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Velocity Correction in Generalized Hohmann and Bi-elliptic Impulsive Orbital Maneuvers Using Energy Concepts

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We applied small tangential impulses due to motor thrusts at peri-apse and apo-apse perpendicular to major axis of the elliptic orbits. Our aim is to obtain a precise final orbit stemming from an initial orbit. We executed these tangential correctional velocities to all the four feasible configurations. The correctional increments of velocities Δv_A & Δv_B at the points A, B for the Hohmann transfer and at the points A, B, C for the Bi-Elliptic transfer induce the precise final orbit. Throughout the treatment we encounter relationships for both cases of transfer that describe the alteration in major axes and eccentricities due to these motor thrusts supplied by a rocket. The whole theory is a correctional improvement process.

Keywords: Orbital mechanics, transfer orbits, velocity corrections

1. Introduction

As it is well known, orbital maneuvers are characterized by a change in orbital velocity. If a velocity increment Δv, which is a vector, is added to a rocket velocity at the points A, B, C, which is also a vector, then new rocket velocity results. If the Δv is added instantaneously, the maneuver is called an impulsive maneuver or transfer orbit [1]. Our process convey features of the estimation theory and differential corrections [2]. The treatment is entirely analytic. We assume that no instantaneous alteration in the radius vector occurs. The central field is gravitational [3]. In our analysis, it is legitimate to use differentials. All the orbits of our eight configurations are elliptic, no circular ones are supposed to be considered. The Hohmann transfer is a two impulse transfer with one transfer orbit, whilst the bi-elliptic is a three impulse transfer with two transfer orbits. For minimum consumption of fuel we do not exceed three transfer impulses, this renders the problem much less sophisticated. Also coplanar vehicle transfer consumes less fuel than the non-coplanar case [4], [5]. We have eight configurations to consider, for these correctional improvements for the generalized Hohmann and bi-elliptic vehicle orbital transfer [6]. We are capable of deriving four identities expressing Δa_1, Δa_T, Δe_1, Δe_T as functions of Δv_A, Δv_B for each generalized Hohmann system. For the bi-elliptic transfer, we deduce three identities for Δa_1, Δa_T, $\Delta a_{T'}$. Moreover we can reveal from the drawings of the four bi-elliptic configurations that $a_T = a_1 + \Delta a_1$, $a_{T'} = a_T + \Delta a_T$, $a_2 = a_{T'} + \Delta a_{T'}$ and that we can evaluate Δv_A, Δv_B, Δv_C as functions of Δa_1.

2. Method and results

2.1. *Generalized Hohmann case*

2.1.1. *First configuration*

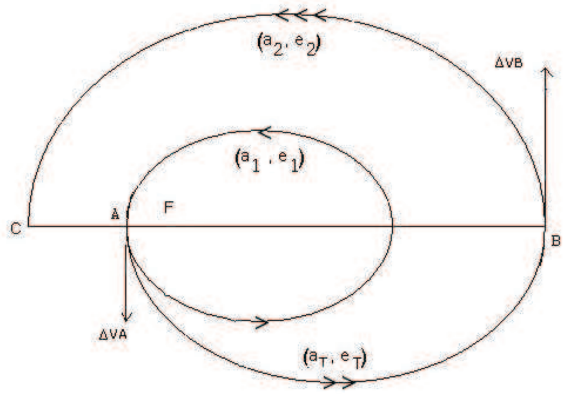

Figure 1

For the first configuration (Fig. 1), we find the following identities:

$$\Delta a_1 = \frac{2 v_A a_1^2 \Delta v_A}{\mu} \tag{1}$$

$$\Delta a_1 = 2a_1 \left(\frac{a_1}{\mu}\right)^{1/2} \left(\frac{1+e_1}{1-e_1}\right)^{1/2} \Delta v_A \tag{2}$$

Similarly,

$$\begin{aligned} \Delta a_T &= \frac{2 v_B a_T^2 \Delta v_B}{\mu} \qquad (3) \\ \Delta a_T &= 2 a_T \left(\frac{a_T}{\mu}\right)^{1/2} \left(\frac{1-e_T}{1+e_T}\right)^{1/2} \Delta v_B \\ a_T &= a_1 + \Delta a_1 \qquad (4) \end{aligned}$$

Put

$$\begin{aligned} b_1 &= a_1 (1-e_1) \\ b_2 &= a_2 (1-e_2) \\ b_3 &= a_1 (1+e_1) \\ b_4 &= a_2 (1+e_2) \end{aligned}$$

We have,

$$\begin{aligned} v_A &= \left\{\frac{\mu (1+e_1)}{a_1 (1-e_1)}\right\}^{1/2} = \left\{\frac{\mu (1+e_1)}{b_1}\right\}^{1/2} \\ v_B &= \left\{\frac{\mu (1-e_T)}{a_T (1+e_T)}\right\}^{1/2} \qquad (5) \end{aligned}$$

From geometry of Fig.1,

$$\begin{aligned} a_1 (1-e_1) &= a_T (1-e_T) = b_1 \quad \text{i.e.} \quad 1-e_T = \frac{b_1}{a_T} \\ a_2 (1+e_2) &= a_T (1+e_T) = b_4 \quad \text{i.e.} \quad 1+e_T = \frac{b_4}{a_T} \qquad (6) \end{aligned}$$

