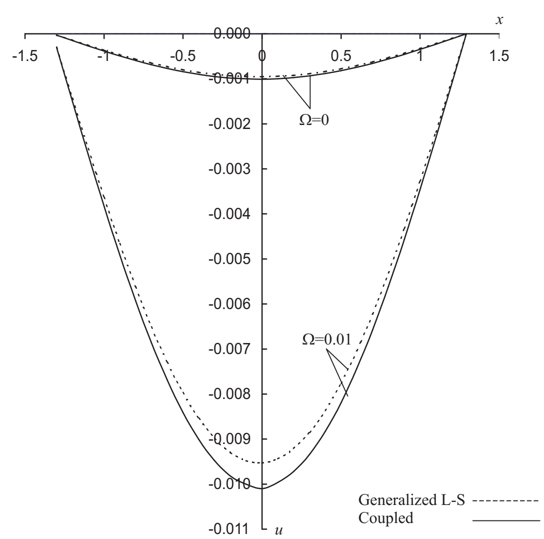

Figure 14 Horizontal displacement distribution u for Problem II on the middle plane at $\alpha = 1.8$

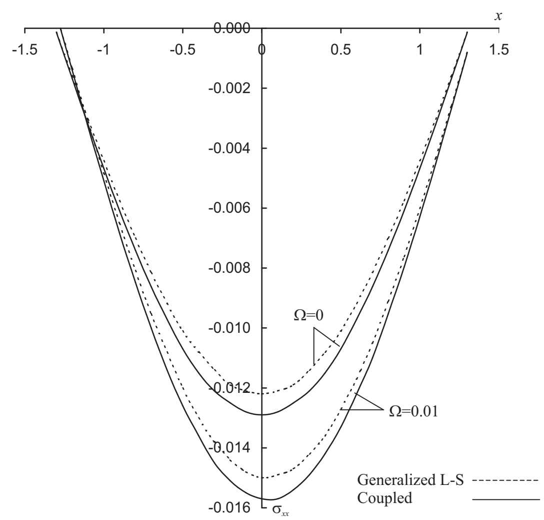

Figure 15 The distribution of stress component σ_{xx} for Problem II on the middle plane at $\alpha = 1.8$

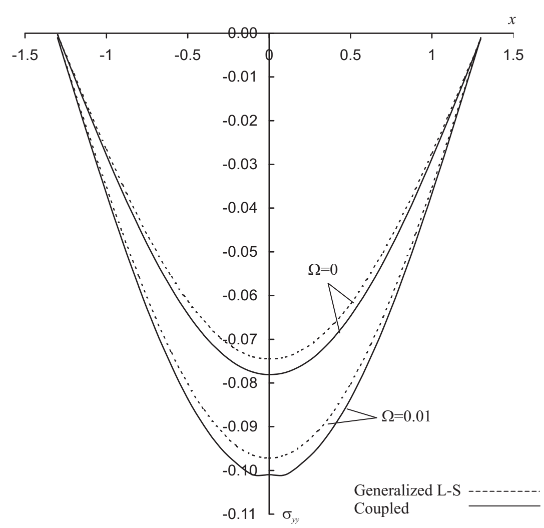

Figure 16 The distribution of stress component σ_{yy} for Problem II on the middle plane at $\alpha = 1.8$