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Stability of Compressible Magnetized Hollow Cylinder Pervaded by Azimuthally Field

Ahmed E. Radwan
Mathematics Department, Faculty of Science, Ain-Shams University, Cairo, Egypt

Nasser E. Elazab and Nahed S. Hussien
Mathematics Department, Faculty of Science, Cairo University, Giza, Egypt

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The stability of compressible magnetized hollow cylinder (gas jet embedded into a liquid) pervaded by varying azimuthally magnetic field has been developed for all symmetric $m = 0$ and asymmetric $m \neq 0$ pertubation modes (m transverse wavenumber). The problem is formulated well, apart from the singular solutions the different variables are determined, the stability criterion is derived and discussed. The axial field in the liquid region is stabilizing for all short and long wavelengths and that effect is independent of m values. In contrast, the azimuthal field in the gas is destabilizing or not according to restrictions and m values. The capillary force along the gas – liquid interface is destabilizing only for m = 0, for small longitudinal wavenumber and stabilizing in the rest. The compressibility has a strong stabilizing effect for all $m \geq 0$ disturbance modes. Here the instability due to azimuthal field and the capillary force could be completely suppressed under restrictions and stability sets in.

Keywords: magnetic field, capillary force, magnetohydrodynamic stability

1. Introduction

The stability of a (gas cylinder immersed into a liquid) hollow jet under the effect of the capillary force is indicated for first time by Chandrasekhar [2], see also Rayleigh [12]. That is as the perturbation is axisymmetric for azimuthal direction. Drazin and Reid [4] gave the dispersion relation in a classical method for such model. In such model it is proposed that the liquid inertia force is predominate over that of the gas jet during perturbation. But one has to keep in mind that the gas pressure in the unperturbed state must be greater than other stresses of the model, otherwise the model will collapse and the gas will be distributed in the liquid layers. Cheng [3] discussed analytically the capillary instability of unbounded hollow jet. Kendall [5] performed very interesting experiments with modern equipment for studying the stability of bounded hollow jet under the action of the inertia and capillary forces. Radwan and Elazab [6] discussed the hydrodynamic stability of viscous hollow jet endowed with surface tension. Radwan [8] examined the effect of the Lorentz force on the capillary instability of an ideal hollow cylinder where the velocity vector is solinoidal. For bounded hollow jet stability we may refer to the recent work of Radwan and Ogail [10]. Also Radwan [8] discussed the stability of a compressible hollow jet pervaded by a uniform magnetic fied in the axial direction.

Here we study the magnetohydrodynamic stability of a compressible (gas jet submerged into a liquid) hollow jet pervaded by azimuthal varying magetic field for all axisymmetric and non-axisymmetric modes of perturbation. Here the technique which will be used is totally different from that used before because the velocity is not solenoidal anymore, i.e. $\nabla \underline{u} \neq 0$.

2. Basic Equations

Consider a gas cylinder of radius R_o embedded into a liquid medium. The liquid is assumed to be non-viscous, non-resistive but it is compressible and pervaded by the uniform magnetic field

$$\underline{H}_o = (0, 0, H_o) \tag{1}$$

The gas cylinder is pervaded by the azimuthally varying magnetic field

$$\underline{H}_o^g = (0, \frac{\beta r}{R_o} H_o, 0) \tag{2}$$

where H_o is the intensity of the magnetic field in the liquid region and β is some parameter satisfying certain condition. In investigating such problem we shall use the cylindrical coordinates (r, φ, z) with the z-axis coinciding with the axis of the gas cylinder.

The model is acting upon the capillary, electromagnetic and pressures gradient forces. Under the present circumstances the fundamental equations required for studying such kind of problems are given as follows, cf. Roberts [12].

