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Capillary Instability of a Streaming Fluid – Core Liquid Jet Under Their Self Gravitating Forces

Ahmed E. Radwan
Mathematics Department, Faculty of Science, Ain-Shams University
Cairo, Egypt
e-mail: ahmed16853@yahoo.com

Emad E. Elmahdy
Mathematics Department, Faculty of Science, Tanta University,
Tanta, Egypt

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The stability criterion of a fluid cylinder (density ρ) embedded into a different fluid (density ρ') is derived and discussed. The model is capillary unstable in the domain $0 < x < 1$ as $m = 0$ where x and m are the axial and transverse wave numbers, while it is stable in all other domains. The densities ratio ρ'/ρ decreases the unstable domains but never suppress them. The streaming increases the unstable domains. Gravitationally, in $m = 0$ mode the model is unstable in the domain $0 < x < 1.0668$ as $\rho' < \rho$, while as $\rho' = \rho$ it is marginally stable but when $\rho' > \rho$ the model is purely unstable for all short and long wavelengths. In $m \neq 0$ modes the self-gravitating model is neutrally stable as $\rho = \rho'$ and ordinary stable as $\rho' < \rho$ but it is purely unstable as $\rho' > \rho$. The streaming destabilizing effect makes the self-gravitating instability worse and shrinks the stable domains. The stability analysis of the model under the combined effect of the capillary and self-gravitating forces is performed analytically and verified numerically. When $\rho' < \rho$ the capillary force and the axial flow have destabilizing influences but the densities ratio ρ'/ρ has a stabilizing effect on the gravitating instability. If $\rho = \rho'$ the streaming is destabilizing but the capillary force is strongly stabilizing and could suppress the gravitational instability. When $\rho' > \rho$ the capillary force improved the gravitational instability and created much domains of stability and moreover the instability of the self-gravitating force disappeared in several cases of axisymmetric disturbances.

Keywords: Capillary, streaming fluid, gravity forces.

1. Introduction

The self-gravitating instability of a fluid cylinder dispersed in self-gravitating infinite liquid of different density has been studied by Radwan (1991). However the investigation of hydrodynamic stability analysis of the interface of two contacted fluids has been started little bit earlier than the comprehensive work of Radwan (1991). Indeed the principle and basic physics of the new type of liquid in air jet are described by Hertz and Hermanrud (1983). The capillary perturbation analysis of a liquid with a thin shell is studied by Petryanov and Shutov (1984) and (1985), see also Mayer and Weihs (1987). For other works see Radwan (2003-2005).

The purpose of the present investigation is elaborating the dynamical oscillation and instability of a streaming fluid cylinder dispersed in a streaming liquid of different density upon the capillary, pressure gradient, inertia and self-gravitating forces of each fluid. Several limiting cases in the literature could be recovered from the present results upon assuming appropriate simplifications.

2. Formulation of the Problem

We consider a fluid cylinder of (radius a) density ρ dispersed in a liquid of density ρ'. Each of these fluids is considered to be incompressible, inviscid and streamed with the uniform velocity.

$$\underline{u}_0 = (0, 0, W)\,. \tag{1}$$

The components of the unperturbed flow (1) are taken along the cylindrical polar coordinates (r, ϕ, z) with the z-axis coinciding with the axis of the fluid core jet. The system is acted upon the self-gravitating, pressure gradient, inertia and capillary forces. The basic equations required for describing the motion of such kind of fluid problems are the combination of the ordinary hydrodynamic equations with those of gravitating Newtonian's theory. Under the present circumstances and for the problem at hand, these basic equations are given by

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u}.\nabla)\underline{u}\right) = -\nabla P - \rho\nabla V\,, \tag{2}$$

$$\nabla.\underline{u} \;=\; 0 \tag{3}$$

$$\nabla^2 V \;=\; 4\pi G\rho\,, \tag{4}$$

$$P_c \;=\; T\,(\nabla.\underline{n}_{\,c})\,. \tag{5}$$

Here $\underline{u}$, ρ, P and V are the fluid velocity vector, fluid density, fluid kinetic pressure and self-gravitating potential; G is the self-gravitating constant. P_c is the curvature pressure due to the surface tension (coefficient T) acting along the interface of the two fluids. $\underline{n}_c$ is a unit outward vector normal to the common boundary fluid-liquid interface given by

$$\underline{n}_c = \frac{\nabla F(r, \phi, z; t)}{|\nabla F(r, \phi, z; t)|} \tag{6}$$

where

$$F\,(r,\, \phi,\, z; t) \;=\; 0 \tag{7}$$

is the equation of the fluid–liquid boundary surface. Similar equations like the system (2)–(4) with dashes over the variables may be written for the liquid outside the fluid cylinder.

