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Nonlinear Instability of Two Superposed Magnetic Fluids in Porous Media Under Vertical Magnetic Fields

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The nonlinear analysis of the Rayleigh - Taylor instability of two immiscible, viscous magnetic fluids in porous media, is performed for two layers, each has a finite depth. The system is subjected to both vertical vibrations and normal magnetic fields. The influence of both surface tension and gravity force is taken into account. Although the motions are assumed to be irrotational in each fluid for small perturbations, weak viscous effects are included in the boundary condition of the normal stress balance. The method of multiple scale expansion is used for the investigation. The evolution of the amplitude is governed by a nonlinear Ginzburg - Landau equation which gives the criterion for modulational instability. When the viscosity and Darcy's coefficients are neglected, the cubic nonlinear Schrődinger equation is obtained. Further, it is shown that, near the marginal state, a nonlinear diffusion equation is obtained in the presence of both viscosity and Darcy's coefficients. Stability analysis and numerical simulations are used to describe linear and nonlinear stages of the interface evolution and then the stability diagrams are obtained. Regions of stability and instability are identified.

Keywords: nonlinearity, Rayleigh - Taylor instability, magnetic fluids, porous media, weak viscous fluids

1. Introduction

The stability problem is very important in many industrial applications, e.g. in the mechanical, geophysical, chemical and nuclear engineering industries. It is well known that stability controls are generally required in precision for finishing processes of coating, laser cutting, and casting production. The further application to the stability analysis involves the coating of a moving solid substrate by a liquid layer, resort to dynamic wetting. Since macroscopic instability can cause detrimental conditions to film flows, and thus, can be very harmful to the quality of final products, it is highly desirable to develop suitable working conditions for homogeneous film growth to be adaptable for various flow configurations and associated time - dependent properties.

The linear stability theories for various film flows have been clearly presented by Lin [1] and Chandrasekhar [2]. Chandrasekhar [2] studied in great detail, the stability of a Rayleigh - Taylor problem and analyzed various stability behaviours of this model. A Rayleigh - Taylor instability occurs, when a low density fluid is supporting a higher density fluid against force. That force may be the force of gravity or, as in the case of supernova remnants, the force of an expanding - hot gas accelerating a cooler denser ambient medium. As the instability develops, the denser fluid tries to make its way around the other fluids resulting in long fingers that reach into the low density fluid. In a Hele - Shaw cell, modelling a quasi - two - dimensional system, the one - dimensional interface is destabilized by the growth of fingers regularly spaced on a line with a well - defined wavelength. This wavelength results from the competition between the stabilizing capillary force, and the destabilizing gravitational force [3]. The Rayleigh - Taylor instability plays an important role in subjects such as astrophysics, fusion and turbulence.

The formation of patterns and shapes in the natural world has long been a source of fascination for both scientists and laymen [4] . Nature provides an endless array of patterns formed by diverse physical, chemical and biological systems. The scales of such patterns range from the growth of bacterial colonies to the large - scale structure of the universe [5]. This enormous range of scales over which pattern formation occurs, and the intriguing fact that they emerge spontaneously from an orderless and homogeneous environment captivate our imagination. Interface dynamics plays a major role in pattern formation. It determines the shape of objects, and therefore, it has important applications in a wide range of interdisciplinary fields : hydrodynamics(convection patterns and shapes of boundaries between fluids), metallurgy (dendritic shapes of crystals), and biology (shapes of plants,cells, etc.).

The development of patterns resulting from the Rayleigh - Taylor instability can be divided into three stages: the early linear stage, where the lengths of the rising and falling fingers are small compared to the wavelength, the middle, weakly - nonlinear stage, and the strongly nonlinear late stage. The linear stage is well described but, to our knowledge, no experiment has been made to verify this behaviour. The nonlinear stages are not fully understood.

Magnetic fluids (or ferrofluids) are colloidal dispersions of single domain nanoparticles in a carrier liquid. The attractiveness of ferrofluids stems from the combination of a normal liquid behaviour with the sensitivity to magnetic fields. This enables the use of magnetic fields to control the flow of the fluid, giving rise to a great variety of new phenomena and to numerous technical applications [6]. One of the most interesting phenomena of pattern formation in ferrofluids is the Rosensweig instability or normal field instability [7]. At a certain intensity of the normal magnetic field, the initially flat surface of a horizontal ferrofluid layer becomes unstable. Peaks appear at the fluid surface, which typically form a static hexagonal pattern at the final stage of the pattern forming process. The reader can find much more information about these fascinating complex fluids in Rosensweig's classic book [6].

Now, let the magnetic fluids be confined to the two - dimensional Hele - Shaw cell. Another instability can appear, if the external field is applied in the direction perpendicular to the cell. This phenomenon, called the labyrinthine instability, occurs above a critical value of the applied field and with a critical wavelength. The threshold value of external magnetic field results from a balance between the destabilizing magnetic dipole - dipole repulsion and the stabilizing surface tension (and possibly gravity in a vertical cell). The effect of magnetic field on the viscosity of ferrofluid was first investigated experimentally by Mctague [8]. His results show that the viscosity increases with the strength of magnetic field and reaches saturation as the magnetization saturates . Shliomis [9], then, proposed a theoretical explanation for this effect by using ferrohydrodynamic equations. Due to this characteristic, ferrofluids have been widely used now in numerous engineering fields such as sealing, lubricant, density separation, ink jet printer, etc. [10]. The widespread use of ferrofluids increases the interest of studying for ferroffluid flows. The theoretical investigation on the Rayleigh - Taylor instability in magnetic fluids was carried out [11 - 13]. Linearly,in the absence of magnetic field, Mikaelian [14] studied the effect of viscosity on Rayleigh - Taylor instability for two finite - thickness fluids in the presence of surface tension. The effect of viscosity on the nonlinear evolution of Rayleigh - Taylor instability in the presence of surface tension was studied by Elgowainy and Ashgriz [15].

The effect of viscosity in producing instability at lower flow velocities, than are required for Kelvin - Helmholtz instabilities and has been considered by Weissman [16], and the similar problem of destabilization due to a flexible damped wall by Landahl [17] . A simple example is discussed by Cairns [18], where he considered an inviscid fluid flowing over a heavier, slightly viscous fluid at rest. The effect of the viscosity is clearly to extract energy from this system, and, it produces instability if the flow velocity of the upper fluid is such as to give rise to a negative energy mode on the interface. As has already been pointed out by Weissman [16], this produces an instability for flow velocities below the critical velocity for Kelvin - Helmholtz instability so that viscosity has a destabilizing effect. Both Weissman [16] and Landahl [17] recognized that the physical mechanism for instability lies, in the fact, that the wave on the interface has negative energy.

In order to make use of the domain perturbation technique, we confine the analysis to considering very weak - viscous effects which are believed to be significant only within a thin vortical surface layer, so that, the motions elsewhere may be resonably assumed irrotational [19]. The viscous effects are incorporated by a novel method of formulating a normal damping stress term in the boundary condition at the free surface. This is based on an assumption that, the overall rate of work done by this damping stress, equals the total rate of dissipation of mechanical energy is given by Lamb [20] (also presented in Batchelor [21]). Thus, the derivations in this problem deal completely with potential flow, so that, the complicated manipulation of the boundary- layer equation for the weak vortical flow can be avoided. Since the field equation governing the irrotational flow is the Laplace equation, modifying the boundary conditions at the surface, should be an acceptable means of including the small viscous effects.

At a free surface, the physical conditions to be satisfied for the stress are that the tangential component is zero, and, that the normal component equates the sum of a constant term and any contribution from surface tension[21]. If the viscous forces are very small in comparison with non- viscous forces, then viscosity only produces a thin - weak vortical layer at the free surface, while the motion remains irrotational throughout the bulk of the fluid. The irrotational motion, however, cannot in general satisfy the condition of zero tangential stress at the free surface. Therefore, the non- zero irrotational tangential stress near the surface drags a thin vortical layer along, making a modification to the velocity field. For viscous potential flow[22], the only place where the viscosity enters is in the normal component of the viscous stress at a free surface. Viscous potential flow analysis gives good approximations to fully viscous flows, where the continuity of the tangential stress at the interface must be negligible [22,23].