Therefore,

$$\frac{1-e_T}{1+e_T} = \frac{b_1}{b_4}; \quad 2a_T = a_1 (1-e_1) + a_2 (1+e_2) = b_1 + b_4 \qquad (7)$$

and

$$\Delta a_T = 2 a_T^{3/2} \left\{\frac{b_1}{\mu b_4}\right\}^{1/2} \Delta v_B \qquad (8)$$

Whence,

$$\begin{aligned} \Delta a_T &= 2 \frac{a_1^{3/2}}{\sqrt{\mu}} \left\{\frac{b_1}{b_4}\right\}^{1/2} \left[1 + 2\left(\frac{a_1}{\mu}\right)^{1/2} \left(\frac{1+e_1}{1-e_1}\right)^{1/2} \Delta v_A\right]^{3/2} \Delta v_B \qquad (9) \\ e_T &= e_1 + \Delta e_1 \qquad (10) \\ \Delta e_1 &= \frac{2 a_1 \left(1-e_1^2\right)}{e_1} \left(\frac{1}{r_1} - \frac{1}{a_1}\right) \frac{\Delta v_A}{v_A}; \quad r_1 = a_1 (1-e_1) = b_1 \qquad (11) \end{aligned}$$

i.e.

$$\Delta e_1 = 2 \left\{\frac{a_1 \left(1-e_1^2\right)}{\mu}\right\}^{1/2} \Delta v_A \qquad (12)$$

From Eq. (10),

$$e_T = e_1 + 2\Delta v_A \left\{ \frac{a_1\left(1-e_1^2\right)}{\mu} \right\}^{1/2} \tag{13}$$

Similarly,

$$\Delta e_T = \frac{2a_T\left(1-e_T^2\right)}{e_T} \left\{ \frac{1}{a_T\left(1+e_T\right)} - \frac{1}{a_T} \right\} \frac{\Delta v_B}{v_B} \tag{14}$$

i.e.

$$\Delta e_T = 2\left\{ \frac{a_T\left(1+e_T\right)}{\mu\left(1-e_T\right)} \right\}^{1/2} \left(e_T - 1\right)\Delta v_B \tag{15}$$

whence,

$$\Delta e_T = 2\left\{ \frac{b_4}{\mu\left(1-e_1\right)} \right\}^{1/2} \left\{ 1 + 2\left\{ \frac{b_3}{\mu\left(1-e_1\right)} \right\}^{1/2} \Delta v_A \right\}^{1/2} \left\{ \left(e_1-1\right) + 2\left\{ \frac{a_1\left(1-e_1^2\right)}{\mu} \right\}^{1/2} \Delta v_A \right\} \Delta v_B \tag{16}$$

2.1.2. *Second configuration*

For the second configuration (Fig. 2), we have the following formulae:

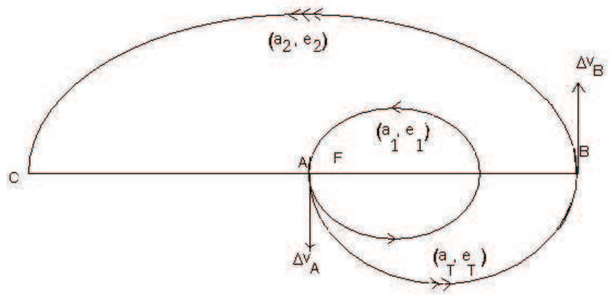

Figure 2

$$a_1\left(1-e_1\right) = a_T\left(1-e_T\right) = b_1; a_2\left(1-e_2\right) = a_T\left(1+e_T\right) = b_2 \tag{17}$$

$$v_A = \left\{ \frac{\mu\,(1+e_1)}{b_1} \right\}^{1/2}; \quad v_B = \left\{ \frac{\mu\,(1-e_T)}{a_T\,(1+e_T)} \right\}^{1/2} \tag{18}$$

$$\Delta a_1 = \frac{2a_1^2 v_A \Delta v_A}{\mu}$$

$$\Delta a_1 = 2a_1^{3/2} \left\{ \frac{(1+e_1)}{\mu\,(1-e_1)} \right\}^{1/2} \Delta v_A \tag{19}$$

$$\Delta a_T = 2a_T^{3/2} \left\{ \frac{b_1}{\mu b_2} \right\}^{1/2} \Delta v_B$$

From Eq. (17), we write:

$$\frac{1-e_T}{1+e_T} = \frac{a_1\,(1-e_1)}{a_2\,(1-e_2)} = \frac{b_1}{b_2}$$

Therefore,

$$\Delta a_T = \frac{2a_1^{3/2}}{\sqrt{\mu}} \left\{ \frac{b_1}{b_2} \right\}^{1/2} \left[1 + 2\left(\frac{a_1}{\mu}\right)^{1/2} \left(\frac{1+e_1}{1-e_1}\right)^{1/2} \Delta v_A \right]^{3/2} \Delta v_B \tag{20}$$

