Mechanics and Mechanical Engineering
Vol. 12, No. 3 (2008) 201–209

General Case of Multi–Products of Axis Vectors and Vectors in an n–dimensional Space

Andrzej Polka
Technical University of Łódź, Division of Dynamics
Stefanowskiego 1/15, 90-924 Łódź, Poland

Received (18 October 2008)
Revised (25 October 2008)
Accepted (15 December 2008)

In the paper basic kinds of multi–products of axis versors and vectors in an n–dimensional space have been identified and named. In addition, two derivative multi–products have been described. A manner of generating products of any number of versors or vectors for any space dimension has been presented. It has been shown that the result of multiplication does not depend on the order of multiplication of vectors, but on the preference of multiplication assumed. Multiplication preference determines the kind of a multi–product. Identification relationships can be determined between different kinds of products, which will be discussed in papers to follow.

Keywords: Multi–product of vectors, even and odd product, first, second and third kind of product

1. Introduction

Vectors and basic operations on vectors, their sum and product, are the elementary problem of mechanics. Operations on vectors have their own specificity. A sum of vectors is defined for any number of them, while a product is subject to quantitative limitations. A calculus of vectors is defined by a scalar product for only two vectors $\mathbf{a}\,\mathbf{b}$. Thus, a question arises why any number of vectors can be added, while it is customary to perform scalar multiplication on only two. The problem is that the scalar multiplication of vectors is performed on the axes of an orthogonal system as a product of versors of the axis (positive unit vectors, lying on the axis), the product of versors of the same axis being equal to 1, whereas the product of versors of different axes being equal to 0. This simple model functions well when multiplying two versors; on the other hand, when three or more versors are multiplied, it loses its uniqueness. With a multi–product of many vectors the product of versors depends on a random order of multiplication and can be equal to 0, 1 or one of the versors.

This ambiguity can be prevented if two possible preferences of choice of multiplication are distinguished and – as a consequence of this assumption – four basic kinds of multi–products of vectors are defined. Such multiplication of vectors is a unique operation, independent of the order of vectors in a product. The present work is an expansion of the subject matter of paper [1], containing basic definitions of multi–products and restricted to a product of three vectors.

2. Assumptions

We are considering a problem a product m of vectors in an orthogonal n-dimensional space.

Let $\mathbf{a}_i$ ($i = 1,...m$) be an i-th vector and let its j-th coordinate in an n–dimensional Cartesian space is a_{ij} $(j = 1,...n)$.

After denoting versors of the coordinate system axis as $\mathbf{e}_j$ we can write the projection of the i-th vector onto the j-th axis of the system $\mathbf{a}_{ij}$ and the vector itself $\mathbf{a}_i$ as

$$\mathbf{a}_{ij} = a_{ij}\mathbf{e}_j \tag{1}$$

$$a_i = \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij}\mathbf{e}_j \tag{2}$$

where the versor of the j-th axis of the system of coordinates
$\mathbf{e}_j = [0,\ 0....j = 1,...0,\ 0]$

3. Description of the Problem

A product m of vectors, described in an n-dimensional space by coordinates a_{ij}, can be written in the following way

$$\mathbf{a}_1\mathbf{a}_2 \ .... \ \mathbf{a}_m = \prod_{i=1}^{m} \mathbf{a}_i = \prod_{i=1}^{m}\sum_{j=1}^{n} \mathbf{a}_{ij} = \prod_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\ \mathbf{e}_j \tag{3}$$

The matrix notation below can be an illustration of equation (3).

Let us introduce a vector of the coordinate system $\mathbf{g}_n$, the vector whose projections onto particular axes of the system are the versors of these axes, that is the vector of the classical form, or in matrix notation

$$\begin{aligned} \mathbf{g}_n &= \sum_{j=1}^{n} \mathbf{e}_j = \mathbf{e}_1 + \mathbf{e}_2 + ... + \mathbf{e}_n \\ \mathbf{g}_n^T &= [1,\ 1, ...1] \end{aligned} \tag{4}$$

A vector of the coordinate system can also be written in such a form of the column matrix of a vector of the coordinate system $\hat{\mathbf{g}}_n$ containing versors of the axis of the system that

$$\hat{\mathbf{g}}_n^T = [\mathbf{e}_1, \mathbf{e}_2, \ \ldots . \mathbf{e}_n] \tag{5}$$

