Mechanics and Mechanical Engineering
Vol. 12, No. 3 (2008) 211–222

Energy Problems of Self–induced Vibrations in Mechanical Systems

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Received (25 October 2008)
Revised (30 October 2008)
Accepted (15 December 2008)

In the paper the author's energy method was used to examine the cooperation of a self–induced system with a linear and non–linear system. The conditions when these can occur were given. Apart from the quantitative research the qualitative research was presented demonstrating the influence of damping and elasticity on amplitude and vibration frequency.

Keywords: Kinematics, dynamics, self–induced systems

1. Introduction

A self–induced vibratory system is an appliance capable of producing non–decreasing vibrations, the characteristic of which are: a source of energy, a valve regulating the energy access to the system, the vibratory system and the valve feedback.

A distinctive feature of the systems is the way of drawing the energy. It allows to distinguish autonomic self–induced systems from non–autonomic systems, where the energy consumption takes place due to external forces openly dependent on time.

In the self–induced system in vibrations dynamic equation, time does not occur in an open way. The source of energy is constant, not dependent on time and the energy supply is regulated by the vibratory system itself (that is why such systems are called self–induced).

Taking into consideration a self-induced system of two degrees of freedom you can single out the following elements, as presented in the block diagram (Fig. 1): element I – source of energy, element II – vibratory system connected with feedback through IV with a regulator II, which controls the energy flow from the source to the vibratory system. Elements I and III can be linear, element II must be non–linear.

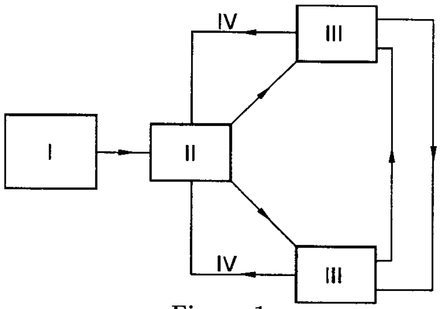

Figure 1

Technology knows many self–induced systems. They are, among others, vibrations of diggers, cutting tools, turbine paddles, airfoils or vibrations of suspension bridges, which may lead to the break of the bridge e.g. Tracon Bridge etc. In such cases steps are taken to eliminate the vibrations. In electrotechnology self–induced systems are used as amplifiers.

The work discusses mechanical systems with frictional contacts in movable joints with particular attention devoted to vibrations forced kinematically, as self-induced vibrations are possible to occur.

The occurrence of the vibrations is conditioned by the form of the frictional contact characteristic. Therefore the selection of the suitable characteristic is essential in the synthesis of mechanical systems with frictional contact. Vibrations occurring along with some characteristics of frictional contacts are self–induced vibrations which, in certain conditions, may cause faster wear, damage or destruction of the self–induced system.

Let us consider the impact of the self–induced system on the linear system and next, on the non–linear system. They are systems of two degrees of freedom. On the basis of the analysis the conditions in which the self-induced vibrations occur will be settled and the influence of damping on vibration amplitude will be discussed. The developed energy method will be used to examine the course of solutions qualitatively and the quantitative asymptotic Krylov–Bogolubov method will be employed to determine the amplitude and the vibration frequency.

2. Power engineering of self–induced vibrations

One of the most general points of view that can be taken in considerations of any physical phenomenon is the *power engineering point of view.* We are, above all, interested in an *energy balance* of a given phenomenon.

The condition necessary for the *fixed vibrations* (i.e. *non–decreasing* and of *non–increasing amplitude*) to occur is the following condition of energy balance equilibrium: the inflow of energy from the source in a period (or a time unit) is exactly equal to the loss of energy during the same period of time.

If:

- loss supplement is insufficient, the vibrations will fade
- energy inflow from the source is excessive, the vibrations will increase

The self–induced mechanism is as follows:

If in the initial phase of the vibratory movement, i.e. along with the lowest amplitudes, the energy inflow is bigger than the loss, the vibration amplitude will increase. If this energy inflow–to–energy loss ratio continued, the amplitude would increase unlimitedly.

In order to obtain a fixed movement, starting from a given amplitude value, the energy loss is required to increase faster than the inflow from the source. In that case the energy balance equilibrium is possible to attain.