With regard to the eccentricities we have,

$$\Delta e_1 = 2\left\{ \frac{a_1\left(1-e_1^2\right)}{\mu} \right\}^{1/2} \Delta v_A \quad \text{and} \quad e_T = e_1 + \Delta e_1$$

whence

$$e_T = e_1 + 2\Delta v_A \left\{ \frac{a_1\left(1-e_1^2\right)}{\mu} \right\}^{1/2} \tag{21}$$

$$\Delta e_T = 2\,(e_T - 1)\,\frac{\Delta v_B}{v_B}$$

i.e.

$$\Delta e_T = 2\left\{ \frac{a_T\,(1+e_T)}{\mu\,(1-e_T)} \right\}^{1/2} (e_T - 1)\,\Delta v_B \tag{22}$$

After some substitutions,

$$\Delta e_T = 2\left\{ \frac{b_2}{\mu\,(1-e_1)} \right\}^{1/2} \left\{ 1 + 2\left\{ \frac{b_3}{\mu\,(1-e_1)} \right\}^{1/2} \Delta v_A \right\} \left\{ (e_1 - 1) + 2\left\{ \frac{a_1\left(1-e_1^2\right)}{\mu} \right\}^{1/2} \Delta v_A \right\} \Delta v_B \tag{23}$$

2.1.3. *Third configuration*

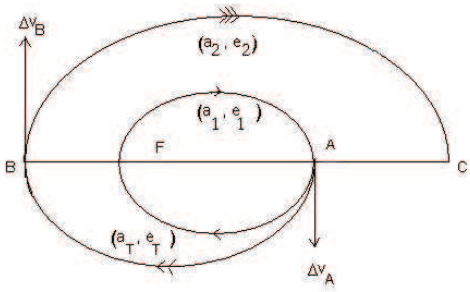

Figure 3

For the third configuration (Fig.3), we have the following identities:

$$a_1\,(1+e_1) = a_T\,(1+e_T) = b_3; \quad a_2\,(1-e_2) = a_T\,(1-e_T) = b_2 \tag{24}$$

$$v_A = \left\{\frac{\mu\,(1-e_1)}{b_3}\right\}^{1/2} \tag{25}$$

$$v_B = \left\{\frac{\mu\,(1+e_T)}{a_T\,(1-e_T)}\right\}^{1/2} \tag{26}$$

We find

$$\frac{1+e_T}{1-e_T} = \frac{a_1\,(1+e_1)}{a_2\,(1-e_2)} = \frac{b_3}{b_2} \tag{27}$$

$$\Delta a_1 = \frac{2a_1^{3/2}}{\sqrt{\mu}}\left(\frac{1-e_1}{1+e_1}\right)^{1/2}\Delta v_A \tag{28}$$

$$a_T = a_1 + \Delta a_1 = a_1\left\{1 + 2\sqrt{\frac{b_1}{\mu\,(1+e_1)}}\Delta v_A\right\} \tag{29}$$

After some substitutions

$$\Delta a_T = \frac{2a_1^{3/2}}{\sqrt{\mu}}\left\{\frac{b_3}{b_2}\right\}^{1/2}\left\{1 + 2\sqrt{\frac{b_1}{\mu\,(1+e_1)}}\Delta v_A\right\}^{3/2}\Delta v_B \tag{30}$$

With respect to the eccentricities

$$\Delta e_1 = \frac{2a_1\left(1-e_1^2\right)}{e_1}\left(\frac{1}{r_1}-\frac{1}{a_1}\right)\frac{\Delta v_A}{v_A}; \quad r_1 = a_1\left(1+e_1\right) = b_3 \tag{31}$$

i.e.

$$\Delta e_1 = -2\left\{\frac{a_1\left(1-e_1^2\right)}{\mu}\right\}^{1/2}\Delta v_A \tag{32}$$

$$e_T = e_1 + \Delta e_1 = e_1 - 2\left\{\frac{a_1\left(1-e_1^2\right)}{\mu}\right\}^{1/2}\Delta v_A \tag{33}$$

Therefore,

$$\Delta e_T = 2\left(1+e_T\right)\left\{\frac{a_T\left(1-e_T\right)}{\mu\left(1+e_T\right)}\right\}^{1/2}\Delta v_B \tag{34}$$

Whence

$$\begin{aligned}\Delta e_T = {} & 2\left\{\frac{b_2}{\mu\left(1+e_1\right)}\right\}^{1/2}\left\{1+2\sqrt{\frac{b_1}{\mu\left(1+e_1\right)}}\Delta v_A\right\}^{1/2} \\ & \left\{\left(1+e_1\right)-2\sqrt{\frac{a_1\left(1-e_1^2\right)}{\mu}}\Delta v_A\right\}\Delta v_B\end{aligned} \tag{35}$$