Let us build a matrix $\mathbf{A}$, whose terms a_{ij} are coordinates of the vectors multiplied, which can be written as: $\mathbf{A} = [a_{ij}]$, $(i = 1,...n;\ j = 1,...m)$

$$A = \begin{bmatrix} a_{11} & a_{12} & ...... & a_{1n} \\ a_{21} & a_{22} & ...... & a_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{m1} & a_{m2} & ...... & a_{mn} \end{bmatrix} \tag{6}$$

and a matrix $\hat{\mathbf{A}}$, whose terms are vectors of projections of the vectors $\mathbf{a}_i$ onto the axes of the system, written as products of the corresponding coordinates and versors of the axis of the form $a_{ij}\mathbf{e}_j$. This matrix can be written as

$\hat{\mathbf{A}} = [a_{ij}\mathbf{e}_j], \quad (i = 1, ...m; \quad j = 1, ...n)$

$$\hat{\mathbf{A}} = \begin{bmatrix} a_{11}\mathbf{e}_1 & a_{12}\mathbf{e}_2 & \ldots\ldots & a_{1n}\mathbf{e}_n \\ a_{21}\mathbf{e}_1 & a_{22}\mathbf{e}_2 & \ldots\ldots & a_{2n}\mathbf{e}_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}\mathbf{e}_1 & a_{m2}\mathbf{e}_2 & \ldots\ldots & a_{mn}\mathbf{e}_n \end{bmatrix} \tag{7}$$

In the column matrix $\hat{\mathbf{a}}$ the vectors being multiplied $\mathbf{a}_i \quad (i = 1, ...m)$ are contained.

$$\hat{\mathbf{a}}^T = [\mathbf{a}_1, \mathbf{a}_2, \ldots\ldots\mathbf{a}_m] \tag{8}$$

The matrix of the vectors multiplied $\hat{\mathbf{a}}$ (8) is a product of the matrix $\mathbf{A}$ (6) and the matrix of the vector of the coordinate system $\mathbf{g}_n$ (5) or a product of the matrix $\hat{\mathbf{A}}$ (7) and the matrix $\mathbf{g}_n$ (4).

$$\hat{\mathbf{a}} = \mathbf{A}\hat{\mathbf{g}}_n = \hat{\mathbf{A}}\mathbf{g}_n \tag{9}$$

Equation (9) describes the system m of vectors. The product of all the columns of the left side of (9) corresponds to the left side of equation (3), while the product of all the rows of the right side of equation (9) corresponds to the right side of equation (3).

4. Solution

Equation (3) on the left side contains a product of vectors of matrix (8), while on the right side it is a product of all the lines of the matrix of projections of the vectors $\hat{\mathbf{A}}$ (7). Following multiplication and ordering, the right side of equation (3) is transformed into a sum m element products containing one element of each line of matrix (7) each. In other words, each element of each line of matrix (7) is multiplied by each element of each line. In this way, a sum n^m of the products is obtained.

Each of them consists of the product m of the scalar values being various coordinates of successive vectors $i = 1, 2, ...m$, and the product of versors of different axes. The sequence of indices of versors of the axis and, at the same time, successive coordinates in each of these products constitutes a variation with repetitions u containing m elements and formed of natural numbers 1, 2, ...n, being numbers of the axes of the system. Thus, following multiplication and ordering, product (3) can be written as a sum containing n^m products of the coordinates a_{iu} and versors $\mathbf{e}_u$

$$\prod_{i=1}^{m} \mathbf{a}_i = \sum_{k=1}^{n^m} \prod_{i=1}^{m} a_{iu}\mathbf{e}_u \tag{10}$$

where

- $u = u_{ki}, \quad u \in N$ – is the i-th term of the k-th variation with repetitions u_k, composed of m elements, formed on an n–element set of natural numbers 1, 2, ...n

- n^m – is the number of m–element variations with repetitions formed on an n–element set.

Matrix $\mathbf{U}$ of dimensions (n^m, m), containing elements u_{ki} (11) contains in successive lines all the m–element variations possible with repetitions u_k of natural numbers formed on an n–element set. Elements u_{ki} of this matrix are indices of product (10).

$$U = \begin{bmatrix} u_{11} & u_{12} & \ldots\ldots & u_{1m} \\ u_{21} & u_{22} & \ldots\ldots & u_{2m} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ u_{n^m 1} & u_{n^m 2} & \ldots\ldots & u_{n^m m} \end{bmatrix} \tag{11}$$

It can be seen that expression (10) is a sum of products of all the elements of any column of the matrix of projections of vectors $\hat{\mathbf{A}}$ (7) when a complete permutation (rearrangement) of all the elements in all the lines is carried out.

