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The Energy Space, Energy Flow and Synchronization

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In this paper the way of transformation of the phase space to obtain the energy space is presented. This new kind of space allows for a new, geometrical view on energy changes in vibrating systems.

Keywords: Energy plane, Kirlian effect, impacts.

1. Introduction

During the motion of vibrating systems, continuous energy changes from the potential energy into the kinetic one and vice versa occur. There can also exist an energy flow between some parts of the system, an energy flow through the system, energy flow synchronization, an energy dissipation, accumulation of energy in the system or some parts of it. An energy flow modelling still arouses interests in the scientific world. Different methods are applied to solve problems connected with energy flow: Statistical Energy Analysis [1, 2, 3], Finite Element Method [4, 5]. But these methods do not allow for a special kind of a geometrical view on energy changes, which could develop our intuitional knowledge on energy flow phenomenon. This intuition is very important especially in modelling the systems, where we still can not measure energy flow, such as telepathy - the problem of energy flow between two electromagnetic systems, bioenergy (observed thanks to GDV method based on Kirlian effect [6, 7, 8]), and so on.

Nowadays one of the most important tools used in dynamical system investigations is the phase space. In the case of vibrating mechanical systems, one can observe the motion of the system, changes of velocities and displacements of some parts of the system, using the phase space. One can conclude about the character of the motion, observe the trajectory of the motion, and it allows for the intuitional, geometrical view on systems dynamics. In this paper the new conception of the energy space is proposed. It allows for a geometrical view on energy changes in mechanical vibrating systems. This space posses all the advantages of the phase space but it shows also an amount of the energy accumulated in the system, the energy changes, flow, dissipation, synchronization, energy attractors.

The construction of the energy space is based on the phase space [13]. It would be better to understand the idea of this construction if we say, that the energy space is almost squeezed and stretched phase space. It is made in the way that if the system has the constant energy, then the trajectory of the motion lies on the multi-dimensional sphere. The radius of this sphere in the proposed energy space is equal to the square root of the total energy accumulated in the system. Thanks to that one can conclude about the energy flow, accumulation and dissipation from the change of the radius of the trajectory.

Some special advantages brings the consideration of the energy planes which are the equivalents to the phase planes. They are the projections of the trajectory onto the energy planes. One can consider the plane showing the energy accumulated in some part of the system. More often it would be the potential energy on the horizontal axis and the kinetic one on the vertical axis. From the image of the trajectory projection onto that plane, one can see energy changes from kinetic into potential, whereas the radius of this projection shows the total energy accumulated in that part of the system. Moreover, one can conclude about all aspects of the motion, like from the phase plane. It is possible thanks to the fact that the kinetic energy is the function of velocity and the potential one – the function of displacement. From that point of view the energy plane is almost the phase plane with displacement on the horizontal axis and velocity on the vertical one.

In that, simple way one can obtain the energy space, which allows for a new, geometrical view on energy changes in vibrating systems.

Proposed energy space can be applied to linear and nonlinear vibrating systems. In the section 2. there are shown the cases of mechanical vibrating systems, with linear or linearized characteristics of springs, with no restrictions on the nonlinearities and discontinuities of the other types, such as nonlinear damping, impacts, friction etc. New kind of maps was introduced in section 2.3. In section 3. the application of the energy space to the systems with nonlinear characteristics of springs is discussed.

2. The energy space for the system with linear characteristics of springs

Consider the $2n$-dimensional phase space $P \subset I\mathbf{R}^{2n}$ of a system with n degrees of freedom. Such a system can be represented by n-masses μ_i, $1 \leq i \leq n$ connected by n springs with stiffness coefficients σ_i, $1 \leq i \leq n$ (Fig. 1). Let x_i be the displacement of the mass μ_i and y_i the velocity of this mass. The velocities y_i and the displacements x_i are the state variables of the phase space then.

Let us transform this phase space in the following way

1. Instead of the displacement co-ordinates x_i of the mass μ_i, put the spring σ_i deflection z_i. If the spring σ_i couples the mass μ_i and μ_j, then $z_i = x_i - x_j$. Of course, if one of the ends of the spring is motionless, then $z_i = x_i$.

2. Depending on the coefficients σ_i and μ_i, squeeze and stretch the space in the directions of z_i and y_i.

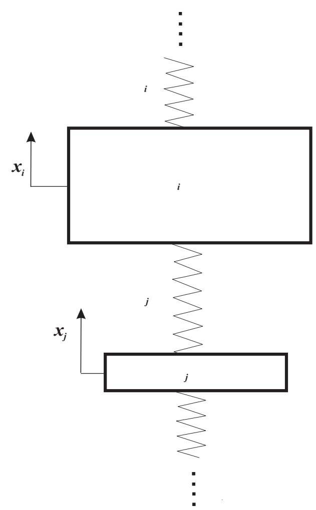

Figure 1 Representation of the system with n degrees of freedom

In these two steps a new energy space can be obtained.

1. In the first step one obtains a new space V with the basis E

$$E = (e_{1x}, e_{1y}, e_{2x}, e_{2y}, \ldots, e_{nx}, e_{ny}), \tag{1}$$

where:

$$e_{ix} = \left[0, ..., 0, \underset{2i-1}{\underset{\uparrow}{1}}, 0, \ldots, 0 \right]^T, \tag{2}$$

$$e_{iy} = \left[0, \ldots, 0, \underset{2i}{\underset{\uparrow}{1}}, 0, \ldots, 0 \right]^T. \tag{3}$$

Let v be an element of the energy space V, with the components

$$z_1, y_1, z_2, y_2, \ldots, z_n, y_n$$

with respect to the basis E.

$$v = [z_1, y_1, z_2, y_2, \ldots, z_n, y_n]^T. \tag{4}$$

Each vector of the space V can be obtained from the linear combination of the basis vectors then

$$\begin{aligned} v &= z_1 e_{1xz} + y_1 e_{1y} + z_2 e_{2xz} + y_2 e_{2y} + ... + z_n e_{nxz} + y_n e_{ny} = \\ &\quad [e_{1x}, e_{1y}, e_{2x}, e_{2y}, ..., e_{nx}, e_{ny}][z_1, y_1, z_2, y_2, ..., z_n, y_n]^T. \end{aligned} \tag{5}$$

