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Kelvin Helmholtz Instability in the Presence of Mass, Heat Transfer, Porous Media, Surface Tension and Magnetic Field

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On the basis of the simplified formulation of Hsieh [6] for interfacial stability problems, a general dispersion relation of Kelvin-Helmholtz stability in the presence of mass, heat transfer, porous media, surface tension and magnetic field is obtained to generalize the problems studied in references [7–10]. It is found that magnetic field, heat transfer, surface tension and streaming have a stabilizing effect, while porous media has a destabilizing effect on unstable modes.

Keywords: Magnetic field, capillary, stability.

1. Introduction

A well-known example of a fluid instability is the Kelvin-Helmholtz type [1–5] caused by the relative motion of a fluid of density $\rho^{(2)}$ by a one of density $\rho^{(1)}$.

For such stability problems, the coupling between dynamics and heat transfer can not be ignored. In particular, the mass transfer across the interface due to the evaporation and condensation processes would play an important role. The transfer of mass and heat across the interface is not only important for the problem of boiling heat transfer, but also in problems of chemical engineering and in geophysics. The presence of magnetic field and porous media, play a significant role in the astrophysical situations such as the theories of sunspot magnetic fields and heating of solar corona [6]. The treatment presented in this paper may be applicable to this general class of problems. A general formulation of the interfacial flow problem with mass and heat transfer was established [7] and the linear Kelvin-Helmholtz stability analysis was carried out. It is found that when the vapor region is hotter than the liquid region, the effect of mass and heat transfer tends to inhibit the growth of the instability. The effect of a uniform magnetic field on the development of Kelvin-Helmholtz stability is presented, for example, in [8–10]. The aim of this paper is to study the general interfacial problem for two fluids with the effects of magnetic field, porous media, heat transfer and surface tension. In Section 2 the formulation of the problem is given and the equilibrium configuration. The equilibrium state is discussed in Section 3. In Section 4 the perturbation equations are explained and the dispersion relation is obtained. Some special cases are recovered and the effects of magnetic field, porous media, heat transfer and surface tension on Kelvin-Helmholtz are explained in Section 5. Finally in the last Section conclusion is given.

2. Formulation of the problem

The general formulation of the interfacial problem with heat and mass transfer, across the interface for two fluids was presented by Hsieh [7]. By applying the general conservation equations of mass, moment and energy across the non stationary interface, a set of interface conditions are derived. We shall now apply the general formulation to the study of the flow of a system of two incompressible, inviscid, porous fluid layers, taking into account a constant magnetic field in x direction in both of the fluids, as shown in Figure 1. The interface is given by

$$S(x, y, z, t) = y - \xi(x, z, t)\,. \tag{1}$$

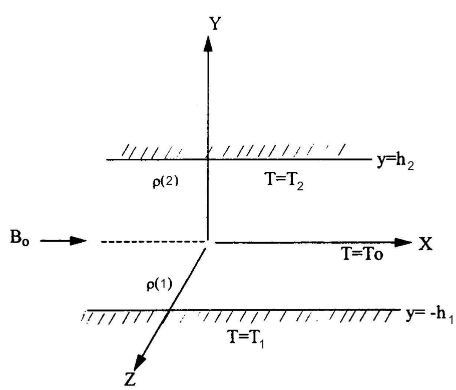

Figure 1 The equilibrium configuration of the fluid system

At equilibrium, the interface is taken to be $y = 0$. Assuming the flows of the fluids to be irrotational and introducing the velocity potentials $\phi(x, y, z, t)$ we have:

$$\nabla^2\phi^{(1)} = 0\,, \quad -h_1 < y < \xi \tag{2}$$

and

$$\nabla^2\phi^{(2)} = 0\,, \quad \xi < y < h_2\,. \tag{3}$$

Let the magnetic potentials be $\psi^{(1)}$ and $\psi^{(2)}$ respectively. Thus in each fluid region we have

$$\nabla^2\psi^{(j)} = 0\,, \quad j = 1, 2\,. \tag{4}$$

The equation of transport of magnetic field is given by

$$\sigma_1\nabla\psi = -\nabla \times (\nabla\phi \times \nabla\psi) \tag{5}$$

The Bernoulli equation takes the form

$$\frac{P^{(i)}}{\rho^{(j)}}+\frac{1}{2}\left(\nabla\phi^{(j)}\right)^2+\frac{1}{2\rho^{(j)}}\left(\nabla\psi^{(j)}\right)^2-\frac{\mu^{(j)}}{\rho^{(j)}k^{(j)}}\phi^{(j)}+gy-\sigma\phi^{(j)}=f^{(j)}\,,\quad j=1,2 \tag{6}$$

