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Impulsive Flow in a Horizontal Layer

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In this article, the problem of slow impulsive flow in a horizontal fluid layer is obtained by the method of Green's function in the inviscid case, and a reduced viscous case is solved by Fourier transform.

Keywords: Impulsive flow, Green's function, viscous damping.

1. Introduction

The surface of the sea or the ocean is always subject to impulsive action of horizontal wind, which results in water waves. These waves propagate with amplitudes varying with depth depending on viscosity and under water currents. Included is a list of references on the subject [1–6].

2. Formulation of the Problem

Consider a horizontal layer of fluid with mass density ρ and viscosity μ extending from $y = 0$ downwards to a horizontal bottom at $y = -B$. The layer is two dimensional and extends horizontally for $\infty > x > -\infty$. If the amplitude of the resulting velocities is sufficiently small, the convective acceleration can be dropped and the equations of motion are

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0\,,$$

for continuity. The momentum is

$$\left.\begin{array}{l} \rho\frac{\partial u}{\partial t} = -\rho\frac{\partial p}{\partial x} + \mu\nabla^2 u \\ \rho\frac{\partial v}{\partial t} = -\rho\frac{\partial p}{\partial y} + \mu\nabla^2 v \end{array}\right\}\,. \qquad (1)$$

Here u and v are velocities components in x and y directions, respectively, t is the time, and p is the pressure. Introducing the stream function ϕ such that,

$$u = \frac{\partial \phi}{\partial y}\,, \quad v = \frac{\partial \phi}{\partial x}\,,$$

and eliminating pressure, the system reduces to

$$\frac{\partial}{\partial t}\nabla^2 \phi = \nu \nabla^2 \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right)\,. \tag{2}$$

2.1. *Solution for the inviscid case:* $(v = 0)$

$$\frac{\partial}{\partial t}\nabla^2 \phi = 0\,.$$

Since the boundary condition is time dependent, a solution is sought by Laplace transform

$$\begin{aligned} \hat{\phi} &= L\phi\,, \\ \nabla^2 \hat{\phi} &= \frac{1}{S}\nabla^2 \phi_0\,, \end{aligned}$$

where ϕ_0 is the initial distribution of ϕ. Since ϕ is specified on the boundary (plane $y = 0$), the solution by Green's function [7] takes the form

$$\hat{\phi}(x_0, y_0, S) = \frac{1}{S}\int_0^\infty \hat{\phi}_{y=0}\, \frac{\partial G}{\partial y}\bigg|_{y=0} dx\,. \tag{3}$$

Here, $G(x, y, x_0, y_0)$ is the Green's function of the problem satisfying

$$\left.\begin{array}{l} \nabla^2 G = \delta(x - x_0)\delta(y - y_0) \\ G|_{y=0}\; G|_{y=B} = 0 \\ G|_{x=\pm\infty} = 0 \end{array}\right\} \tag{4}$$

Using the method of eigen-functions G must take the form:

$$G = \sum_{n=0}^{\infty} F_n(x) \sin \frac{n\pi y}{B}\,.$$

Substituting in (4) and applying orthogonality

$$F_n'' - \frac{n^2\pi^2}{B^2}F_n = \frac{2}{B}\sin\frac{n\pi}{B}y_0\delta(x - x_0)\,.$$

F_n must take the form

$$F_n = K \left\{ \begin{array}{ll} e^{-\frac{n\pi}{B}}(x - x_0) & x > x_0 \\ e^{-\frac{n\pi}{B}}(x_0 - x) & x < x_0 \end{array} \right.$$

The constant K can be determined from the jump condition at x_0

$$F_n'(x_{0+}) - F_n'(x_{0-}) = \frac{2}{B}\sin\frac{n\pi y_0}{B}\,.$$

Giving

$$K = -\frac{1}{n\pi}\sin\frac{n\pi y_0}{B}\,.$$

If we consider

$$\phi_{y=0} = \begin{cases} \delta(t) & 1 > x > -1 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\phi}_{y=0}$$

is given in the graph. Which corresponds to a pulse of limited extent.