2. One can see the second step as a transformation of the space V or a change of the basis vectors of the space V. Let the second step be a change of the basis vectors of the space V. Then the new energy basis E_N of this space is

$$E_N = (b_{1z}, b_{1y}, b_{2z}, b_{2y}, \ldots, b_{nz}, b_{ny}) \tag{6}$$

where:

$$b_{iy} = (\sqrt{\sigma_i})^{-1} \cdot e_{iz} = \begin{bmatrix} 0, \ldots, 0, & \overset{\overset{2i}{\downarrow}}{(\sqrt{\sigma_i})^{-1}} & 0, \ldots, 0, \end{bmatrix}^T , \tag{7}$$

$$b_{iy} = (\sqrt{\mu_i})^{-1} \cdot e_{iy} = \begin{bmatrix} 0, \ldots, 0, & \underset{\underset{2i}{\downarrow}}{(\sqrt{\mu_i})^{-1}} & 0, \ldots, 0, \end{bmatrix}^T . \tag{8}$$

It is easy to prove that the basis vectors are independent linearly.

Let $A_{E_N \leftarrow E}$ be the transition matrix from the basis E to E_N . Then

$$A_{E_N \leftarrow E} = \begin{bmatrix} \sqrt{\sigma_1} & 0 & 0 & ... & ... & 0 \\ 0 & \sqrt{\mu_1} & 0 & ... & ... & 0 \\ 0 & 0 & \sqrt{\sigma_2} & 0 & ... & 0 \\ 0 & 0 & 0 & \sqrt{\mu_2} & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & \sqrt{\sigma_n} & ... \\ 0 & 0 & 0 & ... & ... & \sqrt{\mu_n} \end{bmatrix} . \tag{9}$$

Let v_e be the vector of V with respect to the energy basis E_N. The v_e vector components can be obtained from the components of the vector v, using the transition matrix in the following way:

$$v_e = A_{E_N \leftarrow E} \cdot v\,. \tag{10}$$

Then

$$v_e = [\sqrt{\sigma_1} z_1, \sqrt{\mu_1} y_1, \sqrt{\sigma_2} z_2, \sqrt{\mu_2} y_2, \ldots, \sqrt{\sigma_n} z_{n,} \sqrt{\mu_n} y_{n,}]^T_{E_N} , \tag{11}$$

$$u = [u_{1z}, u_{1y}, u_{2z}, u_{2y}, \ldots, u_{nz}, u_{ny}]^T , \tag{12}$$

$$w = [w_{1z}, w_{1y}, w_{2z}, w_{2y}, \ldots, w_{nz}, w_{ny}]^T , \tag{13}$$

The energy product of the vectors u and w is defined as follows:

$$\langle u, w \rangle = \frac{1}{2} (u_{1z} w_{1z} + u_{1y} w_{1y} + u_{2z} w_{2z} + u_{2y} w_{2y} + \ldots + u_{nz} w_{nz} + u_{ny} w_{ny}) . \tag{14}$$

It is easy to prove that it satisfies three product conditions

1. Linearity $\langle u' + u", w \rangle = \langle u', w \rangle + \langle u", w \rangle$;
2. Symmetry $\langle a \cdot u, w \rangle = a \langle u, w \rangle$;
3. Positiveness $\langle v, v \rangle \neq 0$ for $v \neq 0$, and $0 \cdot 0 = 0$;

for all vectors u',u'',w, u of the space V.

In such an energy space with the energy product, the norm of the vector v_e where:

$$v_e = [\sqrt{\sigma_1}z_1, \sqrt{\mu_1}y_1, \sqrt{\sigma_2}z_2, \sqrt{\mu_2}y_2, \ldots, \sqrt{\sigma_n}z_n, \sqrt{\mu_n}y_n]^T_{E_N} \tag{15}$$

can be obtained in the following way:

$$\begin{aligned} |v_e| &= \sqrt{\langle v_e, v_e\rangle} \\ &= \sqrt{\frac{1}{2}\left[(\sqrt{\sigma_1}z_1)^2 + (\sqrt{\mu_1}y_1)^2 + + \ldots + (\sqrt{\sigma_n}z_n)^2 + (\sqrt{\mu_n}y_n)^2\right]} \\ &= \sqrt{\frac{\sigma_1 z_1^2}{2} + \frac{\mu_1 y_1^2}{2} + \frac{\sigma_2 z_2^2}{2} + \frac{\mu_2 y_2^2}{2} + \ldots + \frac{\sigma_n z_n^2}{2} + \frac{\mu_n y_n^2}{2}} \\ &= \sqrt{E_{p1} + E_{k1} + E_{p2} + E_{k2} + \ldots + E_{pn} + E_{kn}} = \sqrt{\sum_{i=1}^{n} E_{p_i} + E_{k_i}}\,, \end{aligned} \tag{16}$$

where:

E_{pi} – the potential energy accumulated in the i-spring,
E_{ki} – the kinetic energy of the i-mass.

Finally,

$$|v_e|^2 = \sum_{i=1}^{n} E_{p_i} + E_{k_i} = \langle v_e, v_e\rangle\ . \tag{17}$$

It can be seen that the phase space is transformed in a special way. The energy product of the vector v_e by itself equals the total energy accumulated in the system. In other words, the norm of the vector v_e is equal to the square root of the total energy accumulated in the system. Thus, the end of the vector of the system with constant energy moves on the surface of the sphere with the radius equal to the square root of the total energy.

The consideration of the projections of the vector on the energy planes brings some practical advantages of the energy space. As in standard phase space planes all aspects of the kind of the motion can be concluded about from them but these projections also show the amount of the energy accumulated in some parts of the system and the directions of the energy flow between some parts of the system. To make it clearer, consider the system shown in Fig. 2.

2.1. *The application of the energy space in the system with impacts*

The system shown in Fig. 2 has its special application. For some ranges of the system parameters, it works as an impact damper of the motion of the main mass μ (references[8, 9]). The system consists of three oscillators. The external harmonic force excites the main oscillator μ. It is joined with the classical dynamical absorber μ_1. This absorber is allowed to collide with the third oscillator μ_2 which will be called an impact absorber.