2.1.4. *Fourth configuration*

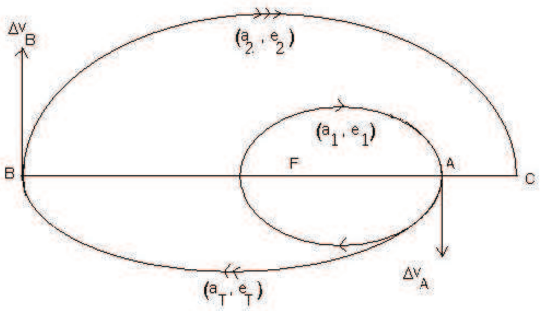

Figure 4

For the fourth configuration (Fig. 4), we have the following equalities:

$$a_1\left(1+e_1\right) = a_T\left(1-e_T\right) = b_3; \quad a_2\left(1+e_2\right) = a_T\left(1+e_T\right) = b_4 \tag{36}$$

$$v_A = \left\{\frac{\mu(1-e_1)}{b_3}\right\}^{1/2} \tag{37}$$

$$v_B = \left\{\frac{\mu(1-e_T)}{a_T(1+e_T)}\right\}^{1/2} \tag{38}$$

$$\Delta a_1 = \frac{2a_1^{3/2}}{\sqrt{\mu}}\left(\frac{1-e_1}{1+e_1}\right)^{1/2}\Delta v_A \tag{39}$$

$$a_T = a_1 + \Delta a_1 = a_1\left\{1 + 2\sqrt{\frac{b_1}{\mu(1+e_1)}}\Delta v_A\right\} \tag{40}$$

But

$$\Delta a_T = \frac{2a_T^2 v_B \Delta v_B}{\mu} \tag{41}$$

Whence after substitution

$$\Delta a_T = \frac{2a_1^{3/2}}{\sqrt{\mu}}\left\{\frac{b_3}{b_4}\right\}^{1/2}\left\{1 + 2\sqrt{\frac{b_1}{\mu(1+e_1)}}\Delta v_A\right\}^{3/2}\Delta v_B \tag{42}$$

As for the eccentricities, we find

$$\Delta e_1 = -2\left\{\frac{a_1\left(1-e_1^2\right)}{\mu}\right\}^{1/2}\Delta v_A \tag{43}$$

$$e_T = e_1 + \Delta e_1$$

i.e.

$$e_T = e_1 - 2\left\{\frac{a_1\left(1-e_1^2\right)}{\mu}\right\}^{1/2}\Delta v_A \tag{44}$$

After substitution and simple reduction, we get

$$\Delta e_T = 2\left\{\frac{b_4}{\mu(1+e_1)}\right\}^{1/2}\left\{1 + 2\sqrt{\frac{b_1}{\mu(1+e_1)}}\Delta v_A\right\}^{1/2} \left\{(e_1-1) - 2\sqrt{\frac{a_1(1-e_1^2)}{\mu}}\Delta v_A\right\}\Delta v_B \tag{45}$$

2.2. ***Generalized bi–elliptic case***

2.2.1. *First configuration*

For the first configuration of bi–elliptic case (Fig. 5), we have the following identities:

$$\begin{aligned} a_1(1-e_1) &= a_T(1-e_T) = b_1 \\ a_T(1+e_T) &= a_{T^{\text{`}}}(1+e_{T^{\text{`}}}) \\ a_2(1-e_2) &= a_{T^{\text{`}}}(1-e_{T^{\text{`}}}) = b_2 \end{aligned} \tag{46}$$

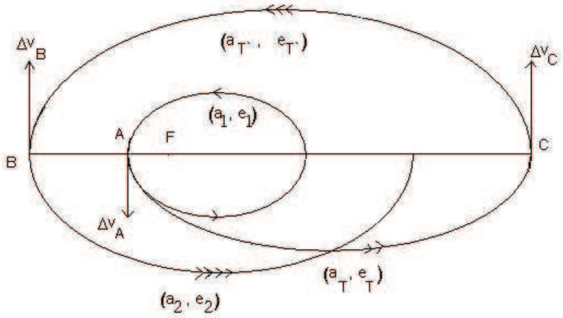

Figure 5

From (1), we get

$$a_T = a_{T'} + \frac{1}{2}(b_1 - b_2) \tag{47}$$

$$e_T = 1 - \frac{b_1}{a_T} \tag{48}$$

$$e_{T'} = \frac{2a_T - b_1 - b_2}{2a_T - b_1 + b_2} \tag{49}$$

At point A

$$\Delta a_1 = \frac{2v_A a_1^2 \Delta v_A}{\mu}; \quad v_A = \left\{\frac{\mu(1+e_1)}{b_1}\right\}^{1/2} \tag{50}$$

$$a_T = a_1 + \Delta a_1 \tag{51}$$

Whence

$$a_T = a_1\left[1 + 2\left\{\frac{b_3}{\mu(1-e_1)}\right\}^{1/2}\Delta v_A\right]$$

Let

$$B = 2\Delta v_A\left\{\frac{b_3}{\mu(1-e_1)}\right\}^{1/2} \tag{52}$$

Therefore

$$a_T = a_1(1+B)$$

$$a_{T'} = a_1(1+B) - \frac{1}{2}(b_1 - b_2) \tag{53}$$

$$e_T = 1 - \frac{b_1}{a_1(1+B)} \tag{54}$$

$$e_{T'} = \frac{2a_1(1+B) - b_1 - b_2}{2a_1(1+B) - b_1 + b_2} \tag{55}$$

At point C:

$$\Delta a_T = \frac{2v_C a_T^2 \Delta v_C}{\mu}$$

i.e.