3. Unperturbed State

In the unperturbed state the self-gravitating potentials interior and exterior the fluid cylinder, respectively, satisfy

$$\nabla^2 V_0^{int} = 4\pi G\rho\,, \tag{8}$$

$$\nabla^2 V_0^{ext} = 4\pi G\rho'\,, \tag{9}$$

where the subscript o characterizes the unperturbed quantities and later those with index 1 are being perturbed quantities. Integrating equations (8) and (9) and identifying the constants of integration upon applying the boundary conditions that the gravitational potential V_0 and its derivative must be continuous across the boundary surface at $r = a$. Apart from the singular solutions we finally obtain

$$V_0^{int} = \pi G\rho\, r^2\,, \tag{10}$$

$$V_0^{ext} = \pi G a^2 \left[\left(\rho'/a^2\right) r^2 + 2\left(\rho' - \rho\right)\ln\left(a/r\right) - \left(\rho' - \rho\right)\right]. \tag{11}$$

It is worthwhile to mention here that if we suppose that $\rho' = 0$, then the solutions (10) and (11) lead to the same results obtained by Chandrasekhar and Fermi (1953).

Now, returning to equation (2) and to the similar one for the liquid outside the fluid cylinder and integrating them, the kinetic pressures of the fluids in the unperturbed state are given by

$$P_0^{int} = -\rho V_0^{int} + C_1\,, \tag{12}$$

$$P_0^{ext} = -\rho' V_0^{ext} + C_2\,, \tag{13}$$

where C_1 and C_2 are arbitrary constants of integration to be determined. By the use of equations (10) and (11) and apply the condition that the pressures must be balanced at $r = a$, we finally obtain

$$P_0^{int} = -\pi G\rho^2 r^2 + C\,, \tag{14}$$

$$P_0^{ext} = \frac{T}{a} + \pi G a^2\left(\rho'^2 - \rho^2\right) - 2\pi G a^2 \rho'\left(\rho' - \rho\right) - \ln\left(\frac{a}{r}\right) - \pi G \rho'^2 r^2 + C\,, \tag{15}$$

where $C(= C_1)$ is an arbitrary constants with which we need not be further concerned.

One has to mention here that both P_0^{int} and P_0^{ext} are variables, in contrast to the case in which a fluid cylinder (cf. Chandrasekhar 1981) or other models (see Radwan 1995 and 1996) acted by the capillary force or/and other forces rather than the self-gravitating force where it is found there that such pressures are constants.

4. Perturbation Analysis

For small departure from the unperturbed state, every variable quantity $Q(r, \phi, z; t)$ can be expanded as

$$Q\left(r, \phi, z; t\right) = Q(r) + \varepsilon Q_1\left(r, \phi, z\right) \tag{16}$$

where Q stands for P, P', V, V', $\underline{u}$, $\underline{u}'$ and P_c.

Here ε is an infinitesimal amplitude of the perturbation at any instant of time t, given by

$$\varepsilon = \varepsilon_0 \exp(\sigma t) \tag{17}$$

where $\varepsilon_0 (= \varepsilon \text{ at } t = 0)$ is the initial amplitude and σ is the temporal amplifications of the perturbation. If $\sigma (= i\omega,\ i = (-1)^{\frac{1}{2}})$ is imaginary then $\omega / 2\pi$ is the oscillation frequency.

Based on the expansions (16), the deformation of the two fluids interface, assuming sinusoidal wave could be described by

$$r = a + R_1, \quad |R_1| << a\,, \tag{18}$$

with

$$R_1 = \varepsilon a \cos kz \cos m\phi \tag{19}$$

here R_1 is the elevation of the surface wave measured from the unperturbed position, m (integer number) is the transverse wave number and k (a real number) is the longitudinal wave number.

By an appeal to the expansions (16) and (18), the basic equations (2)–(5) yield the following perturbation equations

$$\left(\frac{\partial}{\partial t} + W \frac{\partial}{\partial z}\right) \underline{u}_1 \;=\; -\nabla \Pi_1\,, \tag{20}$$

$$\Pi_1 = \frac{P_1}{\rho} + V_1\,, \tag{21}$$

$$\nabla . \underline{u}_1 = 0\,, \tag{22}$$

$$\nabla^2 V_1 \;=\; 0\,, \tag{23}$$

$$P_{1c} = -\frac{T}{a^2} \left(R_1 + \frac{\partial^2 R_1}{\partial \phi^2} + a^2 \frac{\partial^2 R_1}{\partial z^2} \right)\,. \tag{24}$$