The aim of this paper is to study the influence of a homogeneous - magnetic field applied perpendicularly to a free surface between two superposed viscous magnetic fluids in porous media, using the method of multiple scale perturbations. Without an external magnetic field and Darcy's coefficients, the first - order expansion of the perturbations yields the same result as obtained by Mikaelian [14] and Pan et.al.[23] for the viscous decay of the free - oscillation modes.

This paper starts in section (2) with the mathematical formulation of the governing equations and an outline of the perturbation scheme. A detailed derivation of the expression for the viscous normal stress, that, appears in the boundary condition at the free surface is presented in the same section. The results of the first - order expansion are discussed in section (3), where the critical magnetic field is obtained. From the second - and third - order expansions developed in section (4) with the second - harmonic resonance , the nonlinear complex Ginzburg - Landau equation is derived in section (5). From this equation, a nonlinear diffusion equation for stationary waves is obtained and the conditions for the existence of the stability. In section (6), a series of graphs for the determination of regions of stability and instability is given near the marginal state. In the absence of both viscosity and Darcy's coefficients, the nonlinear Schrödinger equation is obtained in section (7). Its stability is discussed both analytically and numerically and stability diagrams are also obtained in the same section (7). A breif summary of the results is presented in section (8).

2. Formulation of the Problem

2.1. *Fundamental Equations*

Consider two incompressible, inviscid magnetic fluids confined between two parallel planes $z = -h_1$ (the lower boundary)and $z = h_2$ (the upper one). Let the interface given by $z = \gamma(x,t)$ where $z = 0$ represents the equilibrium interface. Fluid 1 occupies the region $-h_1 < z < \gamma(x,t)$, while fluid 2 occupies the region $\gamma(x,t) < z < h_2$. Both fluids are subjected to a constant magnetic field,in the z - direction, H_1 and H_2, respectively. The two media are considered as porous. The resistance term $(-\eta \underline{v})$ is only taken into account, where $\underline{v}$ is the filter velocity of the fluid, and η is the Darcy's coefficient which depends on the ratio of the fluid viscosity $\mu^\star$ to the flow permeability K through the voids. The acceleration due to gravity g acts in the negative z- direction. The subscripts 1 and 2 refer to the lower and upper fluids respectively, ρ_1 and ρ_2 are the fluid densities , $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are the magnetic permeabilities, μ_1 and μ_2 are the viscosity coefficients and η_1 and η_2 are the Darcy's coefficients of the two fluids. Between the two fluids there exists a surface tension,

denoted by σ. We assume that there are no free currents at the two phases in the equilibrium state. Therefore, we find that the magnetic induction is continuous at the interface, i.e. $\widetilde{\mu}_1 H_1 = \widetilde{\mu}_2 H_2$.
The interface is represented by the expression

$$F(x, z, t) = z - \gamma(x, t) = 0 \tag{1}$$

for which the outward normal vector is written as

$$\underline{n} = \frac{\nabla F}{|\nabla F|} = [1 + (\frac{\partial \gamma}{\partial x})^2]^{-\frac{1}{2}}, \quad (-\frac{\partial \gamma}{\partial x}, 0, 1) \tag{2}$$

The dynamics of the problem is described by the simultaneous solution of three field equations: Maxwell's equations, Brinkman's equation and the continuity equation.

In formulating Maxwell's equations for the problem, we suppose that the magneto - quasi - static approximation is valid [6]. With a quasi - static model, it is recognized that relevant time rates of change are sufficiently low that contributions due to a particular dynamical process are ignorable. The objective in magnetic fluids is concerned with phenomena in which magnetic energy greatly exceeds electric energy storage and where the propagation times of electromagnetic waves are complete in relatively - short times compared to those of interest to us. Accordingly, in a magneto - quasi - static system with negligible displacement current, Maxwell's equations in the absence of free currents are

$$\nabla . \underline{B} = 0 \qquad \text{and} \qquad \nabla \wedge \underline{H} = \underline{0} \tag{3}$$

where$\underline{B} = \widetilde{\mu} \underline{H}$ is the magnetic induction vector.

The magnetic field can be expressed in terms of a magnetic - scalar potential $\Psi(x, z, t)$ in each of the regions occupied by the fluids, i.e.

$$\underline{H}_j = H_j \underline{e}_z - \nabla \Psi_j, \quad j = 1, 2, \tag{4}$$

where $\underline{e}_z$ is the unit vector along the z- direction.

After that, combining the latter equation (4) with equations (3), considering $\widetilde{\mu}$ is a constant, one finds that the magnetic scalar potentials satisfy Laplace's equations:

$$\nabla^2 \Psi_1 = 0 \quad \text{for} \quad -h_1 < z < \gamma(x, t) \tag{5}$$

$$\nabla^2 \Psi_2 = 0 \quad \text{for} \quad \gamma(x, t) < z < h_2 \tag{6}$$

Flow in a porous medium is described by Darcy's law that relates the movement of fluid to the pressure gradients acting on a parcel of fluid. Darcy bases Darcy's law on a series of experiments, in the mid-19 th century, showing that the flow through a porous medium is linearly proportional to the applied pressure gradient and inversely proportional to the viscosity of the fluid. Fluid motion is governed by a set of nonlinear partial differential equations expressing conservation of mass, momentum and energy.

The Dupuit–Forchheimer equation is a commonly used model. We consider media as initially uniform, so that, motion is of homogeneous fluids in a homogeneous medium. Thus, the equations governing two - dimensional motion of a viscous incompressible fluid through a porous medium, since the porosity of the fluid is considered to have unity, are [24, 25]

$$\rho[\frac{\partial \underline{v}}{\partial t} + (\underline{v}.\nabla)\underline{v}] = -\nabla p - \eta\underline{v} + \mu\nabla^2\underline{v} - \rho g\underline{e}_z \tag{7}$$

and the equation of continuity will be

$$\nabla\underline{v} = 0 \tag{8}$$

where p is the hydrostatic pressure. The pressure should be the same as that in an inviscid fluid because the presence of weak viscous influences the wave frequency but not the pressure within the fluid.

In equation (7), we have two viscous terms. The first, is the usual Darcy term, since $\eta = \mu^*/K$, and, the second is analogous to the Laplacian term that appears in the Navier - Stokes equation. The coefficient μ is an effective viscosity. Brinkman (cf.[25]) set μ and μ^* equal to each other, but in general, they are only approximately equal.

Equations (7) and (8) are general fundamental equations of motion, for flow of a homogeneous fluid, in an isotropic medium with constant permeability and low viscosity limit. These equations can be solved to investigate a magnetic fluid flow through a porous medium. The viscosity is assumed to be small, so that the viscous force ($\mu\nabla^2\underline{v}$) may be dropped from Brinkman equation (7). The viscosity contribution will be formulated from the normal stress boundary condition [26, 27]. The components of the stress tensor σ_{ik} read

$$\sigma_{ik} = -(p + \widetilde{\mu}H^2/2)\delta_{ik} + H_iB_k + \mu(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}) \tag{9}$$

where δ_{ik} is Kronecker's delta.

The Rayleigh -Taylor instability may be analyzed as a viscous - potential flow. The theory of potential flows of an inviscid fluid can be readily extended to a theory of potential flow of slightly viscous fluids which admit the pressure (Bernoulli) function. This means that the flows are assumed to be irrotational, except at the interface. In this case, the perturbation velocity is given by a potential $v = \nabla\Phi$ where the velocity potentials $\Phi_j (j = 1, 2)$ must obey Laplace's equations [19, 28-32]

$$\nabla^2\Phi_1 = 0 \quad \text{for} \quad -h_1 < z < \gamma(x,t) \tag{10}$$

$$\nabla^2\Phi_2 = 0 \quad \text{for} \quad \gamma(x,t) < z < h_2 \tag{11}$$

because the fluids are incompressible.