In the liquid region

$$\rho \frac{d\underline{u}}{dt} = -\nabla P + \mu(\nabla \wedge \underline{H}) \wedge H \tag{3}$$

$$\frac{d\underline{H}}{dt} = (\underline{H} \cdot \nabla)\underline{u} - \underline{H}(\nabla \cdot \underline{u}) \tag{4}$$

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \underline{u}) = 0 \tag{5}$$

$$\rho C_v \frac{dT}{dt} + P(\nabla \cdot \underline{u}) = 0 \tag{6}$$

$$\nabla \cdot \underline{H} = 0 \tag{7}$$

$$P = K\rho^\gamma \tag{8}$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\underline{u} \cdot \nabla) \tag{9}$$

$$\nabla = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}) \tag{10}$$

In the gas region

$$\nabla \cdot \underline{H}^g = 0 \tag{11}$$

$$\nabla \wedge \underline{H}^g = 0 \tag{12}$$

(there is no current) along the gas – liquid interface

$$P_s = S(\nabla \cdot \underline{N}) \tag{13}$$

Here $\rho, \underline{u}, T, C_v$ and P are the liquid mass density, velocity vector temperature, specific heat at constant volume and kinetic pressure where K and γ are constants. μ and $\underline{H}$ are magnetic field permeability and intensity, P_s is the surface pressure due to capillary force, S the surface tension coefficient and $\underline{N}$ is a unit outward vector normal to the gas – liquid interface $f(r, \varphi, z, t) = 0$, given by

$$\underline{N} = \nabla f(r, \varphi, z, t)/|\nabla f(r, \varphi, z, t)| \tag{14}$$

The initial state as $\underline{u}_o = (0, 0, 0)$ is clearly studied, the foregoing basic equations are solved, the boundary conditions are applied and consequently the liquid kinetic pressure P_o is given by

$$P_o = \frac{-S}{R_o} + \frac{\mu H_o^2}{2}(\beta^2 - 1) + P_o^g \tag{15}$$

where P_o^g is the gas constant pressure in the unperturbed state. In absence of the surface tension contribution in equation (15), β must satisfy the restriction

$$\beta^2 \geq 1 - (\frac{2}{\mu H_o^2})P_o^g \tag{16}$$

in order that

$$P_o \quad \geq 0 \tag{17}$$

otherwise the model collapses and the gas will be distributed among the liquid molecules.

3. Linearization Analysis

As the initial state is perturbed, there will be a small departure along the gas – liquid interface. In such a case any physical quantity $Q(r, \varphi, z, t)$ may be expressed as

$$Q(r, \varphi, z, t) = Q_o(r) + \varepsilon_o Q_1(r, \varphi, z, t) + \ldots \tag{18}$$

where the index o characterizes quantities in the unperturbed state while those with index 1 are their increments due to perturbation. Here Q stands for $\rho, \underline{u}, P, \underline{H}, \underline{H}^g, \underline{N}$, T and the radial distance of the gas cylider $\varepsilon(t)$ is the amplitude of the perturbation given by

$$\varepsilon(t) = \varepsilon_o exp(\sigma t) \tag{19}$$

where $\varepsilon_o = \varepsilon$ at $t = 0$ is the initial amplitude while σ is the growth rate or rather the oscillation frequency ω as $\sigma = i\omega$ with $i = \sqrt{-1}$ is imaginary. Based on the expansions (18) and the linear perturbation technique, upon considering sinusoidal propagating wave along the gas – liquid interface, the radial distance of the gas cylinder is given by

$$r = R_o + \varepsilon_o R_1 \tag{20}$$

Here

$$R_1 = exp[i(kz + m\varphi) + \sigma t] \tag{21}$$

is the elevation of the surface wave measured from the unperturbed position, where m (integer) is the azimuthal wave number and k(real) the longitudinal wave number.