Similar system of equations like (20)–(24), but with dashes over the variables, may be written for the liquid exterior the fluid cylinder. based on the linear perturbation technique and in view of the ϕ and z dependence (19), any perturbed quantity $Q_1(r, \phi, z; t)$ may be expressed as an amplitude function of (r) times $\cos(kz) \cos(m\phi)$, viz.,

$$Q_1(r, \phi, z; t) = \varepsilon q_1(r) \cos(kz + \frac{n\pi}{2}) \cos(m\phi + \frac{n\pi}{2}) \tag{25}$$

where n is an integer and q_1 is function in r only. By the use of $(r, \phi, z; t)$ dependence (25), equation (23) turned, to a total second order differential equation in $q_1(r)$ whose solution is given in terms of the ordinary Bessel functions with imaginary argument. Therefore, the non-singular solution of (23) and the similar one interior and exterior the fluid cylinder are given by

$$V_1^{int} = A^{int} I_m(kr) \cos(kz) \cos(m\phi)\,, \tag{26}$$

$$V_1^{ext} = A^{ext} K_m(kr) \cos(kz) \cos(m\phi)\,. \tag{27}$$

Here $I_m(kr)$ and $K_m(kr)$ are the modified Bessel functions of first and second kind of order m while A^{int} and A^{ext} are arbitrary constants. The latter may be determined upon applying the boundary conditions that $V(= V_0 + \varepsilon V_1)$ and its derivative must be continuous across the perturbed interface (18) at $r = a$. These boundary conditions yield

$$A^{int}\, I_m\,(x) \;=\; A^{ext}\; k_m\,(x)\,, \tag{28}$$

$$X(A^{int} I'_m(x) - A^{ext} K'_m(x)) = 4\pi G a^2(\rho' - \rho)\,, \tag{29}$$

where $x(= ka)$ is the dimensionless wave number. Equations (28) and (29) give A^{int} and A^{ext}, so that the change in V^{int} and V^{ext} due to the deformation (16) is given by

$$V_1^{int} = 4\pi G a^2(\rho' - \rho) K_m(x) I_m(kr)\cos(kz)\cos(m\phi)\,, \tag{30}$$

$$V_1^{ext} = 4\pi G a^2(\rho' - \rho) I_m(x) K_m(kr)\cos(kz)\cos(m\phi)\,. \tag{31}$$

By combining equations (20)–(22) and using the expansion (25), we get

$$r^{-1}\frac{d}{dr}\left(r\frac{d\Pi_1}{dr}\right) - \left(\frac{m^2}{r^2} + k^2\right)\Pi_1 = 0\,. \tag{32}$$

Therefore the non-singular solution of equation (23) interior and exterior the fluid cylinder are given by

$$\Pi_1^{int} = B^{int} I_m(kr)\cos(kz)\cos(m\phi)\,, \tag{33}$$

$$\Pi_1^{ext} = B^{ext} K_m(kr)\cos(kz)\cos(m\phi)\,, \tag{34}$$

where B^{int} and B^{ext} are unspecified constants of integration to be determined. These constants may be determined by applying the boundary conditions that the normal component of the velocity vector must be continuous and compatible with the velocity of the deformed fluid-liquid interface (18) at the unperturbed position at $r = a$. Finally, Apart from the singular solution, we obtain for the interior fluid cylinder

$$\underline{u}_1^{int} = \frac{(\sigma + ikW)a^2}{xI'_m(x)}\nabla\left[I_m(kr)\cos(kz)\cos(m\phi)\right]\,, \tag{35}$$

$$P_1^{int} = \rho\left(\frac{-\left(\sigma + ikW\right)^2 a^2)}{xI'_m(x)} + 4\pi G a^2(\rho - \rho')K_m(x)\right) I_m(kr)\cos(kz)\cos(m\phi)\,. \tag{36}$$

While in the liquid region surrounding the fluid cylinder we have

$$\underline{u}_1^{ext} \;=\; \frac{(\sigma +\, i\, kW)\, a^2}{x\, K'_m\,(x)}\;\nabla\;\left[K_m\,(kr)\;\;\cos\,(kz)\;\cos\,(m\phi)\right]\,, \tag{37}$$

$$P_1^{ext} \;=\; \rho'\;\left[\frac{-\;\left(\sigma +\; i\, kW\right)^2\, a^2)}{xK'_m\,(x)} \;+\; 4\pi G a^2\,(\rho - \rho')\, I_m\,(x)\right] K_m\,(kr)\;\cos(kz)\;\cos\,(m\phi)\,. \tag{38}$$

And along the fluid-liquid interface (see equation (24)), we get

$$P_{1c} = \frac{-T}{a}(1 - m^2 - x^2)\cos kz \cos m\phi \,. \tag{39}$$

Moreover, upon applying the condition that the normal component of the total stress must be continuous across the interface (3.18) at the unperturbed position $r = a$, following eigenvalue relation is obtained

$$\begin{aligned}(\sigma + i\,kW)^2 \quad &= \quad 4\pi G\,(\rho + \rho')\left(\frac{1-S}{1+S}\right)\xi_m\,(x)\,\left[(1-S)I_m\,(x)\,K_m\,(x) - \frac{1}{2}\right] \\ &+ \frac{T}{a^3(\rho+\rho')}(1+S)\xi m(x)(1-m^2-x^2)\end{aligned} \tag{40}$$