Therefore, Darcy's law (7) is reduced to an identity provided that the pressure is given by Bernoulli's equation

$$p + \rho\frac{\partial\Phi}{\partial t} + \rho g z + \eta\Phi + \frac{1}{2}\rho|\nabla\Phi|^2 = C \tag{12}$$

where C is a Bernoulli constant.

2.2. *Bonduary Conditions*

Two types of boundary conditions suffice to properly constrain the field equations: conditions on the rigid boundaries $z = -h_1$ and $z = h_2$ and conditions at the dividing surface $z = \gamma(x,t)$. The former expresses the requirements that the tangential components of the magnetic field and the normal fluid velocities tend to be zero at these rigid boundaries, i.e.

$$\frac{\partial \Psi_j}{\partial x} = 0, \quad \text{on} \quad z = (-1)^j h_j, \quad j = 1,2 \tag{13}$$

and

$$\frac{\partial \Phi_j}{\partial z} = 0, \quad \text{on} \quad z = (-1)^j h_{j,} \quad j = 1,2 \tag{14}$$

Interfacial boundary conditions can be divided into Maxwell's magnetic conditions, kinematic condition and normal stress tensor balance.

2.2.1. *Maxwell's magnetic conditions:*

1. The normal components of the magnetic - induction vector are continuous since we assume that there is no free current on the interface. This can be written as $\underline{n} \cdot [|\underline{B}|] = 0$, or

$$[|\widetilde{\mu}\frac{\partial \Psi}{\partial z}|] - \frac{\partial \gamma}{\partial x}\,[|\widetilde{\mu}\frac{\partial \Psi}{\partial x}|] = 0, \quad \text{on} \quad z = \gamma(x,t) \tag{15}$$

where $[| \quad |]$ represents the jump or difference across the interface, i.e. $[|X|] = X_2 - X_1$.

2. The tangential components of the magnetic field should be continuous at the interface, i.e. $\underline{n} \wedge [|\underline{H}|] = \underline{0}$, or

$$[|\frac{\partial \Psi}{\partial x}|] + \frac{\partial \gamma}{\partial x}\,[|\frac{\partial \Psi}{\partial z}|] = \frac{\partial \gamma}{\partial x}\,(H_2 - H_1)\,, \quad \text{on} \quad z = \gamma(x,t) \tag{16}$$

2.2.2. *Kinematic condition*

The normal components of the velocity potential must be compatible with the assumed form of the boundary. Such an equation is

$$\frac{\partial \Phi_j}{\partial z} - \frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial x}\;\frac{\partial \Phi_j}{\partial x}\;, \quad j = 1,2, \quad \text{on} \quad z = \gamma(x,t) \tag{17}$$

2.2.3. *The normal stress tensor*

The interfacial normal stress is balanced at the dividing surface of the system, giving

$$\underline{n} \cdot (\underline{\Pi_2} - \underline{\Pi_1}) = \sigma \;\; \nabla \cdot \underline{n} \quad \text{on} \quad z = \gamma(x,t) \tag{18}$$

where $\underline{\Pi}$ is the force vector acting on the interface, given by

$$\underline{\Pi} = \begin{bmatrix} \sigma_{11} & \sigma_{31} \\ \sigma_{13} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_3 \end{bmatrix} \tag{19}$$

Substituting from (2) and (19) into (18), we obtain

$$[(\frac{\partial \gamma}{\partial x})^2[|\sigma_{11}|] - 2\frac{\partial \gamma}{\partial x}[|\sigma_{13}|] + [|\sigma_{33}|] = \sigma \frac{\partial^2 \gamma}{\partial x^2}[1 + (\frac{\partial \gamma}{\partial x})^2]^{-1/2} \tag{20}$$

on $z = \gamma(x, t)$. By eliminating the pressure by Bernoulli's equation (12) and using (9) we can rewrite the above condition (20) as:

$$\begin{aligned}
&\rho_1 \tfrac{\partial \Phi_1}{\partial t} - \rho_2 \tfrac{\partial \Phi_2}{\partial t} + \tfrac{1}{2}[\rho_1(\nabla\Phi_1)^2 - \rho_2(\nabla\Phi_2)^2] + (\rho_1 - \rho_2)g\gamma + \eta_1\Phi_1 - \eta_2\Phi_2 \\
&-2[|\mu \tfrac{\partial^2 \Phi}{\partial z^2}|] + 4\tfrac{\partial \gamma}{\partial x}[|\mu \tfrac{\partial^2 \Phi}{\partial x \partial z}|] - 2(\tfrac{\partial \gamma}{\partial x})^2[|\mu(\tfrac{\partial^2 \Phi}{\partial x^2} - \tfrac{\partial^2 \Phi}{\partial z^2})|] = \sigma \tfrac{\partial^2 \gamma}{\partial x^2}[1 + (\tfrac{\partial \gamma}{\partial x})^2]^{-3/2} \\
&+2(\tfrac{\partial \gamma}{\partial x})[|\widetilde{\mu} H \tfrac{\partial \Psi}{\partial x}|] - (\tfrac{\partial \gamma}{\partial x})^2[|\widetilde{\mu} H^2|] + 2(\tfrac{\partial \gamma}{\partial x})^2[|\widetilde{\mu} H \tfrac{\partial \Psi}{\partial z}|] - 2(\tfrac{\partial \gamma}{\partial x})[|\widetilde{\mu} \tfrac{\partial \Psi}{\partial x} \tfrac{\partial \Psi}{\partial z}|] \\
&-[|\widetilde{\mu} H \tfrac{\partial \Psi}{\partial z}|] - \tfrac{1}{2}[|\widetilde{\mu}(\tfrac{\partial \Psi}{\partial x})^2|] + \tfrac{1}{2}[|\widetilde{\mu}(\tfrac{\partial \Psi}{\partial z})^2|]
\end{aligned} \tag{21}$$

on $z = \gamma(x, t)$

2.3. *Perturbation Analysis*

To investigate the nonlinear effects on the stability of the system, we employ the method of multiple scales and therefore, we introduce

$$x_m = \epsilon^m x, \quad t_m = \epsilon^m t, \quad (m = 0, 1, 2) \tag{22}$$

$$f(x, z, t) = \sum_{m=1}^{3} \epsilon^m f_m(x_0, x_1, x_2, z, t_0, t_1, t_2) + O(\epsilon^4) \tag{23}$$

where ϵ represents a small parameter characterizing the steepness ratio of the wave and f is any of the variables Φ_j , Ψ_j and $\gamma(x, t)$. While writing the expansion for γ, it will be noted that γ depends only on both x and t and not on z. Also, for the derivatives, we write

$$\frac{\partial}{\partial \beta} = \sum_{m=0}^{2} \epsilon^m \frac{\partial}{\partial \beta_m} + O(\epsilon^3) \tag{24}$$

where β is any of the variables x or t.

The short scale x_0 and the fast scale t_0 denote respectively the wave length and the frequency of the wave. Here t_1 and t_2 represent the slow temporal scales of the phase and the amplitude respectively, whereas the long scales x_1 and x_2 stand for the spatial modulations of the phase and the amplitude. The expression (23) is assumed to be uniformly valid for $-\infty < x < \infty$.

The boundary conditions (15) - (17) and (21) are prescribed at the perturbed surface $z = \gamma(x, t)$. We expand the physical quantities involved in the Maclaurin series about $z = 0$. On substituting (22) - (24) into (5), (6), (10), (11), (13) - (17) and (21), and, equate the coefficients of equal power series in ϵ we obtain the linear and the successive nonlinear partial differential equations of various orders, each of which can be solved with knowledge of the solutions of the previous orders. The procedure is straightforward but lengthy and it will not be included here. The details are available from the authors.