From the viewpoint of the foregoing expansions, the linearized perburbation equations are given as follows

$$\frac{\partial \underline{u}_1}{\partial t} - \frac{\mu}{\rho_o}(\underline{H}_o \cdot \nabla)\underline{H}_1 = -\nabla\Pi_1 \tag{22}$$

$$\frac{\partial \underline{H}_1}{\partial t} = (\underline{H}_o \cdot \nabla)\underline{u}_1 - (\underline{u}_1 \cdot \nabla)\underline{H}_o - \underline{H}_o(\nabla \cdot \underline{u}_1) \tag{23}$$

$$\frac{\partial P_1}{\partial t} = a^2 \frac{\partial \rho_1}{\partial t} \tag{24}$$

$$(\nabla \cdot \underline{H}_1) = 0 \tag{25}$$

$$\frac{\partial \rho_1}{\partial t} = -\rho_o(\nabla \cdot \underline{u}_1) \tag{26}$$

$$\nabla \cdot \underline{H}_1^g = 0 \tag{27}$$

$$\nabla \wedge \underline{H}_1^g = 0 \tag{28}$$

and

$$P_{1s} = \frac{S}{R_o^2}(R_1 + \frac{\partial^2 R_1}{\partial \varphi^2} + R_o^2 \frac{\partial^2 R_1}{\partial z^2} \tag{29}$$

Here

$$\rho\Pi_1 = P_1 + (\mu/2)(\underline{H} \cdot \underline{H})_1 \tag{30}$$

is the total hydromagnetic pressure which is the sum of liquid kinetic pressure P_1 and magnetic pressure $(\mu/2)(\underline{H} \cdot \underline{H})_1$. However the parameter a is the speed of sound in the compressible liquid given by

$$a \quad = (\gamma P_o/\rho_o)^{\frac{1}{2}} \tag{31}$$

By combining equations (24) and (26), after using the time dependence, we get

$$\sigma P_1 = -\rho_o a^2(\nabla \cdot \underline{u}_1) \tag{32}$$

Based on the linear perturfation technique utilized for solving stability problems of cylindrical models, cf. Chandrasekhar [2] and Radwan [7, 8], the perturbed quantities Q_1(r,φ,z,t) may be expressed as

$$Q_1(r, \varphi, z, t) = \varepsilon Q_1^*(r)\exp[i(kz + m\varphi)] \tag{33}$$

By the use of this expansion, the relevant perturbation equations (22) – (32) are solved. Apart from the singular solution, the finite solution is given as follows

$$\underline{H}_1 = (ikH_o/\sigma)\underline{u}_1 + (H_o/\rho_o a^2)\rho_1 \underline{e}_z \tag{34}$$

$$(\sigma^2 + \Omega_A^2)\underline{u}_1 = -\sigma\nabla\Pi_1 + i(\sigma\Omega_A^2/k\rho_o a^2)P_1\underline{e}_z \tag{35}$$

$$\Pi_1 = (\xi/\rho_o)P_1 \tag{36}$$

$$\xi = 1 + [\mu H_o^2/(\rho_o\sigma^2 a^2)](\sigma^2 + k^2 a^2) \tag{37}$$

$$\eta^2 = k^2 + (\sigma^2/a^2\xi) \tag{38}$$

$$\Pi_1 = AK_m(\eta r)\exp[\sigma t + i(kz + m\varphi)] \tag{39}$$

$$\underline{H}_1^g = B\nabla\{I_m(kr)\exp[\sigma t + i(kz + m\varphi)]\} \tag{40}$$

Here $\underline{e}_z$ is a unit vector in z-direction and Ω_A is the Alfven wave frequency

$$\Omega_A = (\mu H_o^2 k^2/\rho_o)^{\frac{1}{2}} \tag{41}$$

defined in terms of H_o. In order to determine the constants A and B of integration, we have to apply appropriate boundaryconditions across the gas – liquid interface at $r = R_o$. Consequently, we have

$$A = \frac{-(\sigma^2 + \Omega_A^2)}{\eta K_m'(y)} \tag{42}$$

and

$$B = \frac{iH_o^m\beta}{xI_m'(x)} \tag{43}$$

where $x = kR_o$ is the dimensionless ordinary longitudinal wavenumber while $y = \eta R_o$ is the dimensionless compressible longitudinal wave-number.