with

$$\xi_m(x) = xI'_m(x)K'_m(x)\left[I_m(x)K'_m(x) - SI'_m(x)K_m(x)\right]^{-1} \,, \tag{41}$$

$$S = \frac{\rho'}{\rho}\,. \tag{42}$$

5. Discussions

5.1. *General Discussions*

The eigenvalue relation (39) is a simple linear combination of capillary eigenvalue relation of a streaming fluid-core liquid cylinder and self-gravitating eigenvalue relation of a streaming fluid-core liquid jet. Indeed this behavior of the simple linear combination has been expected as obtained before by Radwan (1996) and (1997) for different ideal fluid models acted by the electromagnetic or electro-dynamic force in addition to the capillary forces. But this behavior is not true as the fluid is viscous or/and resistive, see Radwan (1991) or/and (1992) respectively.

By means of the relation (39) the characteristics of the present model could be determined. The relation (39) relates the temporal amplification σ or rather the oscillation frequency ω (as $\omega = i\sigma$ is imaginary) with the cylindrical functions $I_m(x)$, $K_m(x)$ and their derivatives and the compound function $\xi_m(x)$, the wave numbers m and x, the densities ratio $S(= \rho'/\rho)$ of fluid-liquid media, the natural parameters W, ρ, ρ', a, G and T and with the fundamental natural quantities $4\pi G(\rho + \rho')^{\frac{-1}{2}}$ and $T/(a^3(\rho + \rho'))^{\frac{-1}{2}}$ as a unit of time.

5.2. *Limiting cases*

The present problem belongs a lot of natural parameters since it is two fluids with separating interface. Also the model is acted by different forces, inertia forces, pressure gradient forces and self-gravitating forces.

Some previous published works may be recovered as limiting cases from the present general result.

A lot of approximation ($G = 0$, $\rho' = 0$, $m = 0$ and $W = 0$) are required to obtain

$$\sigma^2 = \frac{T}{\rho a^3}\frac{xI_1(x)}{I_0(x)}(1 - x^2)\,, \quad I'_0(x) = I_1(x) \tag{43}$$

This relation has been derived by plateau (1897) upon utilizing simple concepts and methods. If we suppose that ($G = 0$, $\rho' = 0$, $m \geq 0$ and $W = 0$), the relation (40) reduces to

$$\sigma^2 = \frac{T}{\rho a^3} \frac{x I'_m(x)}{I_m(x)} (1 - m^2 - x^2)\,. \tag{44}$$

This is classical capillary dispersion relation of a full fluid cylinder embedded into a vacuum which is derived analytically for first time by Rayleigh (1945). If we suppose that $T = 0$, $W = 0$, $\rho' = 0$ and simultaneously $m = 0$, the relation (40) degenerates to

$$\sigma^2 = 4\pi G \rho \frac{x I_1(x)}{I_0(x)} \left(I_0(x) K_0(x) - \frac{1}{2} \right)\,. \tag{45}$$

This is the dispersion relation of a self-gravitating fluid cylinder derived, for first time, by Chandrasekhar and Fermi (1953). The latter authors have used a totally different technique rather than used here. In fact they have used the technique of representing the solenoidal vectors in terms of poloidal and toroidal vector fields in the axisymmetric disturbances.

If we put $T = 0$, $\rho' = 0$, $W = 0$ and at the same time $m \geq 0$, the relation (40) reduces to

$$\sigma^2 = 4\pi G \rho \frac{x I^i_m(x)}{I_m(x)} \left(I_m(x) k_m(x) - \frac{1}{2} \right)\,. \tag{46}$$

This relation, coincides with the self-gravitating dispersion relation of a fluid cylinder surrounded by a self gravitating tenuous medium valid for all axisymmetric $m = 0$ and non-axisymmetric $m \neq 0$ modes of perturbation, is derived by Chandrasekhar (1981).

If we put $T = 0$, $G = 0$, $W \geq 0$ and $m \geq 0$ the relation (40) yields

$$(\rho + ikW)^2 = 0\,. \tag{47}$$

This means that there is no dispersion and consequently there is no self-gravitating hollow jet in reality.

If we postulate that $G = 0$, $\rho = 0$, $W = 0$ and $m = 0$, the stability criterion (40) becomes

$$\sigma^2 = \frac{I}{\rho' a^3} \frac{x K_1(x)}{K_0(x)} (1 - x^2) \tag{48}$$

where use has been made of the relation $K'_0(x) = -K_1(x)$.

The relation (48) coincides with the capillary dispersion relation of a hollow cylinder, valid for axisymmetric mode $m = 0$ only, given for the first time by Chandrasekhar (1981).