The expansion of the right side of equation (10) leads to a conclusion that it is composed of the sum n^m of products, each of which consists of m^2 elements:

The first m elements are scalar coordinates a_{ij}, making up this product,

The next m elements are the versors of axis $\mathbf{e}_j$, accompanying these coordinates.

Thus, the product (9) can be written in the form

$$\prod_{i=1}^{m} \mathbf{a}_i = \sum_{k=1}^{n^m} \prod_{i=1}^{m} a_{iu} \mathbf{e}_u = \sum_{k=1}^{n^m} \left\{ \prod_{i=1}^{m} a_{iu} \prod_{i=1}^{m} \mathbf{e}_u \right\} \tag{12}$$

The calculation of the product of m scalar coordinates a_{iu} is not a problem, while an attempt to multiply m versors $\mathbf{e}_u$ by one another, belonging to n different axes of the coordinate system, makes us aware of the need to define products of the axis versors. The more so that we have to deal with a number n^m variations with repetitions. These products constitute m–element variations with repetitions, built on an n–element set of versors of the system axes.

5. Products of the axis versors

5.1. *Kinds of products*

A product of two versors of the axis is defined as

$$\mathbf{e}_k \mathbf{e}_m = \begin{cases} 0 & \text{if} \quad k \neq m \\ 1 & \text{if} \quad k = m \end{cases} \tag{13}$$

When calculating of a greater number of versors, we can see that the result depends on two factors (aspects).

The first factor is the number of the versors multiplied.

If we have to do with an even number of versors (e.g. $m = v$), then, as a result of multiplication, we obtain a *scalar value*, equal to 0 or 1. Such a product can be called an *even product of v versors.*

If we multiply an odd number of versors (e.g. $m = d$), then, as a result we obtain a *zero vector* or a *vector equal to one of versors.* Such a product will be called an *odd product of d versors.*

The other factor determining the result of a product is the preference of multiplication assumed, that is the priority of the product in a situation when we have to do with the multiplication of several versors, belonging to different or the same axes. Then, the priority can be given to a product of the same ($\mathbf{e}_l\,\mathbf{e}_l = 1$) or different ($\mathbf{e}_l\mathbf{e}_k = 0$) versors. A product in which the priority is given to the multiplication of homogeneous unit vectors will be called *a product of the first kind or a homo–product of versors* and denoted as:

$f^v_{[e]}$ – if it is an even product

$\mathbf{f}^d_{[e]}$ – if it is an odd product

A product in which the priority is given to the multiplication of heterogeneous unit vectors will be *called a product of the second kind or a hetero–product of versors* and will be denoted as:

$s^v_{[e]}$ – if it is an even product

$\mathbf{s}^d_{[e]}$ – if it is an odd product.

And so, for example:

- even products of four versors can be equal to the number zero or one:

  $f^4[\mathbf{e}_l\mathbf{e}_k\mathbf{e}_l\mathbf{e}_k] = (\mathbf{e}_l\mathbf{e}_l)(\mathbf{e}_k\mathbf{e}_k) = 1 \qquad s^4[\mathbf{e}_l\mathbf{e}_k\mathbf{e}_l\mathbf{e}_k] = (\mathbf{e}_l\mathbf{e}_k)(\mathbf{e}_l\mathbf{e}_k) = 0$

  $f^4[\mathbf{e}_l\mathbf{e}_k\mathbf{e}_l\mathbf{e}_u] = (\mathbf{e}_l\mathbf{e}_l)\mathbf{e}_k\mathbf{e}_u = 0 \qquad s^4[\mathbf{e}_l\mathbf{e}_l\mathbf{e}_l\mathbf{e}_l] = (\mathbf{e}_l\mathbf{e}_l)(\mathbf{e}_l\mathbf{e}_l) = 1$

- odd products of three versors disappear or are versors of the axis:

  $\mathbf{f}^3[e_l\mathbf{e}_k\mathbf{e}_l] = (\mathbf{e}_l\mathbf{e}_l)\mathbf{e}_k = \mathbf{e}_k \qquad \mathbf{s}^3[\mathbf{e}_l\mathbf{e}_k\mathbf{e}_l] = (\mathbf{e}_l\mathbf{e}_k\mathbf{e}_l) = 0$