3. The First-Order Problem

The problem of first - order in ϵ represents the linear perturbation of a viscous magnetic fluid in a porous medium, subject to a normal magnetic field. The solution of the first - order problem in the form of progressive waves with respect to the lower scales x_0 and t_0 is obtained as

$$\gamma_1 = Ae^{i\theta} + c.c. \tag{25}$$

$$\Phi_{11} = -\frac{i\omega \cosh k(z+h_1)}{k \sinh kh_1} Ae^{i\theta} + c.c. \tag{26}$$

$$\Phi_{21} = \frac{i\omega \cosh k(z-h_2)}{k \sinh kh_2} Ae^{i\theta} + c.c. \tag{27}$$

$$\Psi_{11} = \frac{H_1(\widetilde{\mu}_2 - \widetilde{\mu}_1)\sinh k(z+h_1)}{\widetilde{\mu}(k)\cosh kh_1} Ae^{i\theta} + c.c. \tag{28}$$

$$\Psi_{21} = \frac{H_1\widetilde{\mu}_1(\widetilde{\mu}_2 - \widetilde{\mu}_1)\sinh k(z-h_2)}{\widetilde{\mu}_2\widetilde{\mu}(k)\cosh kh_2} Ae^{i\theta} + c.c. \tag{29}$$

where $\widetilde{\mu}(k) = \widetilde{\mu}_2 \tanh(kh_1) + \widetilde{\mu}_1 \tanh(kh_2)$ and $\theta = kx_0 - \omega t_0$. In equations (25)-(29), the symbols ω is the frequency of disturbance, k is the wave number, A is an unknown slowly varying function denoted the amplitude of the propagating wave and will be determined later from the solvability conditions, c.c. denotes the complex conjugate of all the preceding terms and $i = \sqrt{-1}$ is the imaginary number. With $\omega = \omega_R + i\omega_I$ the real part of $-i\omega$, ω_I, is called the growth rate and defines whether the disturbances will grow ($\omega_I > 0$) or decay ($\omega_I < 0$). The critical induction occurs at $\omega_I = 0$. The absolute value of the imaginary part of $-i\omega$, $|\omega_R|$, gives the angular frequency of the oscillation if ω_R is different from zero. If $\omega = 0$ the oscillation is neutrally stable.

On substitution the above solutions into the first - order of equation (21), we get the dispersion relation

$$\begin{aligned} D(\omega, k) \;=\;& g(\rho_1 - \rho_2) - kH_1^2\delta_0(k) + \sigma k^2 - \frac{i\omega}{k}[\eta_1 \coth kh_1 + \eta_2 \coth kh_2 + \\ & 2k^2(\mu_1 \coth kh_1 + \mu_2 \coth kh_2)] - \frac{\omega^2}{k}(\rho_1 \coth kh_1 + \\ & \rho_2 \coth kh_2) \end{aligned} \tag{30}$$

where $\delta_0(k) = \widetilde{\mu}_1(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2/\widetilde{\mu}_2\widetilde{\mu}(k)$.

The dispersion relation (30) is reduced to, $D(\omega, k) = 0$,

$$\omega^2 + ia_1\omega - a_2 = 0 \tag{31}$$

where
$a_0 = \rho_1 \coth kh_1 + \rho_2 \coth kh_2$,
$a_1 = \eta_1 \coth kh_1 + \eta_2 \coth kh_2 + 2k^2(\mu_1 \coth kh_1 + \mu_2 \coth kh_2)$,
$a_2 = k[g(\rho_1 - \rho_2) - [kH_1^2\widetilde{\mu}_1(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2/\widetilde{\mu}_2(\widetilde{\mu}_2 \tanh kh_1 + \widetilde{\mu}_1 \tanh_2) + \sigma k^2]$.

It may be noted that the results obtained by Mikaelian[14] and Pan et al. [23] for the Rayleigh - Taylor instability of finite - thickness viscous fluids can be deduced from equation (31) by setting $\eta_{1,2} = 0$ and $H_1 = 0$ and for interfacial instability in porous medium[13] by setting $\mu_{1,2} = 0$ and $h_{1,2} \longrightarrow \infty$.

From the above linear dispersion relation (31), we observe that the constant magnetic field has a destabilizing influence on the wave motion. This theoretical result was first obtained and confirmed experimentally by Cowley and Rosensweig (cf.[6]). In addition, the viscous effects as well as the Darcy's coefficient influence have a destabilizing role.

Once more, we return to the dispersion relation (31), where the parameters η and μ are not equal zero. We know from the Routh - Hurwitz criterion[33], that the necessary and sufficient conditions for stability, for the quadratic equation (31), are

$$a_1 > 0 \quad \text{and} \quad a_2 > 0 \tag{32}$$

since a_0 is always positive.

From above we notice that the condition $a_1 > 0$ is trivially satisfied, while the condition $a_2 > 0$ reduces to

$$H_1^2 < \widetilde{\mu}_2[g(\rho_1 - \rho_2) + \sigma k^2](\widetilde{\mu}_1 \tanh kh_2 + \widetilde{\mu}_2 \tanh kh_1)/[k\widetilde{\mu}_1(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2] \tag{33}$$

For values of $H_1 < H_c$ where

$$H_c^2 = \widetilde{\mu}_2[g(\rho_1 - \rho_2) + \sigma k^2](\widetilde{\mu}_1 \tanh kh_2 + \widetilde{\mu}_2 \tanh kh_1)/[k\widetilde{\mu}_1(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2] \tag{34}$$

the system is linearly stable. For $H_1 > H_c$ the system is unstable. The critical magnetic field H_c is the linear cutoff magnetic field separating stable from unstable disturbances. Since our aim is to study amplitude modulation of the progressive waves, we assume that ω is real and proceed to the higher- order problems.

4. The Second - and Third - Order Problems

4.1. *The Second - Order Problem*

If we now consider the second - order set of equations, we can substitute the solutions of the first - order problem into them and solve the resulting equations. The solutions yield the following uniformly valid second - order wave elevation

$$\gamma_2 = \Lambda\ A^2\ e^{2i\theta} + \Lambda_0\ A\ \overline{A} + c.c. \tag{35}$$

where

$$\begin{aligned}
\Lambda &= \{\omega^2[\rho_2(\coth^2 kh_2 + 0.5csch^2 kh_2) - \rho_1(\coth^2 kh_1 + 0.5csch^2 kh_1)] \\
&\quad +0.5i\omega[(\eta_2 csch^2 kh_2 - \eta_1 csch^2 kh_1) + 4k^2\}\mu_2(\coth^2 kh_2 + csch^2 kh_2) \\
&\quad -\mu_1(\coth^2 kh_1 + csch^2 kh_1)\}] - k^2 H_1^2 \delta_1(k)\}/D(2\omega, 2k) &(36)\\
\Lambda_0(k) &= \{\omega^2(\rho_2 csch^2 kh_2 - \rho_1 csch^2 kh_1) + k^2 H_1^2 \delta_2(k)\}/g(\rho_1 - \rho_2) &(37)\\
\delta_1(k) &= \frac{\delta_0(k)}{\widetilde{\mu}(k)\widetilde{\mu}(2k)(\widetilde{\mu}_2 - \widetilde{\mu}_1)}[2\widetilde{\mu}(k)(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2 + 0.5\widetilde{\mu}(2k)(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2 \\
&\quad -0.5\widetilde{\mu}^2(k)\widetilde{\mu}(2k) - 0.5\widetilde{\mu}_1\widetilde{\mu}_2\widetilde{\mu}(2k)(\tanh kh_2 + \tanh kh_1)^2 \\
&\quad +\widetilde{\mu}_1\widetilde{\mu}_2\widetilde{\mu}(k)(\tanh kh_2 + \tanh kh_1)(\tanh 2kh_2 + \tanh 2kh_1) \\
&\quad -(\widetilde{\mu}_2 - \widetilde{\mu}_1)\widetilde{\mu}(k)(\widetilde{\mu}_2 \tanh kh_1 \tanh 2kh_1 - \widetilde{\mu}_1 \tanh kh_2 \tanh 2kh_2)] &(38)\\
\delta_2(k) &= \frac{\widetilde{\mu}_1(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2}{\widetilde{\mu}_2\widetilde{\mu}^2(k)}(\widetilde{\mu}_1 sech^2 kh_2 - \widetilde{\mu}_2 sech^2 kh_1 &(39)
\end{aligned}$$