where P, ρ, μ, and k are pressure, density, viscosity and the permeability, respectively. $f^{(j)}$ are integration constants and g is the gravitation constant. Boundary conditions at the bottom and top surfaces are

$$\sigma_y\phi^{(1)}=1\,,\quad y=-h_1\,, \tag{7}$$

$$\sigma_y\phi^{(2)}=0\,,\quad y=\quad h_2 \tag{8}$$

and the interfacial conditions between the two fluids are given by

$$\rho^{(1)}\left(\sigma_t S+\left(\nabla\phi^{(1)}\right).(\nabla S)\right)=\rho^{(2)}\left(\sigma_t S+\left(\nabla\phi^{(1)}\right).(\nabla S)\right)\,,\quad y=\xi\,, \tag{9}$$

$$\begin{aligned}\rho^{(1)}\left[\left(\nabla\phi^{(1)}\right).(\nabla S)\right]\left(\sigma_t S+\left(\nabla\phi^{(1)}\right).(\nabla S)\right) &= \\ \rho^{(2)}\left[\left(\nabla\phi^{(2)}\right).(\nabla S)\right]\left(\sigma_t S+\left(\nabla\phi^{(2)}\right).(\nabla S)\right)+ & \\ +\left[P^{(2)}-P^{(1)}-\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]|\nabla S|^2\,, &\qquad y=\xi\,,\end{aligned} \tag{10}$$

$$\rho^{(1)}\left(\sigma_t S+\left(\nabla\phi^{(1)}\right).(\nabla S)\right)=F(\xi)\,, \tag{11}$$

where σ is the surface tension coefficient, L is latent heat and R_1 and R_2 are the principal radii of curvature of the interface. Equations (3)–(4) and (6), express the conservation of mass and momentum, respectively. Equation (11) expresses the condition of local phase equilibrium. Fluid 1 is implied in this equation to represent the vapor phase. The mechanism of heat transfer of fluid is important mainly for its facilitation of the phase change at the interface. Thus it is justifiable to neglect the viscosity in this analysis. Equations (1)–(11) form the basic system of the problem.

3. The Equilibrium State

The equilibrium state is given by the following system:

$$\xi = 0 \tag{12}$$

$$\phi^{(j)} = -U_j x\,,\quad j=1,2\,, \tag{13}$$

$$P^{(j)} = -\rho^{(j)}gy-\frac{B_0^2}{2}\,,\quad j=1,2\,, \tag{14}$$

$$f^{(j)} = -\frac{1}{2}\rho^{(j)}U_j^2\,,\quad j=1,2\,, \tag{15}$$

$$\psi^{(j)} = -B_0 x\,,\quad j=1,2\,, \tag{16}$$

$$G = \frac{K^{(1)}(T_1-T_0)}{h_1}=\frac{K^{(2)}(T_0-T_2)}{h_2}\,, \tag{17}$$

where $K^{(j)}$ is the thermal conductivity.

4. The dispersion relation

Now, let us perturb the interface from $y = 0$ to $y = \exp[i(kx - wt)]$. For small ξ we see that in order to satisfy the boundary conditions where the normal velocities vanish at $y = h_2$ and $y = -h_1$ perturbed velocity potentials are given by

$$\phi^{(1)} = A_1 \cosh k(y + h_1) \tag{18}$$

and

$$\phi^{(2)} = A_2 \cosh k(y - h_2) \tag{19}$$

Neglecting the nonlinear terms, Eqn. (5) takes the form

$$(w - kU_j)\nabla\psi^{(j)} = kB_0\nabla\phi^{(j)}\,, \quad j = 1, 2\,. \tag{20}$$

The expressions of the pressures on $S = 0$ are given by

$$\begin{aligned} P^{(1)} &= -iw\rho^{(1)}A_1 \cosh kh_1 - \rho^{(1)}g\xi - ik\rho^{(1)}U_1A_1 \cosh kh_1 \\ &\quad + \frac{ik^2B_0^2}{w - kU_1}A_1 \cosh kh_1 + \frac{\mu^{(1)}}{k^{(1)}}A_1 \cosh kh_1 \end{aligned} \tag{21}$$

and

$$\begin{aligned} P^{(2)} &= -iw\rho^{(2)}A_2 \cosh kh_2 - \rho^{(2)}g\xi - ik\rho^{(2)}U_2A_2 \cosh kh_2 \\ &\quad + \frac{ik^2B_0^2}{w - kU_2}A_2 \cosh kh_2 + \frac{\mu^{(2)}}{k^{(2)}}A_2 \cosh kh_2\,. \end{aligned} \tag{22}$$