  $\mathbf{f}^3[\mathbf{e}_l\mathbf{e}_k\mathbf{e}_u] = \mathbf{e}_l\mathbf{e}_k\mathbf{e}_u = 0 \qquad \mathbf{s}^3[\mathbf{e}_l\mathbf{e}_l\mathbf{e}_l] = \mathbf{e}_l\mathbf{e}_l\mathbf{e}_l = \mathbf{e}_l$

In formula (12) there appears a product of m versors, belonging to n different axes of the system. This product can be denoted as δ_k and written in the following form

$$\delta_k = \prod_{i=1}^{m} \mathbf{e}_u = \prod_{i=1}^{m} \mathbf{e}_j^{b_{kj}} \qquad \text{where} \qquad \sum_{j=1}^{n} b_{kj} = m \tag{14}$$

The δ_k thus obtained is a *multi–product of* m *versors,* belonging to n different axes and used for the notation of the product of m vectors.

Let us also note that in any k-th multi–product δ_k any j-th versor $\mathbf{e}_j$ may not occurr, which means that $b_{kj} = 0$. Then, it should be assumed that for this j

$$\mathbf{e}_j^0 = 1 \tag{15}$$

5.2. *Definitions of products*

Below we present the manner of determining a multi-product of m versors. Depending on the values assumed by a sequence b_{kj} of index exponents at successive versors of the axis $\mathbf{e}_j$, the multi–product δ_k (14) belongs to the category of the product of the first kind (homo–product) or the second kind (hetero–product), described above. At the same time, we take into consideration whether the multi–product is an even one ($m = v$) or an odd one ($m = d$).

For each k-th multi–product ($\delta_k = \prod \mathbf{e}_u$) (14), corresponding to k-th variation u_k we build a matrix $\mathbf{E}_k$ of versors forming this product. The matrix $\mathbf{E}_k$ has dimensions (m, n). Successive versors $\mathbf{e}_u = [0,\ 0, ...u = 1, ...0,\ 0]$ making up the k-th product are lines of this matrix.

We also form a column matrix $\mathbf{p}$, containing m elements, whose all elements are numbers 1, hence $\mathbf{p}^T = [1,\ 1, ...1,\ 1]$

Now we multiply the matrix

$$\mathbf{E}_k^T \mathbf{p} = \mathbf{b}_k \tag{16}$$

and as a result we obtain a column matrix $\mathbf{b}_k$, containing n elements, such that

$$\mathbf{b}_k^T = [b_{k1}, b_{k2}, ...b_{kn}] \qquad \text{and} \qquad \sum b_{kj} = m$$

Elements of matrix $\mathbf{b}_k$ are numbers $b_{kj} \in N$, which are index exponents of the multi–product of versors δ_k, defined by formula (14).

Making use of equation (14), the four basic kinds of products of versors described above, were defined:

- an even homo–product $f^v_{[e]}$, i.e. an *even product of* $m = v$ versors of *the first kind,* when $m = v,\ v \in \{4,\ 6,\ 8, ...\}$. It will appear only when all the index exponents b_{kj} are even numbers, while it disappears when at least one of the exponents is an odd number. Then, we will introduce the multi–product of versors δ_k as $\delta^v_f = f^v_{[e]}$.

$$f^v_{[e]} = \prod_{j=1}^{n} \mathbf{e}_j^{b_j} \stackrel{f}{=} \begin{cases} 1 & \text{if} \ \underset{j}{\forall}\ b_j \in \{0,\ 2,\ 4, ...\ v\} \\ \\ 0 & \text{if} \ \underset{j}{\exists}\ b_j \in \{1,\ 3,\ 5, ...\ v - 1\} \end{cases} \tag{17}$$

- an even hetero–product $s^v_{[e]}$, i.e. an *even product of* $m = v$ *versors of the second kind*, when $m = v,\ v \in \{4,\ 6,\ 8, ...\}$. It will appear only when one of the index exponents b_{kj} is equal to an even number of multiplied versors v, while the remaining ones are equal to zero. Then, we will introduce $\delta^v_s = s^v_{[e]}$

$$s^v_{[e]} = \prod_{j=1}^{n} \mathbf{e}_j^{b_j} \stackrel{s}{=} \begin{cases} 1 & \text{if} \ \underset{j}{\exists}\ b_j = v \\ \\ 0 & \text{if} \ \underset{j}{\forall}\ b_j < v \end{cases} \tag{18}$$