Since the homogeneous part of the second - order problem has a non trivial solution, because it is the same as the first - order problem, the inhomogeneous problem has a solution if, and only if, the inhomogeneous part is orthogonal to every solution of the adjoint homogeneous problem. Then the solvability condition for the second - order perturbation is

$$-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial t_1}+\frac{\partial D}{\partial k}\frac{\partial A}{\partial x_1}=0 \tag{40}$$

and its complex conjugate relation. If $\partial D/\partial\omega \neq 0$, this equation yields

$$\frac{\partial A}{\partial t_1}+\frac{d\omega}{dk}\frac{\partial A}{\partial x_1}=0 \tag{41}$$

where

$$\frac{d\omega}{dk}=-\frac{\partial D}{\partial k}/\frac{\partial D}{\partial \omega}$$

is the group velocity of the wave packet. It follows, as usual, that the amplitude A depends on the slow variables x_1 and t_1 through the combination $x_1-(d\omega/dk)t_1$.

Note that the asymptotic expansions break down when the denominator of the parameter Λ in equation (36) equals zero, which corresponds to the second - harmonic resonance. We therefore assume that $D(2\omega, 2k)\neq 0$.

4.2. *The Third - Order Problem*

We substitute the first - and second - order solutions into the third - order equations. In order to avoid non - uniformity of the expansion, we, again, impose the condition that secular terms vanish. The elimination of the secular terms imposes the following solvability condition

$$i\left(-\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial t_2}+\frac{\partial D}{\partial k}\frac{\partial A}{\partial x_2}\right)+\frac{1}{2}\frac{\partial^2 D}{\partial\omega^2}\frac{\partial^2 A}{\partial t_1^2}-\frac{\partial^2 D}{\partial\omega\partial k}\frac{\partial^2 A}{\partial x_1\partial t_1}+\frac{1}{2}\frac{\partial^2 D}{\partial k^2}\frac{\partial^2 A}{\partial x_1^2}=G|A|^2A \tag{42}$$

where the coefficients of the linear terms are simply the derivatives of the characteristic function (30),while the coefficient of the nonlinear term is

$$\begin{aligned}
G \;=\;& 2\Lambda\{\omega^2[\rho_1(\coth^2 kh_1+0.5csch^2kh_1)-\rho_2(\coth^2 kh_2+0.5csch^2kh_2)]\\
&+i\omega[\eta_1\coth^2 kh_1-\eta_2\coth^2 kh_2+2k^2(\mu_1csch^2kh_1-\mu_2csch^2kh_2)]\\
&+k^2H_1^2\delta_1(k)\}+\Lambda_0[\omega^2(\rho_1csch^2kh_1-\rho_2csch^2kh_2)+i\omega(\eta_1csch^2kh_1\\
&-\eta_2csch^2kh_2)+2i\omega k^2(\mu_1csch^2kh_1-\mu_2csch^2kh_2)-k^2H_1^2\delta_2(k)]\\
&+2k\{\omega^2[\rho_1\coth kh_1(1-csch^2kh_1)-\rho_2\coth kh_2(1-csch^2kh_2)]\\
&+i\omega(\eta_1\coth kh_1(1-0.5\coth^2 kh_1)+\eta_2\coth kh_2(1-0.5\coth^2 kh_2)\\
&+k^2[\mu_1\coth kh_1(4-\coth^2 kh_1)+\mu_2\coth kh_2(4-\coth^2 kh_2)])\}\\
&+2k^3H_1^2\delta_3(k)-1.5\sigma k^4
\end{aligned} \tag{43}$$

$$\begin{aligned}
\delta_3(k) \;=\;& \frac{\delta_0(k)}{\widetilde{\mu}(k)\widetilde{\mu}(2k)}[2\widetilde{\mu}(k)\widetilde{\mu}(2k)-2(\widetilde{\mu}_2-\widetilde{\mu}_1)^2-3\widetilde{\mu}_1\widetilde{\mu}_2(\tanh kh_1\\
&+\tanh kh_2)(\tanh 2kh_1+\tanh 2kh_2)-(\widetilde{\mu}_2\\
&-\widetilde{\mu}_1)(\widetilde{\mu}_1\tanh kh_2\tanh 2kh_2-\widetilde{\mu}_2\tanh kh_1\tanh 2kh_1)\\
&+2\widetilde{\mu}_1\widetilde{\mu}_2\tanh 2kh_1\tanh 2kh_2(\tanh kh_1+\tanh kh_2)^2]
\end{aligned} \tag{44}$$

5. The Evolution Equations

The solvability conditions (40) and (42) can be simplified and combined together to produce a single equation. By using (41), derivatives in t_1 can be eliminated from equation (42). From (41), let's write

$$\frac{\partial^2 A}{\partial x_1 \partial t_1} = -\frac{d\omega}{dk}\frac{\partial^2 A}{\partial x_1^2} \tag{45}$$

and

$$\frac{\partial^2 A}{\partial t_1^2} = (\frac{d\omega}{dk})^2\ \frac{\partial^2 A}{\partial x_1^2} \tag{46}$$

Substituting (45) and (46) into (42), dividing through by $(-\partial D/\partial\omega)$, and replacing x_m and t_m by $\epsilon^m x$ and $\epsilon^m t$, we have

$$i(\frac{\partial A}{\partial t} + \frac{d\omega}{dk}\frac{\partial A}{\partial x}) + P\frac{\partial^2 A}{\partial x^2} = \epsilon^2 Q|A|^2 A \tag{47}$$

where P (the group velocity rate) and Q (the nonlinear interaction coefficient) are given by

$$P = \frac{1}{2}\frac{d^2\omega}{dk^2} = -\frac{1}{2}[\frac{\partial^2 D}{\partial k^2}(\frac{\partial D}{\partial\omega})^2 - 2\frac{\partial D}{\partial k}\frac{\partial D}{\partial\omega}\frac{\partial^2 D}{\partial k\partial\omega} + \frac{\partial^2 D}{\partial\omega^2}(\frac{\partial D}{\partial k})^2]/(\frac{\partial D}{\partial\omega})^3,$$

and

$$Q = -G/(\frac{\partial D}{\partial\omega})$$

Introducing the transformation.

$$X = \epsilon(x - \frac{d\omega}{dk}t), \quad T = \epsilon^2 t \tag{48}$$

equation (47) reduces to

$$i\frac{\partial A}{\partial T} + P\frac{\partial^2 A}{\partial X^2} = Q|A|^2 A \tag{49}$$

where the coefficient P and Q are complex, so that

$P = P_r + iP_i$ and $Q = Q_r + iQ_i$

Equation (49) is the complex Ginzburg - Landau equation and can be used to study the stability behaviour of the problem. The stability conditions of (50) are discussed by Lange and Newell [34] and Pelap and Faye [35]. Thus, if the solution of this equation is linearly perturbed, the perturbations are stable under the conditions

$$P_r Q_r + P_i Q_i > 0 \quad \text{and} \quad Q_i < 0 \tag{50}$$

Otherwise, the system isunstable.

To discuss the stabilization of the nonlinear system, we shall delimit ourselves to the case of marginal, non- oscillatory state i.e. in the neighbourhood of the linear critical magnetic field (34). We notice that $P_r = Q_r = 0$ in this case, and therefore, the complex Ginzburg - Landau equation (49) is reduced to the nonlinear diffusion equation,

$$\frac{\partial A}{\partial T} + P_i\frac{\partial^2 A}{\partial X^2} = Q_i|A|^2 A \tag{51}$$

where P_i and Q_i are polynomials of the second and third degree in H_1^2, respectively.