However, if $G = 0$, $\rho = 0$, $W = 0$ and $m \geq 0$, the relation (40) gives

$$\sigma^2 = \frac{T}{\rho' a^3} \frac{x K'_m(x)}{K_m(x)} (1 - m^2 - x^2) \tag{49}$$

which is, the same capillary eigenvalue relation of a hollow cylinder valid for all kind of disturbances, given by Drazin and Reid (1980). Also it coincides with Radwan, dispersion relation (1988) if we neglect the magnetic field influence there.

If $G = 0$ and $W = 0$, the relation (40) degenerates the capillary dispersion relation of a fluid core liquid cylinder in the form

$$\sigma^2 = \frac{T}{a^3(\rho+\rho')}(1+S)\,.\xi_m(x)(1-m^2-x^2) \tag{50}$$

Also the self-gravitating eigenvalue relation of a fluid-core liquid cylinder could be obtained from (40), as $T = 0$ and $W = 0$, in the form

$$\sigma^2 = 4\pi G(\rho+\rho')\frac{1-S}{1+S}\xi_m(x)\left[(1-S)I_m(x)K_m(x) - \frac{1}{2}\right]. \tag{51}$$

The capillary dispersion relation of a streaming fluid-core liquid cylinder could be obtained from (40) as $G = 0$, in the form

$$(\rho + ikW)^2 = \frac{T}{a^3(\rho+\rho')}(1+S)\xi_m(x)(1-m^2-x^2)\,. \tag{52}$$

Similarly the dispersion relation of a self-gravitating streaming fluid-core liquid cylinder is given from (40), as $T = 0$, in the form

$$(\sigma + ikW)^2 = 4\pi G(\rho+\rho')\xi_m(x)\frac{1-S}{1+S}\left[(1-S)I_m(x)K_m(x) - \frac{1}{2}\right]. \tag{53}$$

6. Stability analysis

In order to investigate the oscillation and instability states of the present problem we have to identify the influence of

(i) the capillary force on using equation (52),

(ii) the self-gravitating force on using (53), and,

(iii) the combined effect of both the capillary and self-gravitating force on using equation (40).

To do so we need to write down about the behavior of the modified Bessel functions and their derivatives. By the use of the recurrence relation (see Abramowitz and Stegun, 1970)

$$2\ell'_m(x) = \ell_{m-1}(x) + \ell_{m+1}(x)\,, \tag{54}$$

where $\ell_m(x)$ stands for $I_m(x)$ and $(-K_m(x))$ while $\ell'_m(x)$ stands for $I'_m(x)$ and $K'_m(x)$, and using the facts for $x \neq 0$ that

$$I_m(x) > 0\,, \quad K_m(x) > 0\,. \tag{55}$$

We observe that

$$I'_m(x) > 0\,, \quad K'_m(x) < 0\,. \tag{56}$$

Therefore, on utilizing the relations (54) and the inequalities (55) and (56) for the relation (41) we deduce that

$$\xi_m(x) > 0\,. \tag{57}$$

for every non-zero real value of x in all axisymmetric $m = 0$ and non-axisymmetric $m \geq 1$ modes of perturbation.

6.1. *Capillary Instability*

The dispersion relation (52) is discussed analytically, taking into account the inequalities (55)–(57). These discussions reveal that the non-streaming fluid-core liquid jet is capillary unstable in the axisymmetric disturbance $m = 0$ in the domain $0 \leq x < 1$. While it is stable in the domains ($m = 0$ as $1 \leq x < \infty$ and $m \geq 1$ as $0 < x < \infty$), moreover, it is found that the area under the instability curves in the domain $0 < x < 1$ decreases with increasing $S(= \rho'/\rho)$ values but never be suppressed whatever is the large value of S. Since the streaming has a destabilizing influence and that influence is independent of the kind of perturbation. Then the streaming has the influence of increasing the unstable domain $0 < x < 1$ but decreasing the stable domains ($m = 0$ as $1 \leq x < \infty$) and ($m \geq 1$ as $0 < x < \infty$). Therefore, we conclude that the streaming fluid-core liquid jet is capillary unstable not only in the mode $m = 0$ but also in the modes $m \geq 1$ of perturbation.

6.2. *Self-gravitating Instability*

The dispersion relation (53) has been discussed analytically, taking into account the identities (55)–(57). It is found that the self-gravitating stable domains of the non-streaming fluid-core liquid jet could be identified through the determining the sign of the quantity $(1-S)[(1-S)I_m(x)K_m(x) - \frac{1}{2}]$. This in fact, has to be carried out not in axisymmetric mode $m = 0$ but also in the non-axisymmetric modes $m \geq 1$ for different cases $S < 1$, $S = 1$ and $S > 1$ because the densities ratio $S(= \rho'/\rho)$ plays an important role in the stability theory of the present self-gravitating model.