$$\Delta v_C = \frac{\Delta a_T \mu}{2v_C a_T^2} \tag{56}$$

With

$$v_C = \left\{\frac{\mu(1-e_T)}{a_T(1+e_T)}\right\}^{1/2}$$

Whence

$$v_C = \left[\frac{\mu b_1}{a_1(1+B)\{2a_1(1+B)-b_1\}}\right]^{1/2} \tag{57}$$

and

$$\Delta a_T = a_{T^{\text{‘}}} - a_T = -\frac{1}{2}(b_1-b_2) \tag{58}$$

Therefore

$$\Delta v_C = -\frac{(b_1-b_2)\left[\mu\{2a_1(1+B)-b_1\}\right]^{1/2}}{4\sqrt{b_1}\{a_1(1+B)\}^{3/2}} \tag{59}$$

At point B:

$$\Delta v_B = \frac{\mu \Delta a_{T^{\text{‘}}}}{2v_B a_{T^{\text{‘}}}^2} \quad \text{with} \quad v_B = \left\{\frac{\mu(1+e_{T^{\text{‘}}})}{a_{T^{\text{‘}}}(1-e_{T^{\text{‘}}})}\right\}^{1/2} \tag{60}$$

Whence by substitution

$$v_B = \left[\frac{\mu\{2a_1(1+B)-b_1\}}{b_2\left\{a_1(1+B)-\frac{1}{2}(b_1-b_2)\right\}}\right]^{1/2} \tag{61}$$

and

$$a_2 = a_{T^{\text{‘}}} + \Delta a_{T^{\text{‘}}} \tag{62}$$

Hence

$$\Delta a_{T^{\text{‘}}} = a_2 - a_1(1+B) + \frac{1}{2}(b_1-b_2) \tag{63}$$

Whence by substitution and some rearrangement

$$\Delta v_B = \sqrt{\frac{\mu b_2}{2\{2a_1(1+B)-b_1\}}}\frac{\{2a_2-2a_1(1+B)+(b_1-b_2)\}}{\{2a_1(1+B)-(b_1-b_2)\}^{3/2}} \tag{64}$$

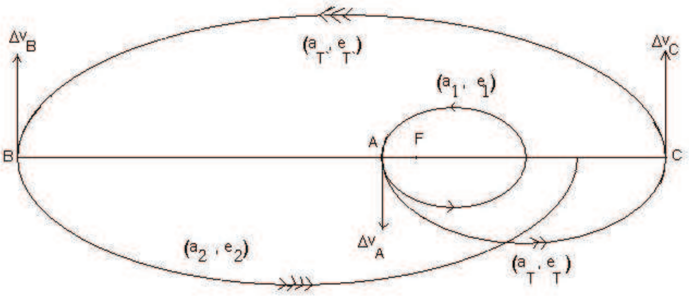

Figure 6

2.2.2. *Second configuration*

For the second configuration of the bi–elliptic (Fig. 6), we get the following equalities:

$$\begin{aligned} a_1 (1 - e_1) &= a_T (1 - e_T) = b_1 \\ a_T (1 + e_T) &= a_{T'} (1 - e_{T'}) \\ a_2 (1 + e_2) &= a_{T'} (1 + e_{T'}) = b_4 \end{aligned} \tag{65}$$

Then

$$a_T = a_{T'} + \frac{1}{2} (b_1 - b_4) \tag{66}$$

$$a_{T'} = a_T - \frac{1}{2} (b_1 - b_4) \tag{67}$$

$$e_T = 1 - \frac{b_1}{a_T} \tag{68}$$

$$e_{T'} = \frac{2a_T - b_1 - b_4}{-2a_T + b_1 - b_4} \tag{69}$$

At point A:

$$v_A = \sqrt{\frac{\mu (1 + e_1)}{a_1 (1 - e_1)}} = \sqrt{\frac{\mu (1 + e_1)}{b_1}} \tag{70}$$

and

$$\Delta a_1 = \frac{2 v_A a_1^2 \Delta v_A}{\mu}$$

whence

$$\Delta v_A = \frac{\sqrt{\mu}}{2 a_1^{3/2}} \sqrt{\frac{1 - e_1}{1 + e_1}} \Delta a_1 \tag{71}$$

But

$$a_T = a_1 + \Delta a_1 \tag{72}$$

$$a_T = a_1 \left[1 + 2\sqrt{\frac{b_3}{\mu(1-e_1)}}\Delta v_A\right] \tag{73}$$

Let

$$B = 2\sqrt{\frac{b_3}{\mu(1-e_1)}}\Delta v_A \tag{74}$$

Therefore

$$a_T = a_1(1+B) \tag{75}$$

$$a_{T^{‘}} = a_1(1+B) - \frac{(b_1 - b_4)}{2} \tag{76}$$

$$e_T = \frac{a_1(1+B) - b_1}{a_1(1+B)} \tag{77}$$

$$e_{T^{‘}} = \frac{2a_1(1+B) - b_1 - b_4}{-2a_1(1+B) + b_1 - b_4} \tag{78}$$

At point C:

$$\Delta a_T = \frac{2a_T^2 v_C \Delta v_C}{\mu} \tag{79}$$

i.e.