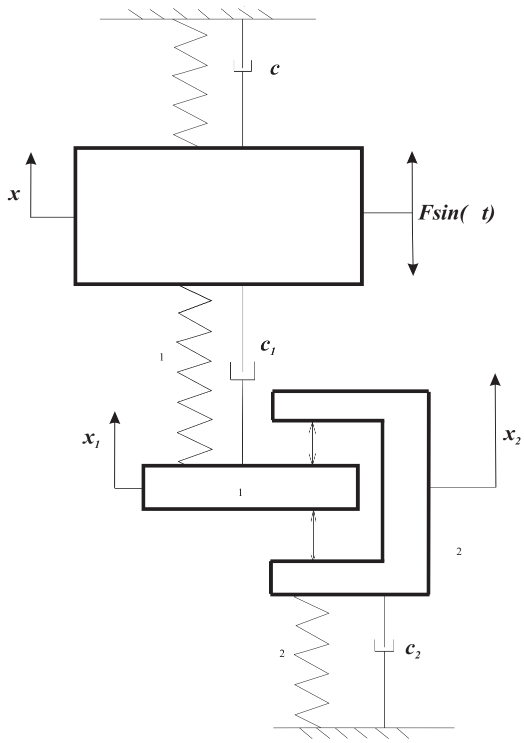

Figure 2 The physical model of the system

In the periods between the impacts the mathematical model of the system is given by six differential equations of the first order:

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= \left(F\sin\eta\tau - cy - c_1\left(y-y_1\right) - \sigma x - \sigma_1\left(x-x_1\right)\right)\cdot\frac{1}{\mu} \\
\dot{x}_1 &= y_1 \\
\dot{y}_1 &= \left(-c_1\left(y_1-y\right) - \sigma_1\left(x_1-x\right)\right)\cdot\frac{1}{\mu_1} \\
\dot{x}_2 &= y_2 \\
\dot{y}_2 &= \left(-c_2y_2 - \sigma_2x_2\right)\cdot\frac{1}{\mu_2}
\end{aligned} \tag{18}$$

where: μ, μ_1, μ_2 – masses (Fig. 2),

σ, σ_1, σ_2 – stiffness coefficients of the springs(Fig. 2),

c, c_1, c_2 – damping coefficients (Fig. 2),

F – amplitude of the external excitation force,

ω – frequency of the external excitation force.

$$\eta = \frac{\omega}{\alpha}\,; \quad \tau = \alpha t\,; \quad \alpha = \sqrt{\frac{\sigma}{\mu}}\,; \tag{19}$$

The impact between the dynamical and impact absorbers is put into the mathematical model of the system over the restitution coefficient.

The phase vector in the standard phase space IR^6 is represented by six components:

$$x;\ y;\ x_1;\ y_1;\ x_2;\ y_2\,. \tag{20}$$

Transform the phase space as follows:

1. Instead of the displacement coordinates x, x_1, x_2, take the deflections of the springs.
$$z = x;\ z_1 = x_1 - x;\ z_2 = x_2\,. \tag{21}$$
2. Depending on the coefficients σ_i and μ_i squeeze and stretch the space in directions of z_i and y_i.

In these two steps a new energy space can be obtained.

In the first step, one obtains a new space V with the basis E:

$$E = (e_z, e_y, e_{1z}, e_{1y}, e_{2z}, e_{2y}) \tag{22}$$

where:

$$\begin{array}{rcl} e_x &=& [1,0,0,0,0,0]^T, \quad e_y = [0,1,0,0,0,0]^T\,, \\ e_{1x} &=& [0,0,1,0,0,0]^T, \quad e_{1y} = [0,0,0,1,0,0]^T\,, \\ e_{2x} &=& [0,0,0,0,1,0]^T, \quad e_{2y} = [0,0,0,0,0,1]^T\,. \end{array} \tag{23}$$

Change the basis vectors of the space V. Then, the new energy basis E_N of this space is:

$$E_N = (b_z, b_y, b_{1z}, b_{1y}, b_{2z}, b_{2y}) \tag{24}$$

where:

$$\begin{array}{rcl} b_z &=& \left[\left(\sqrt{\sigma}\right)^{-1}, 0,0,0,0,0\right]^T, \quad b_y = \left[0, (\sqrt{\mu})^{-1}, 0,0,0,0\right]^T, \\ b_{1z} &=& \left[0,0,(\sqrt{\sigma_1})^{-1}, 0,0,0\right]^T, \quad b_{1y} = \left[0,0,0,(\sqrt{\mu_1})^{-1}, 0,0\right]^T, \\ b_{2z} &=& \left[0,0,0,0,(\sqrt{\sigma_2})^{-1}, 0\right]^T, \quad b_{2y} = \left[0,0,0,0,0,(\sqrt{\mu_2})^{-1}\right]^T. \end{array} \tag{25}$$

The transition matrix $A_{E_N \leftarrow E}$ from the basis E to the basis E_N of the energy space takes the form:

$$A_{E_N \leftarrow E} = \begin{bmatrix} \sqrt{\sigma} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\mu} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\sigma_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\mu_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sigma_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\mu_2} \end{bmatrix}. \tag{26}$$

The coordinates of the vector $v_e = [z_e, y_e, z_{1e}, y_{1e}, z_{2e}, y_{2e}]^T_{EN}$ with respect to the energy basis E_N can be obtained from the vector v with respect to the basis E, using the transition matrix

$$v_e = A_{E_N \leftarrow E} v\,. \tag{27}$$

Then

$$v_e = \left[\sqrt{\sigma}z, \sqrt{\mu}y, \sqrt{\sigma_1}z_1, \sqrt{\mu_1}y_1, \sqrt{\sigma_2}z_2, \sqrt{\mu_2}y_2\right]^T_{E_N} . \tag{28}$$