- a homo–product $\mathbf{f}^d_{[e]}$, i.e. an *odd product of* $m = d$ *versors of the first kind*, when $m = d, d \in \{3,\ 5,\ 7, ...\}$. It will appear when one of the exponents is an odd number, while the remaining index exponents b_{kj} are even numbers. It disappears when more than one of the exponents is an odd number. Then

$\delta_f^d = \mathbf{f}_{[e]}^d$

$$\mathbf{f}_{[e]}^d = \prod_{j=1}^{n} \mathbf{e}_j^{b_j} \stackrel{f}{=} \begin{cases} \mathbf{e}_k \text{ if} \begin{cases} \underset{k\in\{1,\,2,\ldots n\}}{\exists} b_j \in \{1,\,3,\,5,\ldots d\} \\ \text{and} \\ \underset{j\neq k}{\forall} b_j \in \{0,\,2,\,4,\ldots d-1\} \end{cases} \\ \\ 0 \quad \text{if} \begin{cases} \underset{k\in\{1,\,2,\ldots n\}}{\exists} b_j \in \{1,\,3,\,5,\ldots d\} \\ \text{and} \\ b_l \in \{1,\,3,\,5,\ldots d\} \end{cases} \end{cases} \tag{19}$$

- an odd hetero–product $\mathbf{s}_{[e]}^d$, i.e. an *odd product of* $m = d$ *versors of the second kind,* when $m = d$, $d \in \{3,\,5,\,7,\ldots\}$. It will appear only when one of the index exponents b_{kj} is equal to an odd number of multiplied versors d, while the remaining ones are equal to zero. Then $\delta_s^d = \mathbf{s}_{[e]}^d$

$$\mathbf{s}_{[e]}^d = \prod_{j=1}^{n} \mathbf{e}_j^{b_j} \stackrel{s}{=} \begin{cases} \mathbf{e}_k \quad \text{if} \quad \underset{k\in\{1,2,\ldots n\}}{\exists} b_k = d \\ \\ 0 \quad \text{if} \quad \underset{j}{\forall} b_j < d \end{cases} \tag{20}$$

The analysis of formulae (17–20) leads to a conclusion that a product of the second kind is contained in a product of the first kind and this dependence is binding for both even and odd products. Thus, we can introduce a third kind of product of m versors, which is the difference of the homo–product and hetero–product and denote it as

$$t^v = f^v - s^v \quad \text{for even products} \tag{21}$$

$$\mathbf{t}^d = \mathbf{f}^d - \mathbf{s}^d \quad \text{for odd products} \tag{22}$$

The, the two derivative products are equal to

- *an even product of the third kind* $t_{[e]}^v$, $v \in \{4,\,6,\,8,\,\ldots\}$. It will appear when all the index exponents b_{kj} are even numbers, but smaller than v. It disappears when at least one of the exponents is an odd number or equal to v. Then $\delta_t^v = t_{[e]}^v$

$$t_{[e]}^v = \prod_{j=1}^{n} \mathbf{e}_j^{b_j} \stackrel{t}{=} \begin{cases} 1 \quad \text{if} \quad \underset{j}{\forall} b_j \in \{0,\,2,\,4,\ldots v-2\} \\ \\ 0 \quad \text{if} \begin{cases} \underset{j}{\exists} b_j \in \{1,\,3,\,5,\ldots v-1\} \\ \text{or} \\ \underset{j}{\exists} b_j = v \end{cases} \end{cases} \tag{23}$$

- *an odd product of the third kind* $\mathbf{t}_{[e]}^d, d \in \{3,\,5,\,7,\ldots\}$. It will appear when one of the exponents is an odd number, but smaller than d, while all the remaining index exponents b_{kj} are even numbers. It disappears when more than one of the exponents is an odd number or when one of the exponents is equal to d. Then $\delta_t^d = \mathbf{t}^{\mathbf{d}}_{[\mathbf{e}]}$

$$\mathbf{t}^d_{[e]} = \prod_{j=1}^{n} \mathbf{e}_j^{b_j} \overset{t}{=} \begin{cases} \mathbf{e}_k \quad \text{if} \begin{cases} \underset{k\in\{1,2,\ldots n\}}{\exists} \quad b_k \in \{1,\,3,\,5,\ldots d-2\} \\ \text{and} \\ \underset{j\neq k}{\forall}\ b_j \in \{0,\,2,\,4,\ldots d-1\} \end{cases} \\ \\ 0 \quad \text{if} \begin{cases} \underset{k,l\in\{1,2,\ldots n\}}{\exists} \quad b_k,\, b_l \in \{1,\,3,\,5,\ldots d\} \\ \text{or} \\ \underset{j}{\exists}\, b_j = d \end{cases} \end{cases} \tag{24}$$

6. Multi–products of vectors

Having introduced the multi–product of versors of the axis δ_k in the form of equation (14) defined the four basic (17-20) and two derivative (23-24) multi–products of versors of the axis, we can return to the product of m vectors.