$$\Delta v_C = \frac{\mu \Delta a_T}{2 v_C a_T^2} \tag{80}$$

with

$$v_C = \sqrt{\frac{\mu(1-e_T)}{a_T(1+e_T)}} \tag{81}$$

$$\frac{1-e_T}{a_T(1+e_T)} = \frac{b_1}{a_1(1+B)\{2a_1(1+B) - b_1\}} \tag{82}$$

whence

$$v_C = \sqrt{\frac{\mu b_1}{a_1(1+B)\{2a_1(1+B) - b_1\}}} \tag{83}$$

But

$$\Delta a_T = a_{T^{‘}} - a_T = -\frac{(b_1 - b_4)}{2} \tag{84}$$

Therefore

$$\Delta v_C = \frac{(b_4 - b_1)}{4a_T^{3/2}}\sqrt{\frac{\mu(2a_T - b_1)}{b_1}} \tag{85}$$

Or

$$\Delta v_C = \frac{(b_4 - b_1)}{4\left\{a_1(1+B)\right\}^{3/2}}\sqrt{\frac{\mu\left\{2a_1(1+B) - b_1\right\}}{b_1}} \tag{86}$$

At point B:

$$\Delta v_B = \frac{\mu \Delta a_{T^{\text{‘}}}}{2v_B a_{T^{\text{‘}}}^2} \quad \text{with} \quad v_B = \sqrt{\frac{\mu(1 - e_{T^{\text{‘}}})}{a_{T^{\text{‘}}}(1 + e_{T^{\text{‘}}})}} \tag{87}$$

But

$$\frac{1 - e_{T^{\text{‘}}}}{a_{T^{\text{‘}}}(1 + e_{T^{\text{‘}}})} = \frac{2a_T - b_1}{b_4\left\{a_T - \frac{(b_1 - b_4)}{2}\right\}} \tag{88}$$

whence

$$v_B = \left[\frac{\mu(2a_T - b_1)}{b_4\left\{a_T - \frac{(b_1 - b_4)}{2}\right\}}\right]^{1/2} \tag{89}$$

Or

$$v_B = \left[\frac{\mu\left\{2a_1(1+B) - b_1\right\}}{b_4\left\{a_1(1+B) - \frac{(b_1 - b_4)}{2}\right\}}\right]^{1/2} \tag{90}$$

We have

$$a_2 = a_{T^{\text{‘}}} + \Delta a_{T^{\text{‘}}} \tag{91}$$

i.e.

$$\Delta a_{T^{\text{‘}}} = a_2 - a_1(1+B) + \frac{(b_1 - b_4)}{2} \tag{92}$$

Whence after substitution and rearrangement

$$\Delta v_B = \sqrt{\frac{\mu b_4}{2\left\{2a_1(1+B) - b_1\right\}}}\left[\frac{\left\{2a_2 - 2a_1(1+B) + (b_1 - b_4)\right\}}{\left\{2a_1(1+B) - (b_1 - b_4)\right\}^{3/2}}\right] \tag{93}$$

2.2.3. *Third configuration*

For the third configuration of bi–elliptic case (Fig. 7), we find the following relationships:

$$\begin{aligned} a_1(1+e_1) &= a_T(1+e_T) = b_3 \\ a_T(1-e_T) &= a_{T^{\text{‘}}}(1 - e_{T^{\text{‘}}}) \\ a_2(1+e_2) &= a_{T^{\text{‘}}}(1 + e_{T^{\text{‘}}}) = b_4 \end{aligned} \tag{94}$$

$$e_T = \frac{b_3}{a_T} - 1 \tag{95}$$

$$e_{T^{\text{‘}}} = \frac{-2a_T + b_3 + b_4}{2a_T - b_3 + b_4} \tag{96}$$

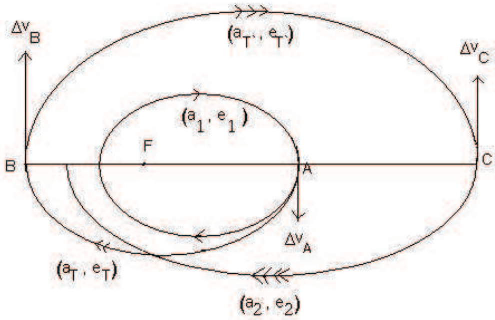

Figure 7

At point A:

$$\Delta v_A = \frac{\mu \Delta a_1}{2 a_1^2 v_A} \quad \text{with} \quad v_A = \sqrt{\frac{\mu (1 - e_1)}{b_3}} \tag{97}$$

i.e.

$$\Delta a_1 = \frac{2 a_1^{3/2}}{\sqrt{\mu}} \left(\frac{1 - e_1}{1 + e_1} \right)^{1/2} \Delta v_A \tag{98}$$

$$a_T = a_1 + \Delta a_1 = a_1 \left[1 + 2 \left\{ \frac{b_1}{\mu (1 + e_1)} \right\}^{1/2} \Delta v_A \right] \tag{99}$$