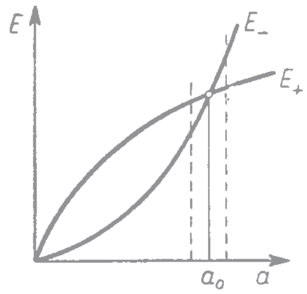

Figure 2

Fig. 2 is a pictorial presentation of this phenomenon.E_+ denotes energy received from the source, E_- denotes energy lost by the vibratory system; they are presented depending on amplitude a. The point of the curves intersection equals to the energy balance equilibrium; the ordinate of this point corresponds to the value of the fixed amplitude a_o.

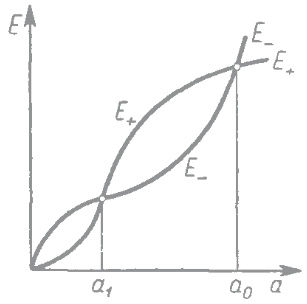

Figure 3

The E_+ and E_- curves can intersect in more points. All the points of stable equilibrium, with the exception of the origin of coordinates, indicate possible stable vibration forms of the system, in which it can generate non-decreasing vibrations.

Considerations regarding E_+ and E_- curves lead to the conclusion that the system must be non-linear to enable their intersection. Therefore, every actual self–induced vibratory system capable of generating vibrations of fixed amplitude must comprise a non–linear element. It should be emphasized here that non–linearity can characterize the vibratory system or load as well as the valve mechanism or the feedback circuit mechanism.

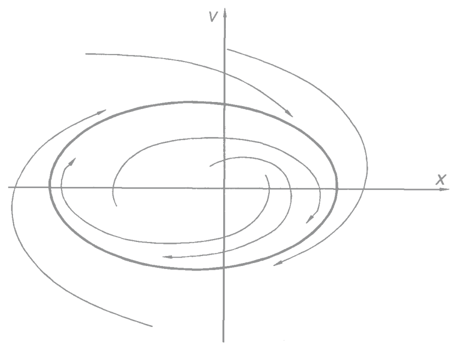

Figure 4

On a phase plane, closed phase trajectory corresponds to fixed vibrations of self–induced system. With the time increase, all trajectories in its certain surrounding head towards it. The above closed trajectory is called a *stable limit cycle* Fig. 4.

The limit cycle must go through the field points in which the system energy increases and through the field points in which the energy decreases, or it must lie on the border of the fields mentioned above.

The self–induced system can also possess a few limit cycles. Such a case is possible to occur when the system energy increases inside a certain ring encircling asymptotically a singular point ($x = v$=0), while on the remaining part of the phase plane it decreases. The singular point is a non–stable balance point, thus periodical self–induced vibrations will occur after any little initial disturbance. Such a self-induced system is called a *soft induction system.*

If the vibrations occur only after sufficiently big initial disturbance to which corresponds phase point position C_1, this kind of the self–inducing system is called a *hard induction system* Fig. 5.

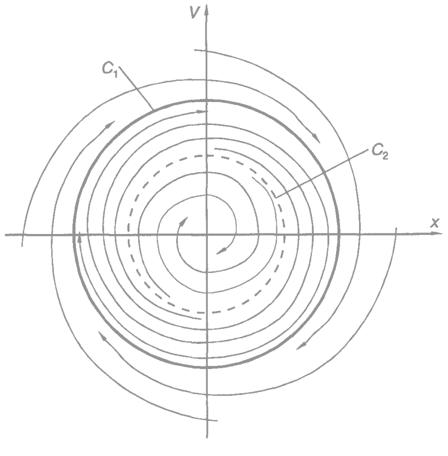

Figure 5

3. Energy analysis of self–induction system

The impact of the self–induced system on the linear or non–linear mechanical system was illustrated by a physical model Fig, 6, where the following denotations were used:

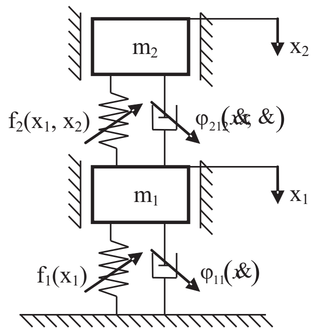

Figure 6
$f_1(x_1)$, $f_2(x_1, x_2)$ – elasticity characteristics
$\varphi_1(\dot{x}_1)$, $\varphi_2(\dot{x}_1, \dot{x}_2)$ – attenuation characteristics

For the physical model Fig. 6, with continuous elasticity and attenuation characteristics non–linear equations were obtained (1). Depending on particular forms of these characteristics the equations can have periodical solutions.