The norm of the vector v_e in the energy space with the energy product is as follows:

$$\begin{aligned} |v_e| &= \sqrt{\langle v_e, v_e\rangle} \\ &= \sqrt{\frac{1}{2}\left[\sqrt{\sigma}\,(z)^2 + (\sqrt{\mu}y)^2 + (\sqrt{\sigma_1}z_1)^2 + (\sqrt{\mu_1}y_1)^2 + (\sqrt{\sigma_2}z_2)^2 + (\sqrt{\mu_2}y_2)^2\right]} \\ &= \sqrt{\frac{\sigma z^2}{2} + \frac{\mu y^2}{2} + \frac{\sigma_1 z_1^2}{2} + \frac{\mu_1 y_1^2}{2} + \frac{\sigma_2 z_2^2}{2} + \frac{\mu_2 y_2^2}{2}} \\ &= \sqrt{E_p + E_k + E_{p1} + E_{k1} + E_{p2} + E_{k2}}\,, \end{aligned} \tag{29}$$

where:

E_p – the potential energy accumulated in the spring σ,
E_k – the kinetic energy of the mass μ,
E_{p1} – the potential energy accumulated in the spring σ_1,
E_{k1} – the kinetic energy of the mass μ_1,
E_{p2} – the potential energy accumulated in the spring σ_2,
E_{k2} – the kinetic energy of the mass μ_2.

$$\begin{cases} \dot{z}_e = & \sqrt{\frac{\sigma}{\mu}}y_e \\ \dot{y}_e = & -\frac{c}{\mu}y_e - \frac{c_1}{\mu}y_e + \frac{c_1}{\sqrt{\mu\mu_1}}y_{1e} - \sqrt{\frac{\sigma}{\mu}}z_e + \sqrt{\frac{\sigma_1}{\mu}}z_{1e} + \frac{F}{\mu}\sin\eta\tau \\ \dot{z}_{1e} = & -\sqrt{\frac{\sigma_1}{\mu_1}}y_{1e} - \sqrt{\frac{\sigma_1}{\mu}}y_e \\ \dot{y}_{1e} = & -\sqrt{\frac{\sigma_1}{\mu_1}}z_{1e} - \frac{c_1}{\mu_1}y_{1e} + \frac{c_1}{\sqrt{\mu\mu_1}}y_e \\ \dot{z}_{2e} = & \sqrt{\frac{\sigma_2}{\mu_2}}z_{2e} \\ \dot{y}_{2e} = & -\sqrt{\frac{\sigma_2}{\mu_2}}y_{2e} - \frac{c_2}{\mu_2}y_{2e}\,, \end{cases}$$

where

$$\begin{aligned} z_e &= \operatorname{sign} x\sqrt{E_p} \\ y_e &= \operatorname{sign} y\sqrt{E_k} \\ z_{1e} &= \operatorname{sign} x_1\sqrt{E_{p1}} \\ y_{1e} &= \operatorname{sign} y_1\sqrt{E_{k1}} \\ z_{2e} &= \operatorname{sign} x_2\sqrt{E_{p2}} \\ y_{2e} &= \operatorname{sign} y_2\sqrt{E_{k2}}\,. \end{aligned}$$

2.2. *Energy flow and synchronization*

As has been mentioned, the consideration of the energy subspaces brings some special advantages. These subspaces are the equivalents of the phase planes of the phase space. In our case there are four interesting planes.

The first one is the plane which is determined by the basis vectors

$$b_z = \left[\left(\sqrt{\sigma}\right)^{-1}, 0, 0, 0, 0, 0\right]^T , \tag{30}$$

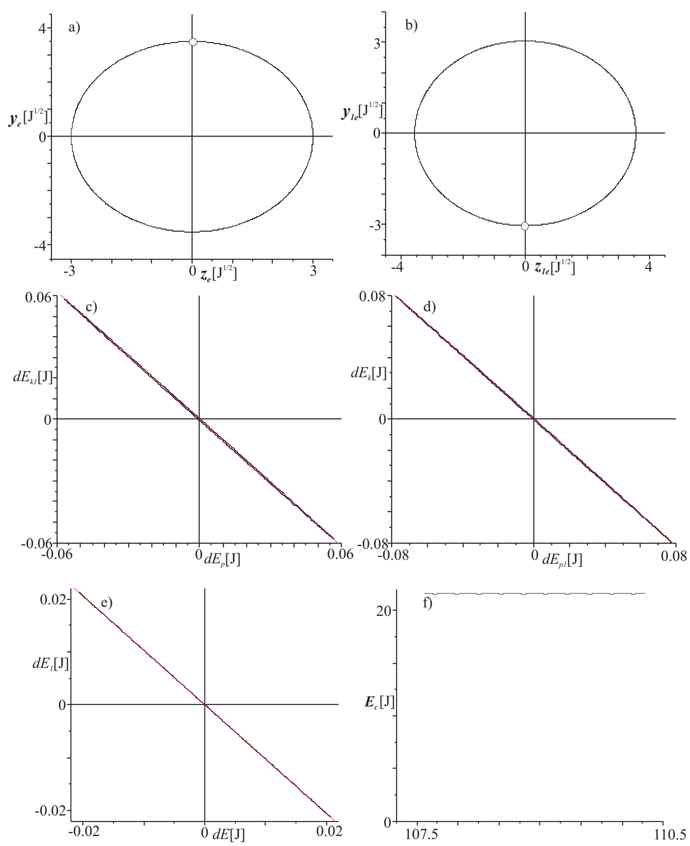

Figure 3 a) The energy plane of the main mass m system; b) The energy plane of the absorber m_1 system ; c), d), e) Differences of the energies in dt intervals f) Total energy of the system. $h = 1.168$, $m = 1$ kg, $m_1 = 0.1$ kg, $s = 100$ N/m, $s_1 = 100$ N/m, $c = 2$ Ns/m, $c_1 = 0.001$ Ns/m, $F = 10$ N

$$b_y = \left[0, (\sqrt{\mu})^{-1}, 0, 0, 0, 0\right]^T . \tag{31}$$

It can be seen from the first two rows of the transition matrix $A_{E_N \leftarrow E}$ that this plane is the squeezed or stretched phase plane $x - \dot{x}$ in the directions given by the basis vectors. The basis vectors b_z and b_y can be considered as the eigenvectors of $A_{E_N \leftarrow E}$ with eigenvalues $\sqrt{\sigma}$ and $\sqrt{\mu}$, respectively. The norm of the vector in that plane gives the energy accumulated in the main mass μ system. In Fig. 3a such a plane can be seen. It shows the motion of the system in the case where impacts do not occur. The value of z_e shows the potential energy of the spring σ and the value of y_e – the kinetic energy of the mass μ, and by means of these coordinates the energies can be calculated. You can see the changes from the potential energy into the kinetic one and vice versa. But see also that the total amount of the energy accumulated in the main mass μ system is not constant. There exists an energy flow between the mass μ and the dynamical absorber μ_1.