The linearized interfacial conditions (9)–(11) lead to

$$\rho^{(1)}\left[i\xi(w - kU_1) + A_1k \sinh kh_1\right] = \rho^{(2)}\left[i\xi(w - kU_2) + A_2k \sinh kh_2\right]\,, \tag{23}$$

$$\begin{aligned} &\rho^{(1)}\left[iwA_1 \cosh kh_1 - g\xi - ikU_1A_1 \cosh kh_1 + \frac{ik^2B_0^2}{\rho^{(1)}(w - kU_1)}\cosh kh_1\right] \\ &+ \frac{\mu^{(1)}}{\rho^{(1)}k^{(1)}}A_2 \cosh kh_1 = \rho^{(2)}\left[-iwA_2 \cosh kh_2 - g\xi - ikU_2A_2 \cosh kh_2 + \right. \\ &\left.\frac{ik^2B_0^2}{\rho^{(2)}(w - kU_2)}A_2 \cosh kh_2 + \frac{\mu^{(2)}}{\rho^{(2)}k^{(2)}}A_2 \cosh kh_2\right] + \sigma k^2\xi\,, \end{aligned} \tag{24}$$

$$\rho^{(1)}\left[i\xi(w - kU_1) + A_1k \sinh kh_1\right] = a\xi\,, \tag{25}$$

where

$$\alpha = \frac{G}{L}\left(\frac{1}{h_1} + \frac{1}{h_2}\right)\,. \tag{26}$$

Elimination of A_1, A_2 and ξ lead to the dispersion relation

$$\begin{aligned} &w^2\left(\rho^{(1)} \coth kh_1 \rho^{(2)} \coth kh_2\right) + w\left[\left(-\frac{\mu^{(1)}}{\rho^{(1)}k^{(1)}} + 2kU_1\rho^{(1)} - i\alpha\right)\coth kh_1 + \right. \\ &\left.\left(-\frac{\mu^{(2)}}{\rho^{(2)}k^{(2)}} + 2kU_2\rho^{(2)} - i\alpha\right)\coth kh_2\right] - \left\{\left[\left(kU_1\rho^{(1)} - i\alpha\right)\left(-\frac{\mu^{(1)}}{\rho^{(1)}k^{(1)}} + kU_1\right)\right.\right. \\ &\left. + k^2B_0^2\right]\coth kh_1 - \left[\left(kU_2\rho^{(2)} - i\alpha\right)\left(-\frac{\mu^{(2)}}{\rho^{(2)}k^{(2)}} + kU_2\right) + k^2B_0^2\right]\coth kh_2 \\ &\left. - \left[\left(\rho^{(2)} - \rho^{(1)}\right)kg - \sigma k^3\right]\right\} - iak^2B_0^2\left[\frac{\coth kh_1}{\rho^{(1)}(w - kU_1)} + \frac{\coth kh_2}{\rho^{(2)}(w - kU_2)}\right] = 0\,. \end{aligned} \tag{27}$$

Eqn.(27) generalizes the one given in reference [7] where there are no porous media and magnetic field, reference [9] where there are no streaming and porous media and reference [10] where there is no porous media.

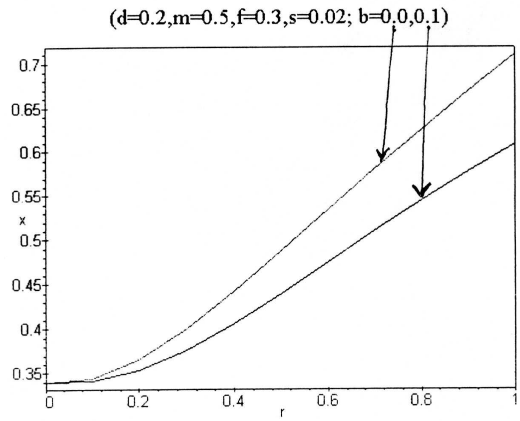

Figure 2 The dimensionless growth rate x as a function of dimensionless wave number r to show the effect of magnetic field