(i) Axisymmetric mode $m = 0$
As $S < 1$, it is found numerically that the model is unstable in the domain $0 < x < 1.0668$ as $S = 0$. However, the area under the instability curves decreases with increasing S (in the range $0 < S < 1$) values but never be suppressed. The oscillation domains depend on the values of S and all these domains are larger than the smallest domain $1.0668 < x < \infty$ which corresponds to the case $S = 0$. As $S = 1$ we have a neutral stability state and this kind of stability is independent of: the kind of perturbation, the values of the axial wave number and of the acting forces.

As $1 < S < \infty$, it is found that the fluid-core liquid jet is unstable not only for long wavelengths but also for very short wavelengths. Since the streaming has a strong destabilizing effect. We found that the destabilizing influence of the streaming will produce unstable states as $S = 1$, increasing the all unstable domains as $S > 1$ and also increasing the unstable domains of the case $S < 1$, and decreasing the stable states as $S < 1$.

We conclude that the streaming fluid-core liquid jet is self-gravitating unstable for all $S(= \rho'/\rho)$ values in the axisymmetric disturbance mode $m = 0$.

(ii) Non-axisymmetric modes $m \geq 1$
In such a case of non axisymmetric perturbation, it is found numerically that

$$I_m(x)K_m(x) < \frac{1}{2} \tag{58}$$

For each non-zero real value of x. Therefore we have the following different cases. As $S < 1$, it is found that the stationary model is purely stable for all $x \neq 0$ values. As $S = 1$ we found that $\sigma = 0$. This mean that there is no dispersion and so the stationary model is marginally stable and this is physically clear since in such a case we have an homogeneous medium of uniform density. As $S > 1$ it is found that non-streaming model is purely unstable for each non-zero real value of x in all non-axisymmetric modes $m \geq 1$. Now, since the streaming has a strong destabilizing influence the streaming has the effect of decreasing the stable domain. In the case $0 < S \leq 1$ and increasing the unstable domain in the case $1 < S < \infty$. We conclude that the streaming fluid-core liquid jet is self-gravitational unstable for all values of the densities ratio $S(= \rho'/\rho)$ in the non-axisymmetric modes $m \geq 1$.

6.3. *Hydro-gravitational Instability*

In this section we intend to investigate the stability of a streaming surrounded by a stream liquid under the combined effects of inertia, pressure gradient, self-gravitating and capillary forces. In order to do so we have to use the dispersion relation (40) in its general form.

By the aid of the previous results available to the capillary instability which comes out from the investigating the relation (52) and those from the relation (53) concerning the self-gravitating instability, we may deduce the following results for the present case. In non-axisymmetric modes of perturbation $m = 1, 2, 3 \ldots$ etc, the model is stable or unstable according to certain restrictions whether it is streaming or not.

In the axisymmetric mode $m = 0$ of perturbation, it is found that the analytical discussions of the relation (40) are not enough to describe the dynamical oscillation and instability states in this important modes.

Keep in mind that there were some cases in which ρ'/ρ, the model of the fluid-core liquid jet is purely self-gravitational unstable not only for long wavelengths but also for very short wavelengths. Therefore, in order to complete and clarify such analysis of (in-)stability as $m = 0$, the dimensionless form of the relation (40)

$$\begin{aligned} \frac{(\sigma + ikW)^2}{4\pi G(\rho' + \rho)} &= \frac{1 - S}{1 + S}\left[(1 - S)\, I_m(x) K_m(x) - \frac{1}{2}\right]\xi_m(x) \\ &\quad + N(1 + 5)\xi_m(x)(1 - m^2 - x^2) \end{aligned} \tag{59}$$

where

$$N = T/(4\pi G a^3(\rho' + \rho)^2) \tag{60}$$

has to be computed in the computer for $m = 0$. This has already been done for different values of the parameters N (cf. equation (59) which is dimensionless parameter due to the capillary and self-gravitating forces) $S(= \rho'/\rho)$ the densities ratio and $W^* = -ikW(4\pi G(\rho' + \rho))^{-\frac{1}{2}}$ is dimensionless streaming velocity). The numerical data, concerning the oscillation states as $\sigma(4\pi G(\rho + \rho'))^{-\frac{1}{2}}$ is imaginary and those concerning the instability states as $\sigma(4\pi G(\rho + \rho'))^{-\frac{1}{2}}$ is real, are collected classified, tabulated and presented graphically. See Figures 1–5. There are many features of interest in these numerical results.