The solution of the nonlinear diffusion equation (51) is valid near the critical magnetic field (i.e the marginal state) and therefore be used to study the stability of the system. From inequalities (50), we find the stability conditions of (51) are [36]

$$P_i < 0 \quad \text{and} \quad Q_i < 0 \tag{52}$$

Thus, if the above conditions (52) are satisfied, the finite deformation of the interface is stable and finite amplitude waves can propagate through the interface. We shall explain the implications of conditions (52) in the following section.

6. Stability Analysis

As shown in section (5) the analysis of stability for finite disturbance depends on conditions (52). Stability depends on the thicknesses $h_{1,2}$, the viscosity coefficients $\mu_{1,2}$ and Darcy's coefficients $\eta_{1,2}$. Stability can therefore be discussed by dividing the $(H_1^2 - k)$ plane into stable and unstable regions. The transition curves are given by the vanishing of P_i and Q_i. These curves are

$$P_i = 0 \tag{53}$$

and

$$Q_i = 0 \tag{54}$$

We observe that Q_i changes sign across the curve $H_1^2 = H_{res}^2$, where

$$H_{res}^2 = \{g(\rho_1 - \rho_2) + 4\sigma k^2\}/[2k\delta_0(2k)] \tag{55}$$

is the second - harmonic resonance at the marginal state. Therefore, equation (55) represents a third - transition curve in the $(H_1^2 - k)$ plane.

The graphs represented by equations (53) - (55) are useful in studying the effects of the normal magnetic field, the viscosity and Darcy's coefficients on the stability of the system, for various values of h_1 and h_2. Also, the linear stability curve (dotted - line) representing relation (34) is given , which is assumed to divide the plane into an unstable region, symbolized by U (above the curve) and a stable region, symbolized by S (below the curve), while the dashed line represents the second - harmonic resonance curve (55). The shaded regions $S_{1,2}$ (stable) and $U_{1,2,3,4,5}$ (unstable) are newly formed regions arising from the nonlinear effects.

Figures 1a - 1d are the stability diagram in the $(logH_1^2 - k$) plane for different values of h_1 and h_2. In these figures, we considered $\mu_{1,2} \neq 0$ and $\eta_{1,2} \neq 0$ (i.e. in the presence of both viscosity and Darcy's coefficients). Figure 1a shows the stability diagram for the case of two semi- infinite fluids, i.e. $h_{1,2} \longrightarrow \infty$. We observe that there are two branches of both the nonlinear interaction parameter $Q_i = 0$ and the group velocity rate $P_i = 0$. The two branches of $P_i = 0$ appearing for small values of k ($k < 0.31cm^{-1}$). These branches produce only one stable region S and four unstable regions $U_{1,2,3,4}$. The one stable region S disappears in figures 1b - 1d. That is because one branch of $Q_i = 0$ disappears, while the other lies above both the linear curve (the dotted curve) and the resonance curve (the dashed curve). For the same reason, new nonlinear stable region S_1 appears.

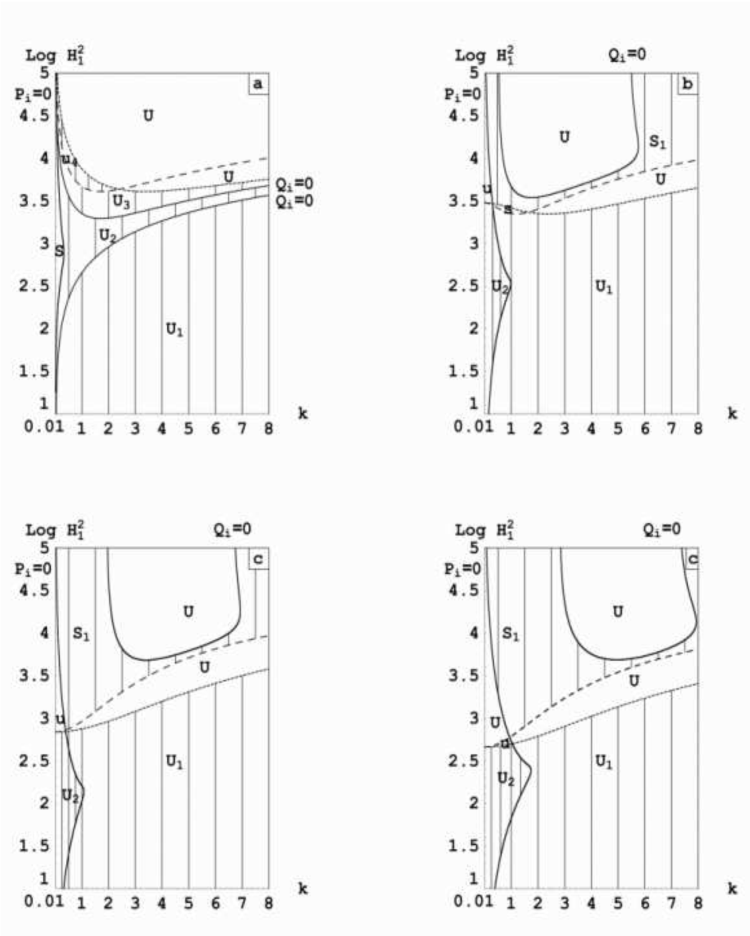

Figure 1 Stability diagram in the $\log H_1^2 - k$ plane for an interface between water and vapor, where $\rho_1 = 0.965\ g/cm^3$, $\rho_2 = 0.585\ g/cm^3$,$\widetilde{\mu}_1 = 1.007$, $\widetilde{\mu}_2 = 1.7$, $\sigma = 64.4\ dyn/cm$, $g = 981\ cm/s^2$, $\mu_1 = 0.28\ cp$, $\mu_2 = 0.0125\ cp$, $\eta_1 = 1.9812\ cp/cm^2$, $\eta_2 = 0.941\ cp/cm^2$, $\omega = 0$;
a) refers to $h_1,\ h_2 \longrightarrow \infty$,
b) to $h_1 = 0.1cm, h_2 = 1cm$,
(c) to$h_1 = h_2 = 0.1cm$,
d) to $h_1 = 0.1cm, h_2 = 0.01cm$.
The dotted-line represents the linear curve while the broken-line represents the second-harmonic resonance curve. The symbols S and U denote stable and unstable regions, respectively, in the linear problem. Shaded regions are newly formed regions due to the nonlinear effects : S_j is stable and U_j is unstable regions.

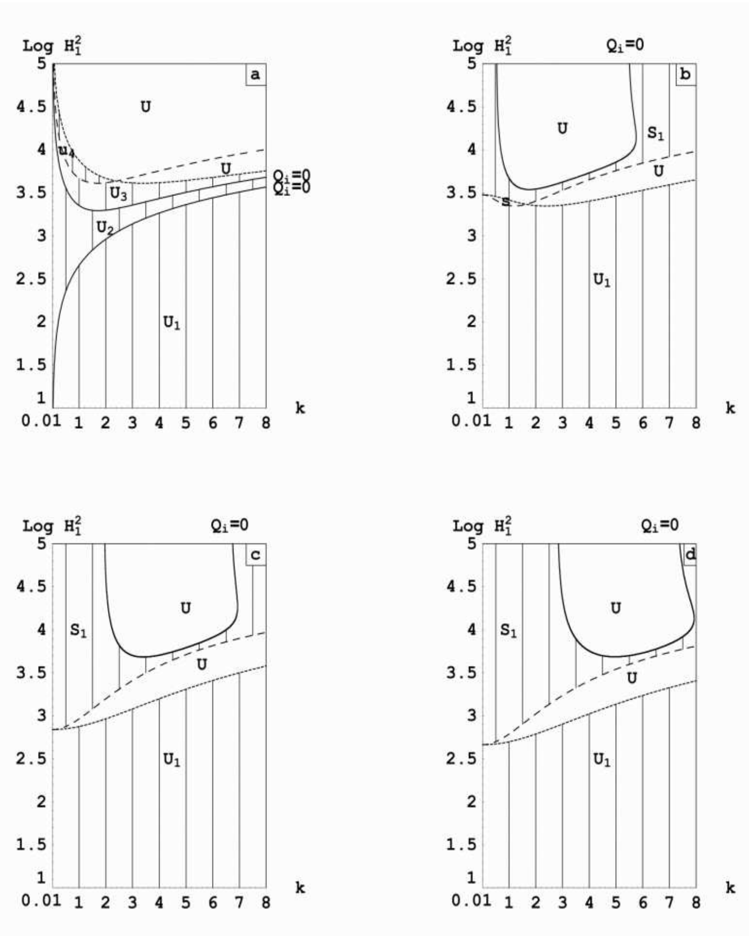

Figure 2 Stability diagram for the same system considered in figure 1 but with $\eta_1 = 0$, $\eta_2 = 0$. The symbols are the same as in figure 1.