Let

$$\xi = 2 \left\{ \frac{b_1}{\mu (1 + e_1)} \right\}^{1/2} \Delta v_A \tag{100}$$

i. e.

$$a_T = a_1 (1 + \xi) \tag{101}$$

At point B:

$$v_B = \sqrt{\frac{\mu (1 + e_T)}{a_T (1 - e_T)}} \tag{102}$$

Therefore

$$\frac{1+e_T}{a_T(1-e_T)} = \frac{b_3}{a_1(1+\xi)\{2a_1(1+\xi)-b_3\}} \tag{103}$$

i.e.

$$v_B = \left[\frac{\mu b_3}{a_1(1+\xi)\{2a_1(1+\xi)-b_3\}}\right]^{1/2}$$

$$\Delta a_T = a_{T'} - a_T = \frac{(b_4-b_3)}{2} \tag{104}$$

$$\Delta v_B = \frac{\mu \Delta a_T}{2 v_B a_T^2} \tag{105}$$

$$\Delta v_B = \frac{\sqrt{\mu}(b_4-b_3)}{4\{a_1(1+\xi)\}^{3/2}}\left\{\frac{2a_1(1+\xi)-b_3}{b_3}\right\}^{1/2} \tag{106}$$

At point C:

$$v_C = \left\{\frac{\mu(1-e_{T'})}{a_{T'}(1+e_{T'})}\right\}^{1/2} \tag{107}$$

After some reductions we get

$$\frac{1-e_{T'}}{a_{T'}(1+e_{T'})} = \frac{2a_T-b_3}{b_4\left\{a_1(1+\xi)+\frac{(b_4-b_3)}{2}\right\}} \tag{108}$$

i.e.

$$v_C = \left[\frac{\mu\{2a_1(1+\xi)\}-b_3}{b_4\left\{a_1(1+\xi)+\frac{(b_4-b_3)}{2}\right\}}\right]^{1/2} \tag{109}$$

$$\Delta a_{T'} = a_2 - a_{T'} \tag{110}$$

i.e.

$$\Delta a_{T'} = \frac{(b_2+b_3)}{2} - a_T \tag{111}$$

$$\Delta v_C = \frac{\mu \Delta a_{T'}}{2 v_C a_{T'}^2} \tag{112}$$

Finally, we get

$$\Delta v_C = \sqrt{\frac{\mu b_4}{2\{2a_1(1+\xi)-b_3\}}}\left[\frac{(b_2+b_3)-2a_1(1+\xi)}{\{2a_1(1+\xi)+(b_4-b_3)\}^{3/2}}\right] \tag{113}$$

2.2.4. *Fourth configuration*

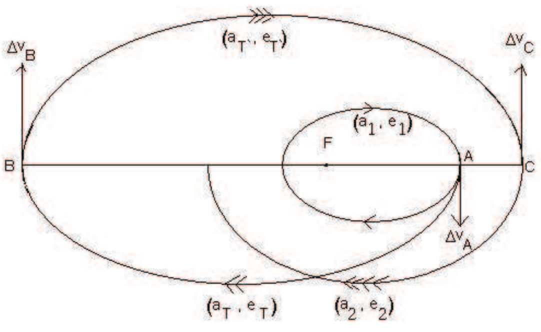

Figure 8

For the fourth configuration of bi-elliptic cas (Fig. 8), we deduce the following identities:

$$\begin{aligned} a_1(1+e_1) &= a_T(1-e_T) = b_3 \\ a_T(1+e_T) &= a_{T'}(1+e_{T'}) \\ a_2(1+e_2) &= a_{T'}(1-e_{T'}) = b_4 \end{aligned} \tag{114}$$

$$2a_{T'} = 2a_T - b_3 + b_4 \tag{115}$$

$$e_T = 1 - \frac{b_3}{a_T} \tag{116}$$

$$e_{T'} = \frac{2a_T - b_3 - b_4}{2a_T - b_3 + b_4} \tag{117}$$

At point A

$$\Delta v_A = \frac{\mu \Delta a_1}{2a_1^2 v_A} \quad \text{with} \quad v_A = \sqrt{\frac{\mu(1-e_1)}{b_3}} \tag{118}$$

i.e.

$$\Delta a_1 = \frac{2a_1^{3/2}}{\sqrt{\mu}} \left(\frac{1-e_1}{1+e_1}\right)^{1/2} \Delta v_A \tag{119}$$

$$a_T = a_1 + \Delta a_1 = a_1 \left[1 + 2\left\{\frac{b_1}{\mu(1+e_1)}\right\}^{1/2} \Delta v_A\right] \tag{120}$$

Let

$$\xi = 2\left\{\frac{b_1}{\mu(1+e_1)}\right\}^{1/2}\Delta v_A \tag{121}$$

i. e.

$$a_T = a_1(1+\xi) \tag{122}$$

At point B

$$v_B = \sqrt{\frac{\mu(1-e_T)}{a_T(1+e_T)}};\quad \Delta v_B = \frac{\mu\Delta a_T}{2v_B a_T^2} \tag{123}$$