In Fig. 3b you can see the energy plane which is determined by the basis vectors

$$b_{1z} = \left[0, 0, \left(\sqrt{\sigma_1}\right)^{-1}, 0, 0, 0\right]^T , \tag{32}$$

$$b_{1y} = \left[0, 0, 0, \left(\sqrt{\mu_1}\right)^{-1}, 0, 0\right]^T . \tag{33}$$

There exist transformations of two kinds which were made on the phase plane $x_1 - \dot{x}_1$to obtain this energy plane. The first one is such that instead of the state variable x_1 we have the spring σ deflection: $z_1 = x_1 - x$. The second kind of transformation is such that the vector which lies on that plane is squeezed or stretched in the directions given by the basis vectors. The basis vectors b_{1z} and b_{1y} can be considered as the eigenvectors of $A_{E_N \leftarrow E}$ with eigenvalues $\sqrt{\sigma_1}$ and $\sqrt{\mu_1}$, respectively. The projection of the vector on that energy plane shows the energy accumulated in the dynamical absorber μ_1 system. The value of z_{1e} shows the potential energy of the spring σ_1 and the value of y_{1e} – the kinetic energy of the mass μ_1, and by means of these coordinates the energies can be calculated.

The energy flow between the main mass μ and the dynamical absorber μ_1 can be seen if you compare Fig. 3a and Fig. 3b. The position of the vector projections on these two energy planes at the same moment of time is marked by a small circle. See that at the same moment the spring σ and spring σ_1 have the maximum of the potential energy, and the same concerns the kinetic energies. Note also that the maximum of the potential energy of the mass μ system equals the maximum of the kinetic energy of the mass μ_1 system and vice versa. The direction of the energy flow can be seen in Fig. 3 (c, d, e), where:

dE_p – the difference of the potential energy of the spring σ in $d\tau$ interval,
dE_k – the difference of the kinetic energy of the mass μ in $d\tau$ interval,
dE_{p1} – the difference of the potential energy of the spring σ_1 in $d\tau$ interval,
dE_{k1} – the difference of the kinetic energy of the mass μ_1 in $d\tau$ interval,
$d\tau$ – 1/1000 of the excitation period,
$dE = dE_p + dE_k$,
$dE_1 = dE_{p1} + dE_{k1}$.

It can be seen, that in each case the dependence is linear. Approximations by the linear function

$$y = \alpha x + \beta .$$

give following results:
c) $\alpha = -1.03$ and $\beta = 1.0e^{-7}$,
d) $\alpha = -1.03$ and $\beta = 1.0e^{-7}$,
e) $\alpha = -1.03$ and $\beta = 9.0e^{-13}$.

It shows some interesting aspects of the continuous energy flow between mass μ system and μ_1 system. All the energy, that flows out the mass μ system is intercepted by μ_1 system, and vice versa (Fig. 3e). The potential energy of the spring σ transforms into kinetic energy of the mass and vice versa (Fig.3c). The same situation takes place with the potential energy of the spring σ_1 and the kinetic energy of the mass μ (Fig. 3d). Additionally, the energy of the whole system is

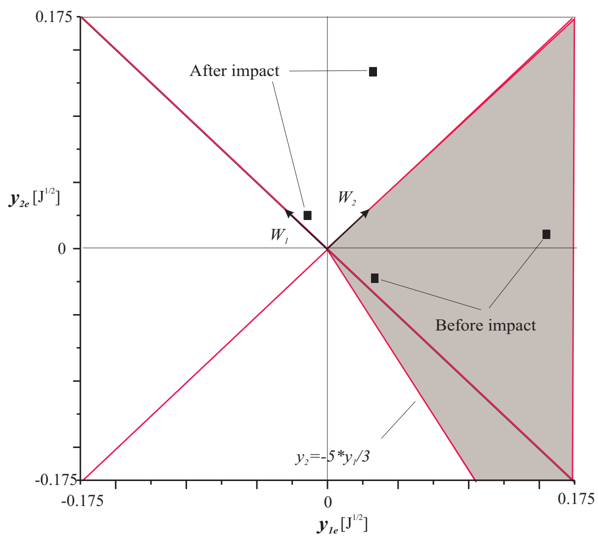

Figure 4 The impact map: $h = 0.618$, $m = 1$ kg, $m_1 = m_2 = 0.1$ kg, $s = 1000$ N/m, $s_1 = 110.25$ N/m, $s_2 = 90.25$ N/m, $c = 6.3$ Ns/m, $c_1 = 0$ Ns/m, $c_2 = 4$ Ns/m, $F = 10$ N, $d = 0.012$ m, $r = 0.7$

constant, what can be seen in Fig. 3f. The energy accumulated in the system flows only between the masses μ and μ_1 systems and the excitation energy is dissipated in the damper c. This energy flow synchronization is possible just only for specially selected parameters. In our case the excitation frequency is equal to the resonance frequency of the system, and the damping coefficient c_1 is close to zero.

2.3. *Energy flow and dissipation during the impact – impact map*

Let π be the plane determined by the basis vectors – Fig. 4. It is better to conclude about all these aspects of impacts from the fourth energy plane (Fig. 5). Let π be the plane determined by the basis vectors:

$$b_{1y} = \left[0, 0, 0, (\sqrt{\mu_1})^{-1}, 0, 0\right]^T , \tag{34}$$

$$b_{2y} = \left[0, 0, 0, 0, 0, (\sqrt{\mu_2})^{-1}\right]^T . \tag{35}$$

The vectors y_{1e} and y_{2e} correspond to the kinetic energies of the masses μ_1 and μ_2 respectively. The position of the vector is marked on this plane just before and after impact. Thus, this plane is a special kind of the impact map. In order to make this map clearer, note that it represents only the points for different types of impact. As the choice criterion, $y_{1e} < y_{2e}$ has been applied. It is possible because impacts in both the absorber stops are symmetrical. The consideration of the position and norm of the vector on this map allows one to conclude about the energy flow between the dynamical and impact absorbers and also about the energy dissipation during each collision. The energy dissipation is included into the mathematical model of the system over the restitution coefficient.