$$\left.\begin{array}{l} m_1\ddot{x}_1 + f_1(x_1) + f_2(x_1, x_2) + \varphi_1(\dot{x}_1) + \varphi_2(\dot{x}_1, \dot{x}_2) = 0 \\ m_2\ddot{x}_2 - f_1(x_1) - f_2(x_1, x_2) - \varphi_2(\dot{x}_1, \dot{x}_2) = 0 \end{array}\right\} \tag{1}$$

The obtained system of equations (1) will describe decreasing vibrations, if it is a positive dissipation system and the elasticity characteristics will be positive. The conditions will be determined by the inequalities (2) and (3).

$$\frac{\partial H_1}{\partial x_1} > 0 \qquad \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{vmatrix} \geq 0 \tag{2}$$

$$\Phi_1\dot{x}_1 + \Phi_2\dot{x}_2 \geq 0 \tag{3}$$

The inequalities (2) indicate that elasticity characteristics derivatives are positive, and the inequality (3) shows that the system possesses positive dissipation.

4. Cooperation of self–induction system with linear system

The interaction of self–induction system was illustrated by the model in Fig. 3, where the denotations were introduced (4).

$$\left.\begin{array}{l} f_1(x_1) = k_1 x_1 \\ f_2(x_1, x_2) = k_2(x_1 - x_2) \\ \varphi_1(\dot{x}_1) = -\alpha\,\dot{x}_1 + \beta\,\dot{x}_1^3 \\ \varphi_2(\dot{x}_1, \dot{x}_2) = l(\dot{x}_1 - \dot{x}_2) \end{array}\right\} \tag{4}$$

For the model taken (Fig. 4), with continuous characteristic of a non–linear silencer $\varphi_1(\dot{x}_1) = -\alpha\,\dot{x}_1 + \beta\,\dot{x}_1^3$, where $\alpha > 0$ and $\beta > 0$, the non–linear equations were obtained (5) which depending on the springs and silencers characteristics can have periodical solutions. Such a periodical solution is the limit regime (limit cycle) of self–induction vibrations.

$$\begin{aligned} &m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) - \alpha\,\dot{x}_1 + \beta\,\dot{x}_1^3 + l(\dot{x}_1 - \dot{x}_2) = 0 \\ &m_2\ddot{x}_2 + k_2(x_1 - x_2) - l(\dot{x}_1 - \dot{x}_2) = 0 \end{aligned} \tag{5}$$

The obtained system of equations (5) will describe decreasing vibrations if it is a dissipation system and the coefficients of the system elasticity are positive. The conditions are described by the inequalities (2) and (3). In the case considered the expressions of $H_1,\ H_2,\ \Phi_1,\ \Phi_2$ are equal (6).

$$\left.\begin{aligned} &H_1 = k_1x_1 + k_2(x_1 - x_2) \\ &H_2 = -k_2(x_1 - x_2) \\ &\Phi_1 = -\alpha\,\dot{x}_1 + \beta\,\dot{x}_1^3 + l(\dot{x}_1 - \dot{x}_2) \\ &\Phi_2 = -l(\dot{x}_1 - \dot{x}_2) \end{aligned}\right\} \tag{6}$$

After the substitution (6) in (2) and (3) and the execution of proper operations the inequalities were obtained:

$$\left.\begin{aligned} &k_1 + k_2 > 0 \\ &k_1 \cdot k_2 \geq 0 \end{aligned}\right\} \tag{7}$$

$$\left[(l - \alpha) + \beta\dot{x}_1^2\right]\dot{x}_1^2 - 2l\dot{x}_1\dot{x}_2 + l\dot{x}_2 \geq 0 \tag{8}$$

The inequalities (7) are always fulfilled, while the inequality (8) will proceed if

$$4l^2\dot{x}_1^2 - 4l\left[(l - \alpha) + \beta\dot{x}_1^2\right]\dot{x}_1^2 < 0 \tag{9}$$

which means with

$$\dot{x}_1^2 > \frac{\alpha}{\beta} \tag{10}$$

From the energy analysis, it appears that during the cooperation of the self–induction system with the linear system the self–induced vibrations of the system may occur if the inequality is not fulfilled (8). The balance position is the non–stable position.