Therefore,

$$v_B = \left[\frac{\mu b_3}{a_1(1+\xi)\{2a_1(1+\xi)-b_3\}}\right]^{1/2} \tag{124}$$

$$\Delta a_T = a_{T'} - a_T = \frac{(b_4-b_3)}{2} \tag{125}$$

After some rearrangements, we acquire

$$\Delta v_B = \frac{\sqrt{\mu}(b_4-b_3)}{4\{a_1(1+\xi)\}^{3/2}}\left\{\frac{2a_1(1+\xi)-b_3}{b_3}\right\}^{1/2} \tag{126}$$

At point C

$$v_C = \left\{\frac{\mu(1+e_{T'})}{a_{T'}(1-e_{T'})}\right\}^{1/2};\quad \Delta v_C = \frac{\mu\Delta a_{T'}}{2v_C a_{T'}^2} \tag{127}$$

After substitution, we get

$$\frac{1+e_{T'}}{a_{T'}(1-e_{T'})} = \frac{2a_T-b_3}{b_4\left\{a_1(1+\xi)+\frac{(b_4-b_3)}{2}\right\}} \tag{128}$$

i.e.

$$v_C = \left[\frac{\mu\{2a_1(1+\xi)\}-b_3}{b_4\left\{a_1(1+\xi)+\frac{(b_4-b_3)}{2}\right\}}\right]^{1/2} \tag{129}$$

$$\Delta a_{T'} = a_2 - a_{T'}$$

i.e.

$$\Delta a_{T'} = \frac{(b_2+b_3)}{2} - a_T \tag{130}$$

Finally, we get

$$\Delta v_C = \sqrt{\frac{\mu b_4}{2\{2a_1(1+\xi)-b_3\}}}\left[\frac{(b_2+b_3)-2a_1(1+\xi)}{\{2a_1(1+\xi)+(b_4-b_3)\}^{3/2}}\right] \tag{131}$$

Table 1 Generalized Hohmann System

Fig.	Δv_A	Δv_B	Δa_1	Δa_T	Δe_1	Δe_T
1	0.1141	0.0702	0.2321	0.1475	0.2282	0.2044
2	0.0732	0.1138	0.1489	0.2364	0.1464	0.3128
3	0.1012	0.1441	0.1990	0.3246	0.2024	0.2996
4	0.1736	0.0751	0.3414	0.1823	0.3471	0.1492

Table 2 Generalized bi–elliptic System

Fig.	Δa_1	Δa_T	$\Delta a_{T'}$	Δv_C	Δv_B
1	0.2321	0.3414	0.0497	0.1532	0.0134
2	0.1489	0.1990	0.1758	0.0934	0.0567
3	0.1990	0.3246	0.0010	0.0030	0.1441
4	0.3414	0.3247	0.1423	0.0331	0.1983

3. Numerical Results

We consider the case of "Earth – Mars" transfer orbit, where [8], $a_1 = 1.0000$, $e_1 = 0.0167$, $a_2 = 1.5237$, $e_2 = 0.0934$, the subscript 1 refers to Earth and 2 refers to Mars. In our calculations, we put $\mu = 1$ (canonical system).

4. Discussion

We did not investigate the problem pragmatically when the primary mass is situated in the right focus. But by intuition we shall have the same results, and there will be eight feasible configurations, four for the Hohmann transfer and four for the bi–elliptic transfer. We deal with a correctional problem, in which our aim is to obtain a precise final transferred orbit. This is acquired by the application of two differential increments of velocity at points A, B for the Hohmann transfer, and three differential increments of velocity at points A, B, C for the bi–elliptic transfer. These differential increments are produced by motor thrusts of a rocket. The terminal and the transfer orbits are all elliptic. The significance of the analysis lies in its simplicity and correctness of the deduced formulae.

For the generalized Hohmann case we assigned the differential corrections Δa_1, Δa_T, produced by the differential variations $\Delta v_A \,\&\, \Delta v_B$ in terms of a_i, e_i $(i = 1, 2)$, Δv_A, Δv_B.

With regard to the eccentricity correction Δe_1, Δe_T, we assigned the velocity corrections Δv_A, Δv_B that give rise to the two infinitesimal variations Δe_1, Δe_T.

In addition we write down the expressions for $a_T = a_1 + \Delta a_1$, $e_T = e_1 + \Delta e_1$ in terms of a_1, e_1, Δv_A, since we deal with a differential variation of velocity at peri–apse.

As for the bi–elliptic generalized transfer, we have three infinitesimal impulses at points A, B, C. We deduced the correction Δa_1 due to the differential change in velocity at point A, Δv_A , from which we could find a$_T$, $a_{T'}$, e_T, $e_{T'}$ expressed in terms of Δv_A.

For the terminal points A, B of the transfer orbits, we can have the relationships between Δa_T, Δv_C at point C and Δv_B & $\Delta a_{T^{\text{‘}}}$ at point B. Whence we could determine Δv_C, Δv_B expressed in terms of a_i, $e_i\,(i = 1, 2)$ and Δv_A.

We extended the two tables of Art 3 of reference [9], to include the numerical results of all four feasible configurations, namely adding case (3)&(4).

The above treatment is a first time publication, using energy concepts, in the literature of the subject.

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