Let $v_{e\pi}$ be the projection of the v_e on the plane π just before the impact, and $v'_{e\pi}$ after it.

The transformation of the vector $v_{e\pi}$ during each impact is given by the matrix

$$A_{E_N}: \quad v'_{e\pi} = A_{EN} v_{e\pi} \,,$$

where:

$$A_{E_N} = \begin{bmatrix} \frac{\mu - r}{\mu + 1} & \frac{1 + r}{\mu + 1} \\ \frac{\mu + \mu r}{\mu + 1} & \frac{1 - \mu r}{\mu + 1} \end{bmatrix} \tag{36}$$

$$\mu = \frac{\mu_1}{\mu_2}$$

and r is the restitution coefficient.

In the case when $\mu = 1$, which means that the masses of the dynamical and impact absorbers are equal, the eigenvalues of the A_{E_N} matrix are:

$$\lambda_1 = -r \lambda_2 = 1 \,, \tag{37}$$

and the eigenvectors are:

$$w_1 = [-1, 1]^T \, w_2 = [1, 1]^T \,, \tag{38}$$

respectively.

The directions of the eigenvectors are shown in Fig. 4. Note that during impact the vector $v_{e\pi}$ is transformed only in the direction given by the eigenvector w_1.

The energy dissipation in time of each collision can be found from the change of the norm of the energy space vector:

$$|v_{e\pi}| - |v'_{e\pi}| \tag{39}$$

The maximum dissipation of the energy takes place when the vector $v_{e\pi}$ has the same direction as the eigenvector w_1. Then

$$v'_{e\pi} = \lambda_1 \cdot v_{e\pi} \tag{40}$$

Taking in the consideration that

$$|v_{e\pi}| = \sqrt{E_{K1} + E_{K2}} \tag{41}$$

where:

E_{K1} – the kinetic energy of the mass μ_1 before impact,

E_{K2} – the kinetic energy of the mass μ_2 before impact,

and

$$|v'_{e\pi}| = \sqrt{E'_{K1} + E'_{K2}} \,. \tag{42}$$

where:

E'_{K1} – the kinetic energy of the mass μ_1 after impact,

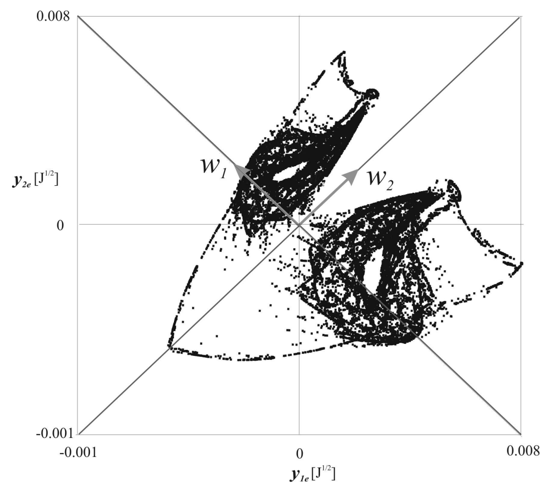

Figure 5 The impact map; $\eta = 1.165$, $\mu = 1$ kg, $\mu_1 = \mu_2 = 0.1$ kg, $\sigma = 1$ N/m, $\sigma_1 = 0.1$ N/m, $\sigma_2 = 0.1$ N/m, $c = 0.04$ Ns/m, $c_1 = 0.01$ Ns/m, $c_2 = 0.02$ Ns/m, $F = 0.002$ N, $\delta = 0.0216$ m, $r = 0.5$

E'_{K2} – the kinetic energy of the mass μ_2 after impact,

one can find easily that the energy relation after and before impact assumes the form:

$$\frac{E'_{K1} + E'_{K2}}{E_{K1} + E_{K2}} = \lambda_1^2 = r^2 \, . \tag{43}$$

The closer the direction of the vector $v_{e\pi}$ is to the second eigenvector w_2, the less energy dissipation occurs. In the case when the directions of $v_{e\pi}$ and w_2 are the same, there is no energy dissipation during the collision. The velocities of the masses μ_1 and μ_2 are equal then and in practice we do not know if impact occurs or not. It is the so called grazing collision and it causes chaotic motion of the system (Fig. 5).

The transformation matrix A_{E_N} allows one also to divide the impact map into two kinds of fields: the first one for the case when the energy flows during impact from the dynamical to impact absorber and the second when the energy flows in the opposite direction.

Consider the matrix A_{E_N} in the case $\mu_1 = \mu_2$:

$$A_{E_N} = \begin{bmatrix} \frac{1-r}{2} & \frac{1+r}{2} \\ \frac{1+r}{2} & \frac{1-r}{2} \end{bmatrix} \tag{44}$$

Let:

$$v_{e\pi} = [v_1, v_2] \, T \text{ and } v'_{e\pi} = [v'_1, v'_2]^T \, . \tag{45}$$

Then

$$v'_{e\pi} = A_{E_N} \cdot v_{e\pi}, \tag{46}$$

so

$$\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} \frac{1-r}{2} & \frac{1+r}{2} \\ \frac{1+r}{2} & \frac{1-r}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} . \tag{47}$$

The energy flow from the dynamical to impact absorber is given by the condition:

$$-v_1 \leq v_1' \leq v_1\,. \tag{48}$$

Consider :

1. $v_1' = v_1$
   The condition is satisfied when the directions of $v_{e\pi}$ and the eigenvector $w_2$are the same.