It lies above the resonance curve and below the curve $Q_i = 0$. Also for smaller values of k we see that the curve $P_i = 0$ has two branches lying below the linear curve and creating a stable regions S (figure 1a) and an unstable region U_2 (figures 1$b-$1d). This new unstable region U_2 increases with the decrease of h_2. This means that the system is destabilizing for smaller values of k with smaller values of H_1^2.

Figures 2a-2d represent the same system considered in figures 1a -1d, but when $\eta_{1,2} = 0$ and $\mu_{1,2} \neq 0$ (i.e. in the absence of Darcy's coefficient). We observe that the curve $Q_i = 0$ has two branches as in figure 2a or one branch as in figures 2b-2d, while the curve $P_i = 0$ disappears. We also observe that the resonance curve lies above the linear curve. The system is divided into one stable region S_1 and five unstable regions $U, U_{1,2,3,4}$. The stable region S_1 increases with the decrease of h_2, while the regions $U_{2,3,4}$ disappear in figures 2b-2d. The unstable region U_1 increases with the decrease of h_2, and lies below the linear curve (figures 2b-2d) and below the lower branch of $Q_i = 0$ (figure 2a). The behaviour of the system is similar to that in figures 1.

7. Nonlinear Schrődinger Equation

A special case occurs when the viscosity and Darcy's coefficients are negligible, by taking $\mu_{1,2}$ and $\eta_{1,2}$ as equal to zero in the evolution equation (49). In this case, P_i and Q_i in equation (49) vanish. Therefore, (49) is reduced to the nonlinear Schrődinger equation[11, 37]

$$i\frac{\partial A}{\partial T} + P_r\frac{\partial^2 A}{\partial X^2} = Q_r|A|^2 A \tag{56}$$

where

$$\begin{aligned}
P_r \;=\; & \frac{1}{2}\{\{\frac{2\omega}{k}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)\}^{-1}\{2\sigma - H_1^2[k\delta_0''(k) + 2\delta_0'(k)] \\
& -\frac{2\omega^2}{k^3}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) - \frac{2\omega^2}{k^2}(\rho_1 h_1 csch^2 kh_1 + \rho_2 h_2 csch^2 kh_2) \\
& -\frac{2\omega^2}{k}(\rho_1 h_1^2 \coth kh_1 csch^2 kh_1 + \rho_2 h_2^2 \coth kh_2 csch^2 kh_2)\} \\
& +2\{\frac{2\omega}{k}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)\}^{-2}\{\frac{2\omega}{k^2}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) \\
& +\frac{2\omega}{k}(\rho_1 h_1 csch^2 kh_1 + \rho_2 h_2 csch^2 kh_2)\}\{2\sigma k - H_1^2[k\delta_0'(k) + \delta_0(k)] \\
& +\frac{\omega^2}{k^2}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) + \frac{\omega^2}{k}(\rho_1 h_1 csch^2 kh_1 + \rho_2 h_2 csch^2 kh_2)\} \\
& +\{\frac{2\omega}{k}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)\}^{-3}\{\frac{-2}{k}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)\}[2\sigma k \\
& -H_1^2\{k\delta_0'(k) + \delta_0(k)\} + \frac{\omega^2}{k^2}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2) + \frac{\omega^2}{k}(\rho_1 h_1 csch^2 kh_1 \\
& +\rho_2 h_2 csch^2 kh_2)]^2\}
\end{aligned}$$

$$\begin{aligned}
Q_r \;=\; & -\{\frac{2\omega}{k}(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)\}^{-1}\{2[\omega^2\{\rho_2(\coth^2 kh_2 + 0.5 csch^2 kh_2) \\
& -\rho_1(\coth^2 kh_1 + 0.5 csch^2 kh_1)\} - k^2 H_1^2 \delta_1(k)]^2[g(\rho_1 - \rho_2) - 2kH_1^2\delta_0(2k) \\
& +4\sigma k^2 - \frac{2\omega^2}{k}(\rho_1 \coth 2kh_1 + \rho_2 \coth 2kh_2)]^{-1} + [g(\rho_1 - \rho_2)]^{-1} \\
& \times[\omega^2(\rho_2 csch^2 kh_2 - \rho_1 csch^2 kh_1) + k^2 H_1^2 \delta_2(k)][\omega^2(\rho_2 csch^2 kh_2 - \rho_1 csch^2 kh_1) \\
& -k^2 H_1^2 \delta_4(k)] + 1.5\sigma k^4 - 2k^3 H_1^2 \delta_3(k) - 2k\omega^2[\rho_1 \coth kh_1(1 - csch^2 kh_1) \\
& +\rho_2 \coth kh_2(1 - csch^2 kh_2)]\}
\end{aligned}$$

and

$$\omega^2 = k[g(\rho_1 - \rho_2) - kH_1^2\delta_0(k) + \sigma k^2]/(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)$$

Equation (56) describes the modulation of a one - dimensional weakly nonlinear dispersive wave in the presence of an externally applied magnetic field and absence of both the viscosity and Darcy's coefficients. It is well known that the solutions of equation (56) are stable if

$$P_r Q_r > 0 \tag{57}$$

Thus, if the above condition is satisfied, the finite deformation of the interface is stable and finite - amplitude waves can propagate through the surface. Therefore, the analysis of the stability of the system for finite disturbance depends on this condition (57) . In general, this condition depends on $k,\ g,\ \sigma,\ H_1,\ h_{1,2},\ \rho_{1,2}$ and $\widetilde{\mu}_{1,2}$. The critical values of these parameters required for stability may be obtained from replacing the equality in (57), by equality, namely

$$P_r Q_r = 0 \tag{58}$$

The stability diagram in the $(log H_1^2 - k)$ plane is divided into stable and unstable regions bounded by the curves:

$$P_r = 0 \tag{59}$$

and

$$Q_r = 0 \tag{60}$$

where $P_r = 0$ and $Q_r = 0$ are polynomials of the second and third degree in H_1^2, respectively . We observe from equation (60), Q_r changes sign across the curve $H_1 = H^*_{res}$ where

$$\begin{aligned} H^{*2}_{res} \;=\; & \{g(\rho_1 - \rho_2)[2(\rho_1 \coth 2kh_1 + \rho_2 \coth 2kh_1) - (\rho_1 \coth kh_1 \\ & + \rho_2 \coth kh_1)] + 2\sigma k^2[(\rho_1 \coth 2kh_1 + \rho_2 \coth 2kh_1) \\ & - 2(\rho_1 \coth kh_1 + \rho_2 \coth kh_1)]\}/\{2k[(\delta_0(k)(\rho_1 \coth 2kh_1 \\ & + \rho_2 \coth 2kh_1)) - \delta_0(2k)(\rho_1 \coth kh_1 + \rho_2 \coth kh_1)]\} \end{aligned} \tag{61}$$

is the second - harmonic internal resonance.