The case $0 < S < 1$

When $S = 0.2$ and $W^* = 0.5$: corresponding to $N = 0$, 0.25, 0.5, 1.0, and 2.0 the unstable domains are

$$0 < x < 1.86302\,,$$
$$0 < x < 1.35498\,,$$
$$0 < x < 1.23970\,,$$
$$0 < x < 1.16430\,,$$
$$0 < x < 1.09939\,,$$

and their maxima mode of instability are

$$\sigma_{max} = 0.6645,\ 0.72486,\ 0.7779,\ 0.8613 \text{ and } 0.9932\,.$$

The stable domains are

$$1.86302 \le x < \infty\,,$$
$$1.35498 \le x < \infty\,,$$
$$1.23970 \le x < \infty\,,$$
$$1.16430 \le x < \infty\,,$$
$$1.09939 \le x < \infty\,,$$

where the equalities correspond to the marginal stability, see Figure 1. Thus we

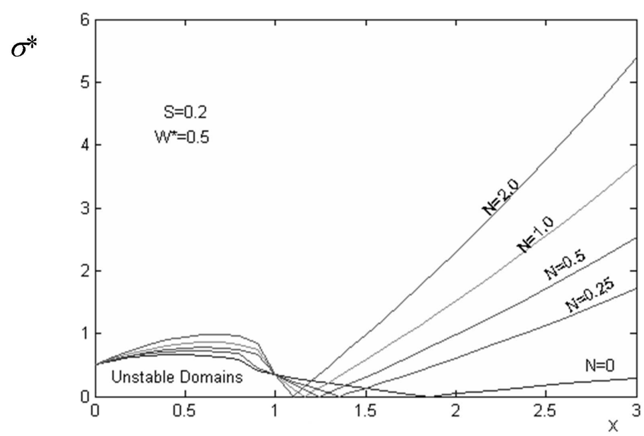

Figure 1 σ^* versus x

can say that each of the axial flow and the capillary force has destabilizing influence but the densities (fluid to liquid) ratio has strong stabilizing influence on the self-gravitating instability.

The case $S = 1$

It is well known in this case that the self-gravitating force has no influence at all on the stability of the fluid-core liquid jet. This is obvious from the dimensionless relation (59) where the factor $(1 - S)$ is occurred in the term contributed to the

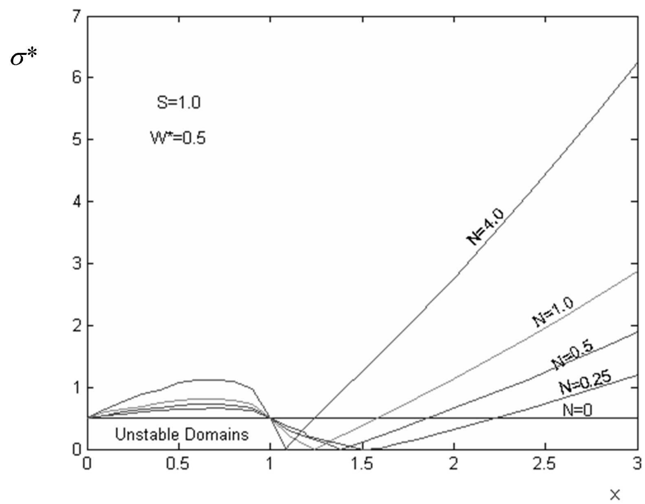

Figure 2 σ^* versus x

self-gravitating forces. Therefore, one has to keep in mind that the stability analysis here will be due to the influence of the inertia, pressure gradient and capillary forces.

When $S = 1.0$ and $W^* = 0.5$: corresponding to $N = 0$, 0.25, 0.5, 1.0 and 4.0 it is found that the unstable domains are

$$\begin{gathered} 0 < x < \infty\,, \\ 0 < x < 1.5856\,, \\ 0 < x < 1.3830\,, \\ 0 < x < 1.2389\,, \\ 0 < x < 1.0870\,, \end{gathered}$$

and their maxima mode of instability are

$$0.5,\ 0.6562,\ 0.7209,\ 0.81243 \text{ and } 1.12484.$$

See Figure 2. As $N = 0.25$, 0.5, 1.0 and 4.0 the stable domains are found to be

$$\begin{gathered} 1.5856 \le x < \infty\,, \\ 1.3830 \le x < \infty\,, \\ 1.2389 \le x < \infty\,, \\ 1.0870 \le x < \infty\,, \end{gathered}$$

see Figure 2. When $S = 1.0$ and $W^* = 0.7$: corresponding to $N = 0$, 0.25, 0.5, 1.0 and 4.0 we may find similar results as above. See Figure 3.

<u>The case $1 < S < \infty$</u>

When $S = 2.0$ and $W^* = 0.5$: corresponding to $N = 0.25$, 0.5, 1.0, and 2.0 it is