2. $v_1' = -v_1$. then
$$-v_1 = \frac{1-r}{2}v_1 + \frac{1+r}{2}v_2\,,$$
   and we obtain
$$v_2 = -\frac{3-r}{1+r}\cdot v_1\,. \tag{49}$$

For the case shown in Fig. 4 $r = 0.5$ and then

$$v_2 = -\frac{5}{3}\cdot v_1\,. \tag{50}$$

As a result, the energy flow from the dynamical to the impact absorber occurs: if $v_1 > 0$

$$-\frac{5}{3}v_1 \leq v_2 \leq v_1\,; \tag{51}$$

if $v_1 < 0$

$$v_1 \leq v_2 \leq -\frac{5}{3}v_1. \tag{52}$$

The field of energy flow in this direction is marked in Fig. 4 by the grey colour. Note that it is just for $y_{1e} < y_{2e}$, which was the criterion of the impact choice. It can be seen that during both impacts the energy flows from the dynamical to impact absorber.

3. The energy space for the system with nonlinear characteristic of spring

Consider the system shown in Fig. 6 with nonlinear spring of the mass μ system. The mathematical model of the system is given by four differential equations of the first order:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \left(F\sin\eta\tau - cy - c_1\left(y - y_1\right) - \sigma x^3 - \sigma_1\left(x - x_1\right)\right)\cdot\frac{1}{\mu} \\ \dot{x}_1 = y_1 \\ \dot{y}_1 = \left(c_1\left(y - y_1\right) + \sigma_1\left(x - x_1\right)\right)\cdot\frac{1}{\mu_1}\,. \end{cases} \tag{53}$$

Because of nonlinearity of the spring σ the transformation of the phase space, to obtain the energy space, in direction of x is nonlinear. Thus the description given in paragraph 2 can be applied only after linearization of the system, in neighbourhood of selected points, such as a critical points. In general case we have to show the

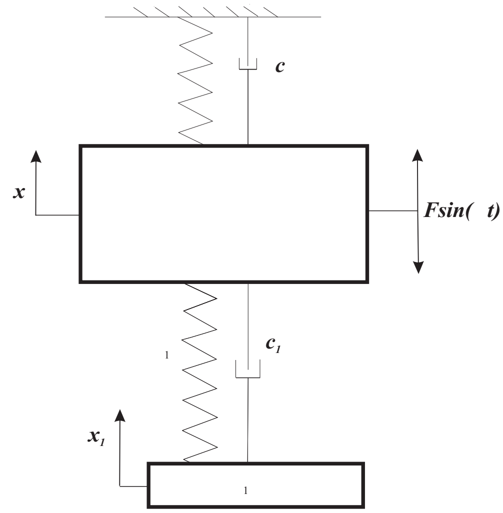

Figure 6 The physical model of the system

transformation of the space, as function $f : \mathbf{R}^4 \to \mathbf{R}^4$, that transforms the phase space the following way:

$$f(x, y, x_1, y_1) = \begin{cases} \sqrt{\frac{\sigma}{2}} x^2 sign(x) = z_e \\ \sqrt{\frac{\mu}{2}} y = y_e \\ \sqrt{\frac{\sigma_1}{2}} (x_1 - x) = z_{1e} \\ \sqrt{\frac{\mu_1}{2}} y_1 = y_{1e} \,. \end{cases} \tag{54}$$

To simplify the description note, that in the energy space there are square roots of the potential or kinetic energies on each axis.

$$\begin{cases} \dot{z}_e = & 2\frac{\sqrt[4]{\sigma}}{\sqrt{\mu}} z_e^{0.5} y_e \\ \dot{y}_e = & -\frac{c}{\mu} y_e - \frac{c_1}{\mu} y_e + \frac{c_1}{\sqrt{\mu\mu_1}} y_{1e} - \sqrt[4]{\frac{4\sigma}{\mu^2}} z_e^{1.5} + \sqrt{\frac{\sigma_1}{\mu}} z_{1e} + \frac{F}{\sqrt{2\mu}} \sin \eta\tau \\ \dot{z}_{1e} = & -\sqrt{\frac{\sigma_1}{\mu_1}} y_{1e} - \sqrt{\frac{\sigma_1}{\mu}} y_e \\ \dot{y}_{1e} = & -\sqrt{\frac{\sigma_1}{\mu_1}} z_{1e} - \frac{c_1}{\mu_1} y_{1e} + \frac{c_1}{\sqrt{\mu\mu_1}} y_e \,. \end{cases}$$

3.1. *Energy flow and synchronization*

To simplify description of the energy flow analysis for system with nonlinear spring, let us to say, that considered case is similar to the one shown in Section 2.2. All the energy, that flows out the mass μ system is intercepted by μ_1 system, and vice versa (Fig. 7e) In that case approximations by the linear function

$$y = \alpha x + \beta$$

give results:

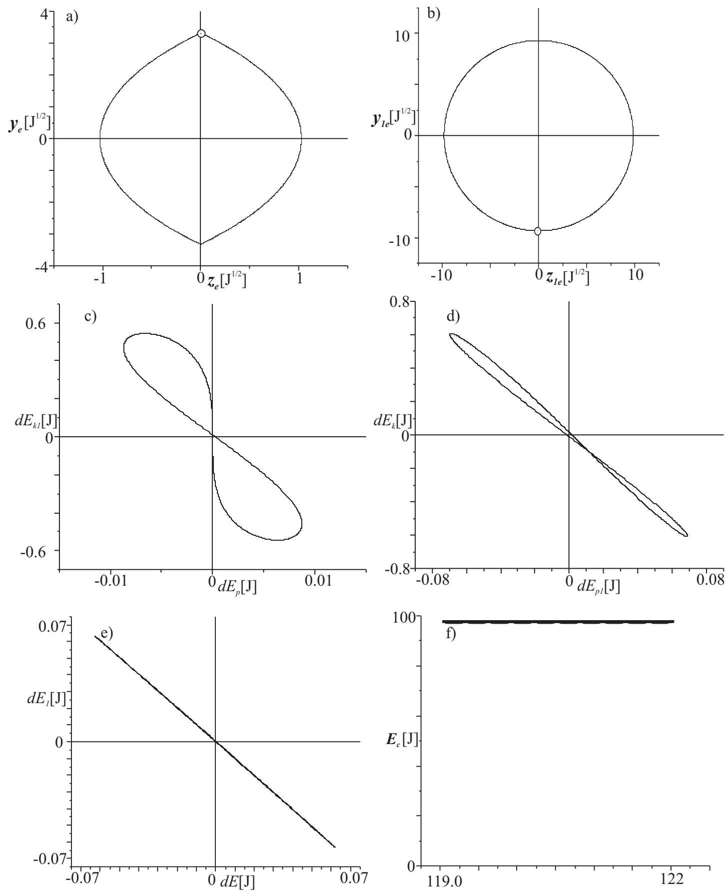

Figure 7 a) The energy plane of the main mass m system; b) The energy plane of the absorber m_1 system ; c), d), e) Differences of the energies in dt intervals f) Total energy of the system. $h = 1.057$, $m = 1$ kg, $m_1 = 0.1$ kg, $s = 100$ N/m, $s_1 = 100$ N/m, $c = 2$ Ns/m, $c_1 = 0.001$ Ns/m, $F = 10$ N

e) $\alpha = -1.01$ and $\beta = 0.002$.

The energy accumulated in the system flows only between the masses μ and μ_1 systems and the excitation energy is dissipated in the damper c. The energy of the whole system is constant, what can be seen in Fig. 7f.

What differs linear and nonlinear case is the energy flow synchronization. In nonlinear system there exists different type of the energy flow between the main mass μ and the dynamical absorber μ_1. what can be seen in Fig. 7c, 7d. Each dependence is not linear. What is interesting the energy of the dynamical absorber μ_1 is constant what can be seen in Fig. 7b (circle).

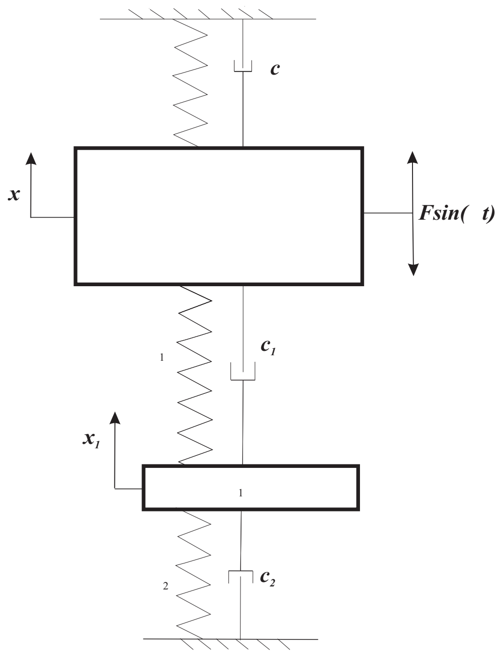

Figure 8 The example of the coupling

4. The application of the energy space in the system with more complicated coupling

In case of the more complicated coupling between the masses, especially when the number of the springs is bigger than the number of the masses (Fig. 8), the dimension of the energy space is different from the dimension of the phase space. It results from the first step of transformation of the phase space to obtain the energy one. In case which is shown in Fig. 8 one should consider the potential energy of three springs. Thus the displacement coordinates x, x_1 of the phase space change into three springs deflections coordinates $z = x$, $z_1 = x_1 - x$, $z_2 = x_1$. After this step of the transformation coordinates of the vector v are as follows:

$$v = [z, y, z_1, z_2, y_1]^T \, , \tag{55}$$

where:

y – velocity of the mass μ,

y_1 – velocity of the mass μ_1.

In that way the dimension of the energy space is five not four. In the case which is considered the second step of the transformation to obtain the energy space is

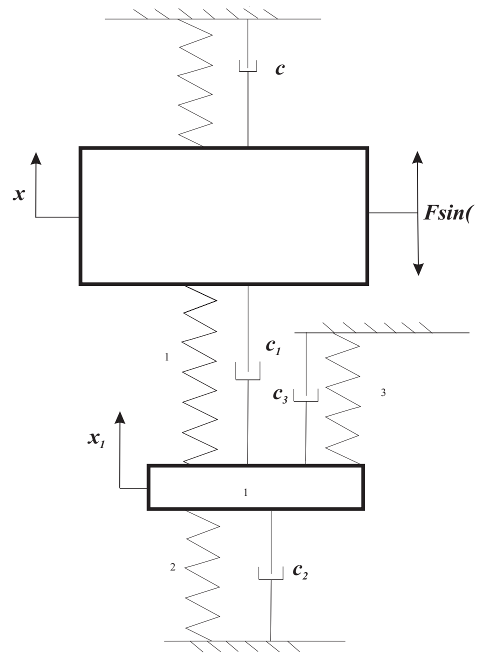

Figure 9 The example of the coupling

given by the following matrix:

$$A_{E_N \leftarrow E} = \begin{bmatrix} \sqrt{\sigma} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\mu} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\sigma_1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\sigma_2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\mu_1} \end{bmatrix} . \tag{56}$$

The coordinates of the vector $v_e = [z_e, y_e, z_{1e}, y_{1e}, z_{2e}, y_{2e}]^T_{E_N}$ with respect to the energy basis E_N can be obtained from the vector v with respect to the basis E, using this transition matrix

$$v_e = A_{E_N \leftarrow E}\ v\,. \tag{57}$$

In case of the coupling which is shown in Fig. 9 one should at first calculate the reduced stiffness of the springs σ_2 and σ_3 ($\sigma_r = \sigma_2 + \sigma_3$) and then follow the pattern given above.

5. Conclusions

The way of transformation of the phase space to obtain the energy space has been shown. It has been proved that this new kind of space allows for concluding about the energy state of a vibrating system. The norm of the vector in that space is equal to the square root of the total energy accumulated in the system. The projection of the vector space on energy subspaces show the amount of the energy that accumulates in some parts of the system. It has been shown that using this kind of spaces, all aspects of the kind of motion can be concluded about, like from the phase space and, moreover, the energy state, accumulation, flow and dissipation can be observed. Different types of the energy flow synchronization were shown. New kind of maps was introduced. It was shown, that the energy space allows for a new, geometrical view on energy changes in vibrating systems.

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