Moreover, upon applying the balance of the normal component of the total stress tensor across the gas – liquid interface at $r = R_o$. This condition, yields

$$\begin{aligned}\sigma^2 \quad = \quad & \frac{-S}{\rho_o R_o^3}(1 - m^2 - x^2)\frac{yK_m'(y)}{K_m(y)} \\ & + \frac{\mu H_o^2}{\rho R_o^2}[-x^2 + (m^2\beta^2\frac{I_m(x)}{xI_m'(x)} - \beta^2)\frac{yK_m'(y)}{K_m(y)}]\end{aligned} \tag{44}$$

4. Discussions

Equation (44) is the stability criterion of magnetohydrodynamic of a gas cylinder immersed into non-viscous, non-resistive compressible liquid pervaded by azimuthal varying magnetic field. It relates the temporal amplification σ with the wavenumbers m, x and y, the modified Bessel functions I_m & K_m of the first and second kind of order m and their derivatives and with the parameters T, ρ_o, R_o, μ, H_o and β of the problem. The relation (44) posses the natural fundamental quantity $(\rho_o R_o^2/(\mu H_o^2))^{1/2}$ as well as $(\rho_o R_o^3/S)^{1/2}$ as a unit of time.

Upon using the general dispersion relation (44), several stability criteria documented previously by others could be obtained under a lot of restrictions.

For the classical capillary dispersion relation of ordinary hollow jet we (assume $\beta = 0, H_o = 0$ and a tends to infinity) get

$$\sigma^2 = \frac{-S}{\rho R_o^3} \frac{x K'_m(x)}{K_m(x)} (1 - m^2 - x^2) \tag{45}$$

This relation is given by Drazin and Reid (1980).

For axisymmetric perturbation as $m = 0$, the foregoing relation (45) yields

$$\sigma^2 = \frac{S}{\rho R_o^3} \frac{x K_1(x)}{K_0(x)} (1 - x^2), \; K'_0(x) = K_1(x) \tag{46}$$

This relation has been indicated by Chandrasekhar [2] in comparising the stability of a hollow jet with that of a full liquid jet. Also the relation (46) coinciding with the relation derived by Radwan and Elazab [6] as we neglect the contribution of viscosity there.

As the lquid is incompressible as a tends to infinity we have y tends to x and in this case the stability criterion of hydromagnetic ordinary jet is given by

$$\begin{aligned} \sigma^2 \;&=\; \frac{-S}{\rho_o R_o^3} \frac{x K'_m(x)}{K_m(x)} (1 - m^2 - x^2) \\ &+ \frac{\mu H_o^2}{\rho R_o^2} \{-x^2 + \beta^2 (\frac{m^2 I_m(x)}{x I'_m(x)} - 1) \frac{x K'_m(x)}{K_m(x)}\} \end{aligned} \tag{47}$$

As $S = 0$ and $a \to \infty$, we get

$$\sigma^2 = \frac{\mu H_o^2}{\rho R_o^2} [-x^2 + \beta^2 (\frac{m^2 I_m(x)}{x I'_m(x)} - 1) \frac{x K'_m(x)}{K_m(x)}] \tag{48}$$

which is the magnetodynamic dispersion relation of an incompressible hollow jet pervaded by azimuthal varying magnetic field for all axisymmetric and non-axisymmetric perturbation.

5. Stability Discussions

In order to study the stability analysis of the present model and its behaviour analytically, we have to discuss the properties of the modified Bessel functions and their derivatives.