Having taken into consideration (14) in (12) we obtain

$$\prod_{i=1}^{m} \mathbf{a}_i = \sum_{k=1}^{n^m} \prod_{i=1}^{m} a_{iu}\mathbf{e}_u = \sum_{k=1}^{n^m} \prod_{i=1}^{m} a_{iu}\,\delta_k \tag{25}$$

Expression (14) is a multi–product of versors δ_k, formed for k-th variation with repetitions u_k, described in matrix (11). As mentioned above, the kind and value of the product of m vectors depends on their number and the preference of multiplication of the axis versors. Thus, the selection of the kind of the multi–product of versors δ_k determines the kind of the multi–product of vectors. Thus, using formula (25) and substituting the multi–products of versors defined earlier, we obtain corresponding multi–products of vectors. These are, as for the versors, the following products:

- the even multi–product of the first kind of v vectors – f^v
- the even multi–product of the second kind of v vectors – s^v
- the odd multi–product of the first kind of d vectors – $\mathbf{f}^d$
- the odd multi–product of the second kind of d vectors – $\mathbf{s}^d$

$$\prod_{i=1}^{m} \mathbf{a}_i = \begin{cases} \overset{f}{=} f^v & \text{if} \quad \text{m} = v \wedge \delta_k = \delta_f^v \quad \text{look at (17)} \\ \overset{s}{=} s^v & \text{if} \quad \text{m} = v \wedge \delta_k = \delta_s^v \quad \text{look at (18)} \\ \overset{f}{=} \mathbf{f}^d & \text{if} \quad \text{m} = d \wedge \delta_k = \delta_f^d \quad \text{look at (19)} \\ \overset{s}{=} \mathbf{s}^d & \text{if} \quad \text{m} = d \wedge \delta_k = \delta_s^d \quad \text{look at (20)} \end{cases} \tag{26}$$

and their corresponding differences:

- the even product of the third kind of v vectors - t^v

- the odd products of the third kind of d vectors - $\mathbf{t}^d$

$$\prod_{i=1}^{m} \mathbf{a}_i = \begin{cases} \stackrel{t}{=} t^v & \text{if} \quad \text{m} = v \wedge \delta_k = \delta_t^v \quad \text{look at (23)} \\ \stackrel{t}{=} \mathbf{t}^d & \text{if} \quad \text{m} = d \wedge \delta_k = \delta_t^d \quad \text{look at (24)} \end{cases} \tag{27}$$

7. Conclusions

We return here to the question put in the beginning – what will the multiplication of more than two vectors by one another result in? Is it possible – by multiplying m vectors by one another – obtain a result whose value will be independent of the order in which they are multiplied? It results from the reasoning presented above that it is possible.

In the paper, kinds of products of versors and vectors have been defined, calculated in the orthogonal system of axes, determining an n-dimensional space.

These products – of the first kind, the second kind, and their difference , products of the third kind, both even and odd ones, are – for definite m vectors – constant and their value does not depend on the order in which they are arranged in the matrices $\mathbf{A}$ (6), $\hat{\mathbf{A}}$ (7) and $\hat{\mathbf{a}}$ (8).

References

[1] **Polka, A.**: Multi–Products of Unit Vectors and Vectors. Basic Notions, *Mechanics and Mechanical Engeeniering*, V. 12, 2, **2008**

Nomenclature

$\mathbf{a}_{ij}$	the projection of an i-th vector onto a j-th axis of the system of coordinates
$\mathbf{g}_n$	the vector of the system of coordinates
δ_k	the multi-product of the axis versors
f^v	the first kind of product of even ($m = v$) vectors
$\mathbf{f}^d$	the first kind of product of odd ($m = d$) vectors
s^v	the second kind of product of even (v) vectors
$\mathbf{s}^d$	the second kind of product of odd (d) vectors
t^v	the third kind of product of even vectors
$\mathbf{t}^d$	the third kind of product of odd vectors