5. Cooperation of self–induction system with no–linear system

The impact of the self–induction system on the non–linear mechanical system was illustrated by the model in fig.2.2 but the elasticity and attenuation characteristics were taken as non–linear (11).

$$\left.\begin{aligned} &f_1(x_1) = k_1x_1 + \gamma_1x_1^3 \\ &f_2(x_1, x_2) = k_2(x_1 - x_2) + \gamma_2(x_1 - x_2)^3 \\ &\varphi_1(\dot{x}_1) = -\alpha\,\dot{x}_1 + \beta\,\dot{x}_1^3 \\ &\varphi_2(\dot{x}_1, \dot{x}_2) = l(\dot{x}_1 - \dot{x}_2) + \mu(\dot{x}_1 - \dot{x}_2)^3 \end{aligned}\right\} \tag{11}$$

With the assumptions taken (11), the system of equations (1) for the model in Fig. 3 takes the following form:

$$\begin{aligned}&m_1\ddot{x}_1 + k_1x_1 + \gamma_1x_1^3 + k_2\left(x_1 - x_2\right) + \gamma_2\left(x_1 - x_2\right)^3 - \alpha\,\dot{x}_1 + \beta\,\dot{x}_1^3\\ &+l\left(\dot{x}_1 - \dot{x}_2\right) + \mu\left(\dot{x}_1 - \dot{x}_2\right)^3 = 0\\ &m_2\ddot{x}_2 - k_2\left(x_1 - x_2\right) - \gamma_2\left(x_1 - x_2\right)^3 - l\left(\dot{x}_1 - \dot{x}_2\right) - \mu\left(\dot{x}_1 - \dot{x}_2\right)^3 = 0\end{aligned} \tag{12}$$

In the equations (12) the functions of elasticity and attenuation take the following form:

$$\left.\begin{aligned}&H_1 = k_1x_1 + \gamma_1x_1^3 + k_2\left(x_1 - x_2\right) + \gamma_2\left(x_1 - x_2\right)^3\\ &H_2 = -k_2\left(x_1 - x_2\right) - \gamma_2\left(x_1 - x_2\right)^3\\ &\Phi_1 = -\alpha\,\dot{x}_1 + \beta\,\dot{x}_1^3 + l\left(\dot{x}_1 - \dot{x}_2\right) + \mu\left(\dot{x}_1 - \dot{x}_2\right)^3\\ &\Phi_2 = -l\left(\dot{x}_1 - \dot{x}_2\right) - \mu\left(\dot{x}_1 - \dot{x}_2\right)^3\end{aligned}\right\} \tag{13}$$

After substitution (13) into (2) and (3) and the execution of certain operations the following was obtained:

$$k_1 + k_2 + 3\gamma_1x_1^2 + 3\gamma_2\left(x_1 - x_2\right)^2 > 0 \tag{14}$$

$$\left(k_1 + 3\gamma_1x_1^2\right)\left[k_2 + 3\gamma_2\left(x_1 - x_2\right)^2\right] \geq 0 \tag{15}$$

$$-\alpha\,\dot{x}_1^2 + \beta\,\dot{x}_1^4 + l\left(\dot{x}_1 - \dot{x}_2\right)^2 + \mu\left(\dot{x}_1 - \dot{x}_2\right)^4 \geq 0 \tag{16}$$

After substitution $u = x_1,\ v = x_1 - x_2$ the inequalities (14) and (15) can be written in the form of (17) and (18)

$$-3\gamma_1u^2 - 3\gamma_2v^2 < k_1 + k_2 \tag{17}$$

$$\left(k_1 + 3\gamma_1u^2\right)\left(k_2 + 3\gamma_2v^2\right) \geq 0 \tag{18}$$

Depending on the sign $\gamma_1,\ \gamma_2$ the following cases can be considered, for which the inequalities (17) and (18) will be fulfilled.

Case 1: $\gamma_1 > 0$ *i* $\gamma_2 > 0$. The inequalities (17) and (18) are fulfilled on the whole plane u, v.

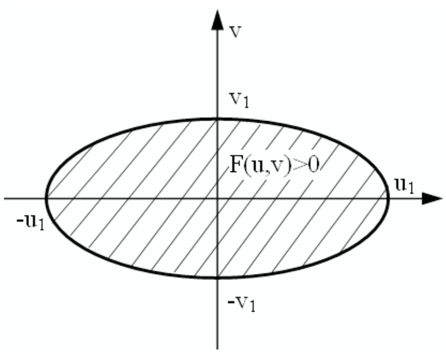

Figure 7

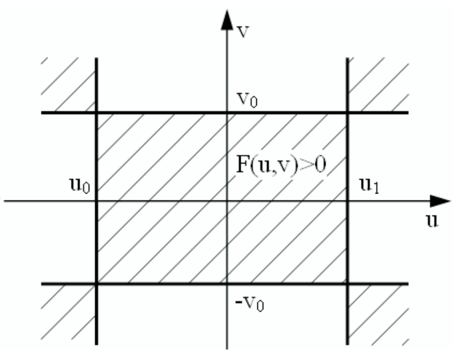

Figure 8

Case 2: $\gamma_1 < 0$ *i* $\gamma_2 < 0$. The inequality (17) is fulfilled in the field lined in Fig. 7, the inequality (18) in the field lined in Fig. 8, and the quantities marked on axes u, v amount to