Consider the resurrence relations (see Abramowitz and Stegun [1])

$$2I'_m(x) = I_{m-1}(x) + I_{m+1}(x) \tag{49}$$

$$2K'_m(x) = -K_{m-1}(x) - K_{m+1}(x) \tag{50}$$

In view of the relations (49) and (50) and the fact for non-zero value of x that

$$I_m(x) > 0 \tag{51}$$

$$K_m(x) > 0 \tag{52}$$

we may see that

$$I'_m(x) > 0 \tag{53}$$

$$K'_m(x) < 0 \tag{54}$$

Therefore, for $x \neq 0$ and $y \neq 0$, we have

$$yK'_m(y)/K_m(y) < 0 \tag{55}$$

$$I_m(x)/(xI'_m(x)) > 0 \tag{56}$$

The capillary instability a compressible hollow jet as $H_o = 0$ could be discussed via the dispersion relation

$$\sigma^2 = \frac{-S}{\rho_o R_o^3} \frac{yK'_m(y)}{K_m(y)}(1 - m^2 - x^2) \tag{57}$$

In view of the recurrence relations (49) and (50) and the inequalities (51) – (56), the sign of σ^2 is depending on the sign of the quantity $(1 - m^2 - x^2)$ based on the values of m and x. Therefore, we have

1. for $m = 0$:

$$\frac{\sigma^2}{S/(\rho_o R_o^3)} \begin{cases} < 0 \text{ as } 1 \leq x < \infty \\ > 0 \text{ as } 0 < x < \infty \end{cases} \tag{58}$$

2. for $m \geq 1$:

$$\frac{\sigma^2}{S/(\rho_o R_o^3)} < 0 \text{ as } 0 < x < \infty \tag{59}$$

This means that $S/(\rho_o R_o^3)^{1/2}$ is imaginary in the axisymmetric perturbation as $1 \leq x \leq \infty$ and it is so in the non-axisymmetric perturbation as $0 < x < \infty$. While is real only in the axisymmetric mode $m = 0$ as $0 < x < 1$. Consequently, the hollow jet is capillary stable:

1. in the non-axisymmetricperturbation for all short and long wavelengths
2. in the axisymmetric mode $m = 0$ for short wavelengths $\lambda \leq 2\pi R_o$ which are shorter than the eireumerence of the gas jet.

While the present model is capillary unstable only in the axisymmetric mode as the wavelength $\lambda > 2\pi R_o$ is greater than the circumference of the gas jet. It is remarkable that the transition from stability states to those of instability in the axisymmetric mode $m = 0$ occurred as $\lambda = 2\pi R_o$. i.e. at $x = 1$.

We have to mention here for the problem under consideration that:

$$y > x \,\text{since}\, 1 < q < \infty \tag{60}$$

so

$$I_m(y) > I_m(x) \tag{61}$$

$$K_m(x) > K_m(y) \tag{62}$$

$$K'_m(x) < K'_m(y) \tag{63}$$

$$I'_m(y) > I'_m(x) \tag{64}$$

Therefore, for the given different values of y based on the assumed values of x, we may see that the compressibility increases the stable domains of the hollow jet under the capillary force effect.

We conclude that the compressibility has a stabilizing tendency. So it increases the classical capillary stable domains and simultanuously decreases the unstable domain ($0 < x < 1$ in $m = 0$).

The magnetohydrodynamic stability of the hollow jet could be investigated upon using (see equation (47) as $S = 0$) the relation

$$\sigma^2 = \frac{\mu H_o^2}{\rho R_o^2}\{-x^2 + \beta^2(\frac{m^2 I_m(x)}{xI'_m(x)} - 1)\frac{yK'_m(y)}{K_m(y)}\} \tag{65}$$

The effect of the magnetic pervaded in the liquid region is represented by the term $(-x^2)$ following the natural quantity $(\mu H_o^2/\rho_o R_o^2)$ in equations (65) and (48). It has a stabilizing effect whether the fluid is compressible or not and that effect is independent of the kind of perturbation for all long and short wavelengths.