Figure 3a shows the stability diagram for the two semi - infinite magnetic fluids case (i.e. $h_{1,2} \longrightarrow \infty$). In this case, the condition $P_r = 0$ gives

$$H_1^2 = [3\sigma^2 k^4 + 6\sigma g k^2(\rho_1 - \rho_2) - g^2(\rho_1 - \rho_2)^2]/(4k^3\sigma\delta_0) \tag{62}$$

where

$$\delta_0 = \widetilde{\mu}_1(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2/\widetilde{\mu}_2(\widetilde{\mu}_2 + \widetilde{\mu}_1)\ ,$$

while the condition $Q_r = 0$ gives

$$\begin{aligned} & 2g(\rho_1 - \rho_2) + 0.5\sigma k^2 + 2(\rho_1 - \rho_2)^2\{g(\rho_1 - \rho_2) + \sigma k^2 \\ & -k\delta_0 H_1^2[1 + (\widetilde{\mu}_2 - \widetilde{\mu}_1) \\ & (\rho_1 + \rho_2)/(\widetilde{\mu}_2 + \widetilde{\mu}_1)(\rho_1 - \rho_2)]\}^2/\{(\rho_1 + \rho_2)^2[g(\rho_1 - \rho_2) - 2\sigma k^2]\} = 0 \end{aligned} \tag{63}$$

We may observe that the curve

$$k^2_{res} = g(\rho_1 - \rho_2)/2\sigma \tag{64}$$

is the second - harmonic resonance. Then the curve given by equation (64) is independent of the magnetic field. Also, this curve does not appear in this figure (3a), while each of the curves $Q_r = 0$ and $P_r = 0$ has only one branch. The branch of the curve $P_r = 0$ lies above the linear curve and creates a stable region S_1. The branch of $Q_r = 0$ cuts the linear curve at $k \simeq 0.5\ cm^{-1}$ to produce a stable region S_2, which lies above the curve $Q_r = 0$. Therefore, the system is stabilizing for larger values of H_1^2 and vice versa.

In figures 3b and 3c, we see that the curve of the second harmonic resonance does not appear because the values of the magnetic field H_1^2 are negative. We also see that there are two branches of $Q_r = 0$ and only one branch of $P_r = 0$. These branches produce two stable regions S_1 and S_2 above the linear curve and two unstable regions U_1 and U_2. One of them (U_1) lies between $P_r < H_1^2 < Q_r$, while the other (U_2) lies among the linear curve, the curve $Q_r = 0$ and the curve $P_r = 0$. The resonance curve appearing in $3d$, and it lies above the linear curve and cuts the curve $P_r = 0$ to produce three stable regions S_1, S_2 and S_3. The behaviour of the system is similar to that in figures 3b and 3c, but with an increase in the size of the region U_2 and a decrease of the region U_1. The reason is that the branch $P_r = 0$ shifts downwards.

8. Conclusion

The multiple scale method [37] is used to study the linear and nonlinear instability of an interface between two magnetic fluids of finite depths under a normal magnetic field, including the effect of slight viscosity and Darcy's coefficients. The two fluids have different densities, magnetic permeabilities, weak viscosities and Darcy's coefficients. They are taken to be incompressible and the motion is assumed to be irrotational for the small perturbations.

Although the motions are assumed to be irrotational, weak viscous effects are included in the boundary condition of the normal stress tensor balance but the tangential stress condition is neglected. Without an external field and a Darcy's coefficient, the first- order expansion of the multiple scales method yields the same result as that obtained by Mikaelian [14], Joseph et al. [22] and Pan et al. [23] for the viscous decay of the free oscillation problem. Moreover, for two semi - infinite fluids (i.e. $h_{1,2} \longrightarrow \infty$) in porous medium (i.e. $\mu_{1,2} = 0$) the linear approximation yields the same result as that obtained by El - Dib and Ghaly [13]and by Sharma and Bhardwaj [38].

Here, we have found in the linear approximation that, the magnetic field and both viscosity and Darcy's coefficients, have a destabilizing effect. We have also found that the effective interfacial tension succeeds in stabilizing perturbations of certain wave numbers (small wavelength perturbations) which were unstable in the absence of effective interfacial tension, for unstable configuration.

From the nonlinear evolution equation in the nonlinear approximation, we have obtained a complex Ginzburg- Landau equation. When the linearized magnetic field is assumed to be nearly equal to the critical magnetic field, the nonlinear diffusion equation is derived. Furthere, it is shown that, in section 7, a nonlinear Shrődinger - equation is obtained in the absence of both viscosity and Darcy's coefficients.

From the numerical discussion it is evident that, besides the effect of the variation of h_1 and h_2 ($h_{1,2}$ are the depths of the two fluids), the viscosity and Darcy's coefficients, play an important role in the nonlinear stability criterion of the problem. In view of the discussion on the effect of the choices of viscosity and Darcy's coefficients in section 6, the results obtained may be more relevant to reality than it appears. It seems, in general, that both the viscosity and Darcy's coefficients are a nonlinear phenomenon and it is best understood via nonlinear analysis.

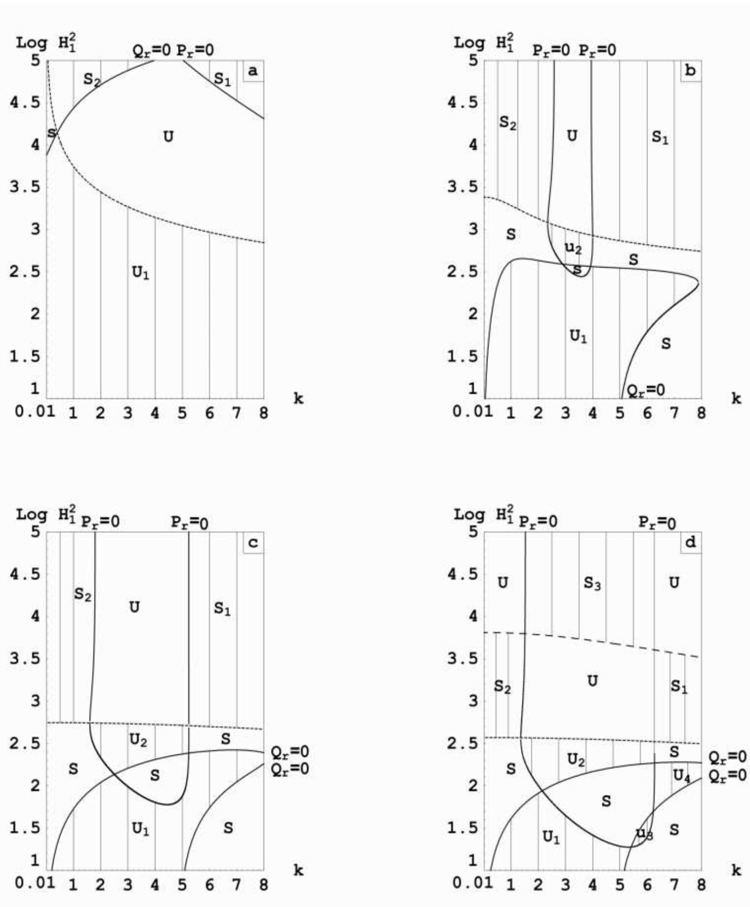

Figure 3 Stability diagram in the $\log H_1^2 - k$ plane for an interface between water and ethyl ether, where $\rho_1 = 1.0\ g/cm^3$, $\rho_2 = 0.71\ g/cm^3$, $\widetilde{\mu}_1 = 1.007$, $\widetilde{\mu}_2 = 1.7$, $\sigma = 10.5\ dyn/cm$ and $g = 981\ cm/s^2$.
a) refers to $h_1, h_2 \longrightarrow \infty$,
b) to $h_1 = 0.1cm, h_2 = 1cm$,
c) to $h_1 = h_2 = 0.1cm$,
d) to $h_1 = 0.1cm, h_2 = 0.01cm$.
The symbols are the same as in figure 1.

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