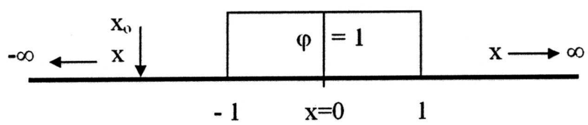

Figure 1 Pulse of limited extent

We indicate that as

$$\hat{\phi} = \int \frac{\partial\hat{\phi}}{\partial x}dx + \frac{\partial\hat{\phi}}{\partial y}dy$$

we have

$$1 = -\int_{-1}^{1}\hat{v}dx$$

accordingly. The average $\hat{v} = -\frac{1}{2}$, i.e., the velocity producing the impulse is

$$v - -\frac{1}{2}\delta(t)$$

and $u = 0$.

The integral (3) gives the solution for an arbitrary location of x_0 as shown

$$\phi(x_0, y_0, S) = -\frac{1}{\pi S}\sum_{n=0}^{\infty}\frac{1}{n^2}\sin\frac{n\pi y_0}{B}\left[e^{-\frac{n\pi}{B}(1-x_0)} - e^{-\frac{n\pi}{B}(-1-x_0)}\right]\,.$$

Inverting Laplace transform, we get,

$$\phi(x, y) = -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{n^2}\sinh\frac{n\pi}{B}\sin\frac{n\pi y}{B}e^{-\frac{n\pi[x]}{B}}\,.$$

And the corresponding velocity components are

$$\begin{aligned} u &= -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{n}\sinh\frac{n\pi}{B}\cos\frac{n\pi y}{B}e^{-\frac{n\pi[x]}{B}}\,, \\ v &= -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{n}\sinh\frac{n\pi}{B}\sin\frac{n\pi y}{B}e^{\frac{n\pi[x]}{B}}\,. \end{aligned} \qquad (5)$$

A steady flow field exists and are expressed in convergent series.

2.2. *The viscous case*

Taking Laplace transform of (2)

$$\nabla^2 \left[S\hat{\phi} - v\left(\hat{\phi}_x + \hat{\phi}_y\right) = 0 \right] .$$

And consider same previous example with $\nabla^2 \phi_0 = 0$, and restrict ourselves to the case

$$S\hat{\phi} - v\left(\hat{\phi}_x + \hat{\phi}_y\right) = 0 \,.$$

Taking Fourier transform in $x \to \infty$

$$\left(S + i\omega\right) \hat{\phi} = \hat{\phi}_y$$

i.e.

$$\hat{\phi} = \hat{\phi}_{y=0}\; e^{\left(\frac{S}{v} + i\omega\right) y} \,.$$

If we consider same pulse

$$\hat{\phi}_{y=0} = \frac{2\sin\omega}{\omega}$$

and

$$\hat{\phi} = \frac{2\sin\omega}{\omega} e^{\left(\frac{S}{v} + i\omega\right) y}$$

Inverting both transforms

$$\phi = \begin{cases} \delta\left(t - \frac{y}{v}\right) & \text{for} \quad y - x < 1 \\ 0 & \text{for} \quad y - x > 1 \end{cases}$$

The delta function can be approximated physically by

$$\delta(t) = \frac{1}{\sigma} e^{-\sigma^2 t^2} \,.$$

When σ is a suitable sufficiently small parameter

$$\phi = \frac{1}{\sigma} e^{-\sigma\left(\frac{y}{v}\right)^2}$$

for $y - x < 1$.

The resulting flow will have

$$\begin{aligned} v &= 0 \\ u &= \frac{2}{v}\left(t - \frac{y}{v}\right)\; e^{-sigma\left(t - \frac{y}{v}\right)^2} \,. \end{aligned}$$

The value of u is positive for $t > \frac{y}{v}$ and negative for $t < \frac{y}{v}$ and vanishes instantaneously for $t = \frac{y}{v}$ or along the layer $y = vt$ and as the viscosity increases, the velocity decreases linearly.

3. Conclusion

The obtained solution is valid for simple cases of the inviscid and the viscous cases. More complicated problems can be handled along these lines. The series obtained for the inviscid problem is convergent due to the negative n exponential and the approximation of the delta function by the Gaussian is valid physically for small σ.

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