The effect of the magnetic field pervaded into the gas cylinder is represented by the terms including the parameter β following $(\mu H_o^2/\rho R_o^2)$ in equation (65) which are

$$\beta^2(\frac{m^2 I_m(x)}{xI'_m(x)} - 1)\frac{yK'_m(y)}{K_m(y)} \tag{66}$$

In axisymmetric perturbation $m = 0$, its effect is represented by the term $\beta^2(-yK'_0(y)/K_0(y))$ that has destabilizing influence since $K_0(y) < 0$.

In the non-axisymmetric mode $m \geq 1$, the magnetic field in the gas cylinder has destabilizing effect in the term $\beta^2(-yK'_m(y)/K_m(y))$ while it has a stabilizing effect in the term $\beta^2[m^2 I_m(x)/(xI'_m(x))][yK'_m(y)/K_m(y)]$.

Therefore, the magnetic field pervaded in the gas cylinder is purely stabilizing in the axisymmetric mode while in the non-axisymmetric mode it is stabilizing or not according to restriction.

Based on the foregoing disussions, the model of hollow jet is magnetodynamic stable in the axisymmetric mode m = 0. While in the non-axisymmetric modes the hollow jet is magnetodynamic stable or not according to restriction.

It is found here also that the compressibility has stabilizing tendency. So its effect is to increases the stable domains as $m = 0$ and decreases the unstable domains as $m \geq 1$.

By combining the results of the discussions of the relations (57) and (65), the electromagnetic force overcomes the capillary unstable domains in $m = 0$ while in $m \geq 1$ there will be stable and unstable domains.

In discussing the behaviour of the present model under the effect of surface pressure along the gas − liquid interface, it is found via the numerical that the instability curves show a non-zero growth rate for zero wavenumber. In order to interpret such observations we may discuss the following asymptotic behaviour.

For very long wavelengths ($x << 1$), the expansion of the modified Bessel functions of first and second kind of zero order (cf. Abramowitz and Stegun [1]) are given by

$$I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + ... \tag{67}$$

and

$$\begin{aligned} K_0(x) &= \frac{x^2}{2} + \frac{3}{128}x^4 + \frac{11}{13824}x^6 + ... \\ &\quad -(1 + \frac{x^2}{4} + \frac{x^6}{64} + ...)\ln(\alpha x/2) + ... \end{aligned} \tag{68}$$

where α is Euler's constant.

By substituting from equation (67) and (68) and the derivatives $I_o'(x)$ and $K_o'(x)$ into equation (57) as $m = 0$ for $x << 1$, we get

$$\sigma_o^2 = \frac{S}{\rho_o R_o^3}\{\frac{-1}{\ln(\alpha x/2)} + \frac{3}{2\ln(\alpha x/2)}x^2 + ...\} \tag{69}$$

The discussion of this asymptotic relation confirms that σ^2 tends to zero with a vertical tangent as x tends to zero for very long wavelengths, $x < 1$. This behavior is very different from that of a liquid cylinder (emmersed in a gas of zero inertia) where σ_o for liquid approaches zero linearly as x tends to zero (no dispersion!). moreover, it is remarkable that unstable domains of the present model are much greater than those of liquid jet. In spite of the curves for the hollow and full liquid jet interset at $x = 0$ and at $x = 1$.

For very long wavelengths $x << 1$, in the case of the lowest non-axisymmetric perturbation mode m = 1, the capillary dispersion relation (57) gives

$$\sigma_1^2 = \frac{-S}{\rho_o R_o^3}[1 - (\ln\gamma/2)x^2 + ...]x^2 \tag{70}$$

which is the same approximation as for the liquid cylinder. There is a neutral stability for infinite wavelengths (as $x \to 0$) while the model is stable for all other wavelengths.

In the case of higher non-axisymmetric perturbation modes $m \geq 2$, the eigenvalue relation (57) yields

$$\sigma_m^2 = \frac{S}{\rho_o R_o^3}[m(1 - m^2) - \frac{(1 + 3m)}{2}x^2 + ...]$$

which is the same, except for a change in sign, for a full liquid jet embedded into a gas medium of negligible inertia.

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