$$u_o = \sqrt{\frac{k_1}{-3\gamma_1}}, \qquad v_o = \sqrt{\frac{k_2}{-3\gamma_2}}$$
$$u_1 = \sqrt{\frac{k_1+k_2}{-3\gamma_1}}, \qquad v_1 = \sqrt{\frac{k_1+k_2}{-3\gamma_2}}$$

Function $F(u, v) = k_1 + k_2 + 3\gamma_1 u^2 + 3\gamma_2 v^2$

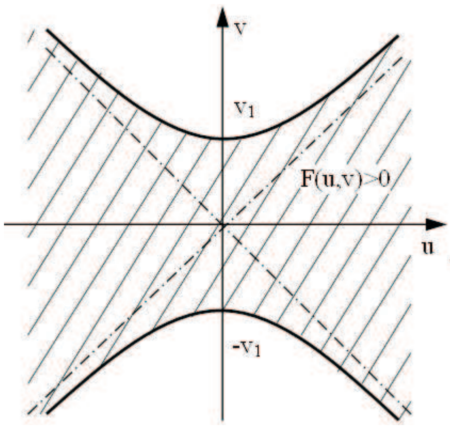

Figure 9

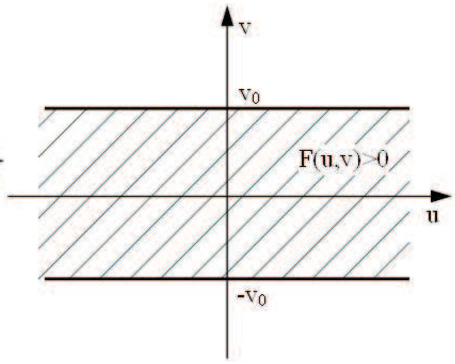

Figure 10

Case 3: $\gamma_1 < 0$ *a* $\gamma_2 > 0$. The inequality (17) is fulfilled in the field lined in Fig.9 and the inequality (18) in the field lined in Fig. 10.

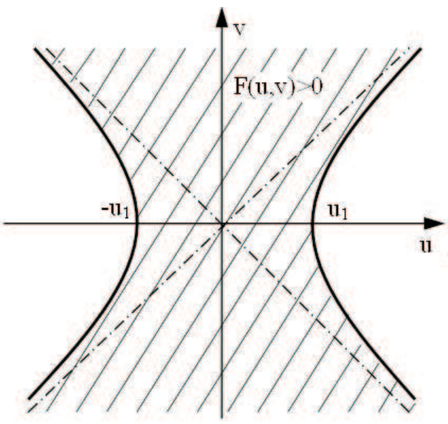

Figure 11

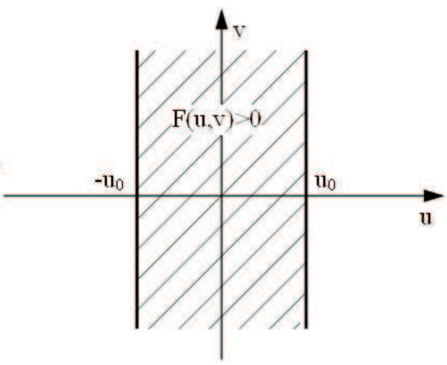

Figure 12

Case 4: $\gamma_1 > 0$ *a* $\gamma_2 < 0$. The inequality (17) is fulfilled in the field lined in Fig.11 and the inequality (18) in the field lined in Fig. 12.