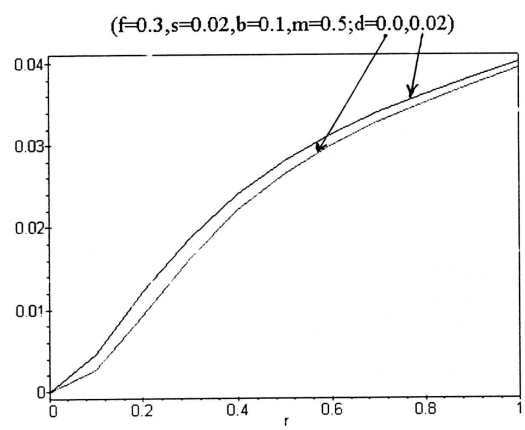

Figure 3 The dimensionless growth rate x as a function of dimensionless wave number r to show the effect of porous media

For simplicity and without lose of generality, let us, now, put in Eqn.(27):

$$h_1 = h_2 = h\,, \quad \rho^{(1)} = 2\rho^{(2)}\,, \quad U_1 = 0\,, \quad U_2 = U\,, \\ \mu^{(1)} = \mu^{(2)} = \mu\,, \quad K^{(1)} = K^{(2)} = \kappa\,. \tag{28}$$

and, also, the dimensionless quantities in the form

$$x = w\sqrt{\frac{h}{g}} \quad r = kh\,, \quad s = \frac{\sigma}{\rho g h}\,, \quad m = \frac{\alpha}{\rho}\sqrt{\frac{h}{g}}\,,$$
$$d = \frac{\mu}{\rho\kappa}\sqrt{\frac{h}{g}}\,, \quad b = \frac{B_0^2}{\rho g h}\,. \tag{29}$$

Then, the Eqn.(27) is equivalent to

$$3x^2 - 2x(d - rf + im) - \left[(2r^2b + rfd - K^2f^2) + im\left(\frac{3}{2}d - rf\right)\right.$$
$$\left. + (r^3 s + r)\tanh r\right] - imr^2 b\left(\frac{1}{2x} + \frac{1}{x - rf}\right) \quad = \quad 0\,. \tag{30}$$

The roots of Equation (30) are obtained by using Maple V and the dimensionless growth rates in terms of wave numbers are plotted for unstable mode to declare the effects of various parameters.

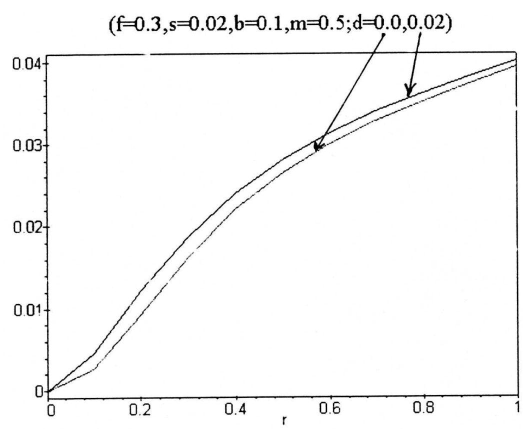

Figure 4 The dimensionless growth rate x as a function of dimensionless wave number r to show the effect of heat transfer

5. Summary and conclusions

We have presented analytical and numerical results on Kelvin-Helmholtz instability in the presence of heat transfer, magnetic field, porous media and surface tension. The dispersion relation is obtained as an algebraic equation of the fourth order with complex coefficients. According to the mathematical shape of the disturbance, the negative imaginary parts determine the conditions of stability. The roots are

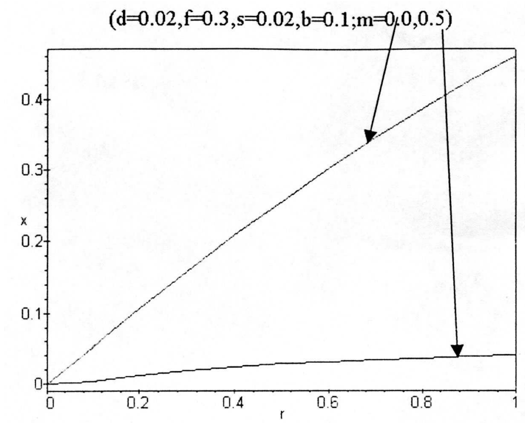

Figure 5 The dimensionless growth rate x as a function of dimensionless wave number r to show the effect of surface tension

calculated by using Maple V. The dimensionless growth rates in terms of the dimensionless wave numbers are drown to show the effects of various parameters. From Figures 2, 4 and 5 we find that, magnetic field, heat transfer and streaming have a stabilizing effect, while from Figure 3 we find that porous media has a destabilizing effect. This problem can be applied in several applications, for example, to the hall heroult cells and that is will be the scope of a subsequent paper.

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