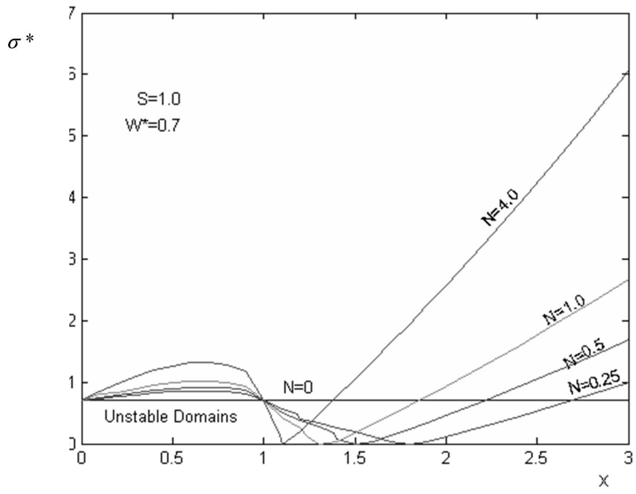

Figure 3 σ^* versus x

found that the unstable domains are

$$0 < x < 2.28995\,,$$
$$0 < x < 1.84664\,,$$
$$0 < x < 1.53930\,,$$
$$0 < x < 1.32987\,,$$

and their maxima mode of instability are

$$1.0412,\ 1.0327,\ 1.0508 \text{ and } 1.1125\,.$$

One has to write down here, as $N = 0$ that the model is purely self-gravitating unstable and its maxima mode of instability is sited near infinity.

Corresponding to $N = 0.25$, 0.5, 1.0 and 2.0 the stable domains are

$$2.28995 \le x < \infty\,,$$
$$1.84664 \le x < \infty\,,$$
$$1.53930 \le x < \infty\,,$$
$$1.32987 \le x < \infty\,,$$

where the equalities corresponding to marginal stability. See Figure 4 for visualization.

When $S = 2.0$ and $W^* = 0.7$: corresponding to $N = 0.25$, 0.5, 1.0 and 2.0 it is found that the unstable domains are

$$0 < x < 2.48880\,,$$
$$0 < x < 2.00387\,,$$
$$0 < x < 1.65875\,,$$
$$0 < x < 1.41553\,,$$

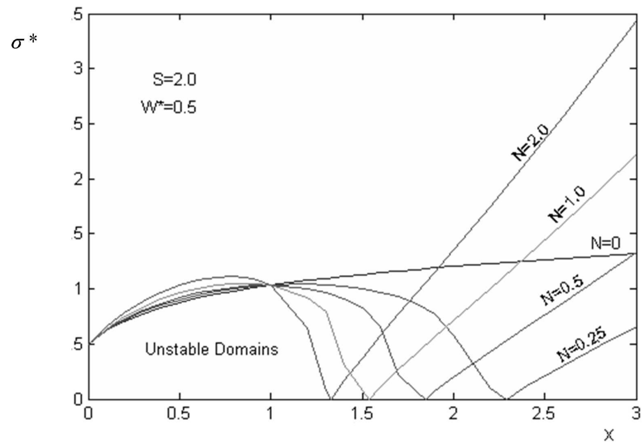

Figure 4 σ^* versus x

and their maxima mode of instability are

$$1.2421,\ 1.2327,\ 1.2508 \text{ and } 1.3125\,.$$

The fluid-core liquid cylinder is infinity unstable as $N = 0$. The neighbour stable domains are

$$2.48880 \leq x < \infty\,,$$
$$2.00387 \leq x < \infty\,,$$
$$1.65875 \leq x < \infty\,,$$
$$1.41553 \leq x < \infty\,,$$

where the equalities correspond to the marginal stability states. See Figure 5. Therefore, for this case in which $S = 2.0$, we conclude that the axial flow has a

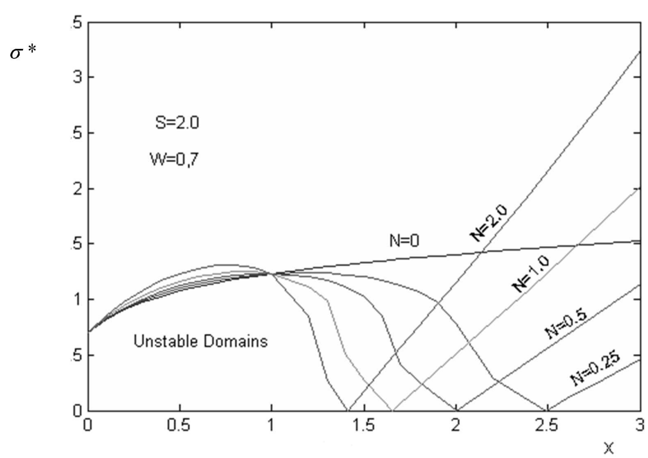

Figure 5 σ^* versus x

destabilizing influence for all N values while the capillary force due to the curvature pressure has stabilizing influence on the gravitational instability in particular for large values of x. The model of the fluid-core liquid cylinder is infinity unstable for all W^* values as $N = 0$ i.e., in the absence of the surface tension. Any how the existence of the capillary force besides the self-gravitating force created some stability states in the dangerous case ($m = 0$ with $S > 1$) whether the model is streaming or not. In fact this important conclusion is vital for our problem because in the case $T = 0$ it is found that the model is purely unstable and the maxima mode of instability of each case is sited at infinity. Therefore the capillary force developed and modified the self-gravitating instability in particular as $S > 1$.

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