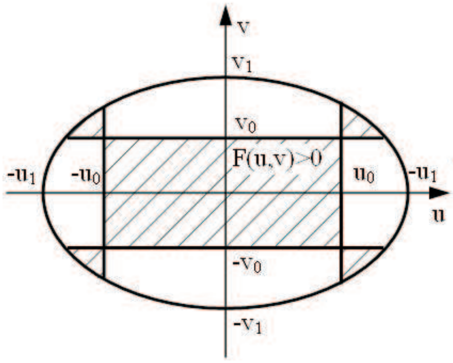

Figure 13

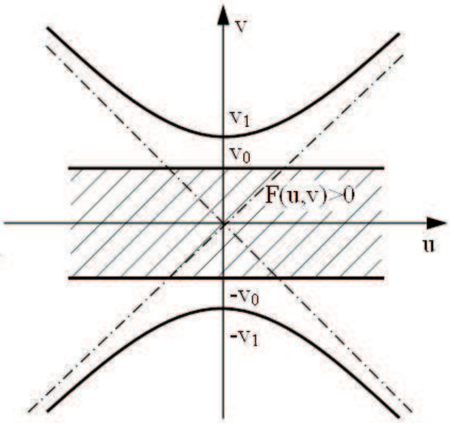

Figure 14

On the basis of the carried out analysis it is possible to find the field where the inequalities (17) and (18) are fulfilled simultaneously. Fading vibrations occur in this field if the inequality (16) is fulfilled at the same time.

For case 1, the field in which the inequalities (17) and (18) are fulfilled simultaneously remains the same.

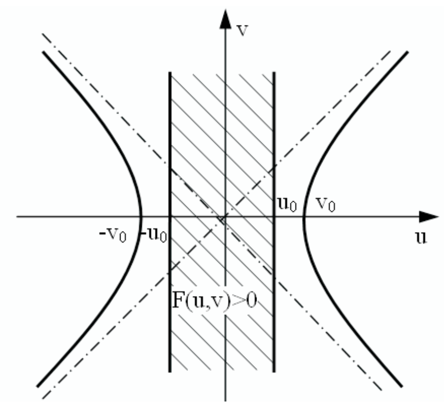

Figure 15

For case 2, the field in which the inequalities (17) and (18) are fulfilled simultaneously is the field lined in Fig. 13.

For case 3, it is the field lined in Fig. 14, and for case 4 the field lined in Fig. 15.

The inequality (16) after substitution $u^* = \dot{x}_1^2$, $v^* = (\dot{x}_1 - \dot{x}_2)^2$ will take the following form

$$-\alpha u^* + \beta u^{*2} + lv^* + \mu v^{*2} \geq 0 \tag{19}$$

After the expression is brought to a canonical form

$$F\left(u^{*},\, v^{*}\right) = -\alpha u^{*} + \beta u^{*2} + l v^{*} + \mu v^{*2}$$

It will take the following form

$$\frac{\beta z^{2}}{R} + \frac{\mu w^{2}}{R} \geq 1 \tag{20}$$

Where: $z = u^{*} - \frac{\alpha}{2\beta}, \quad w = v^{*} + \frac{l^{2}}{2\mu}, \quad R = \frac{\alpha^{2}}{4\beta} + \frac{l^{2}}{4\mu}$

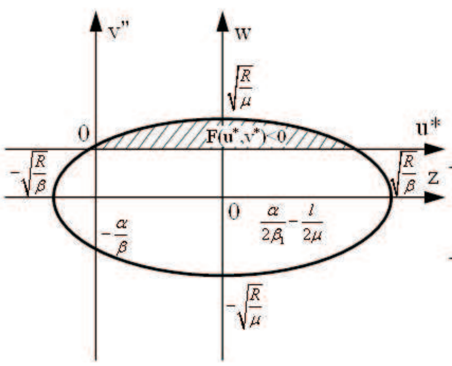

Figure 16

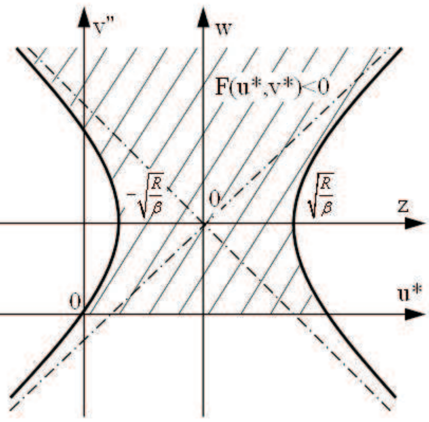

Figure 17

The inequality (19) is not fulfilled for $\mu > 0$ in the field lined in fig.5.10, for $\mu < 0$ and $R > 0$ in Fig. 17, and for $\mu > 0$ and $R < 0$ in Fig. 18.

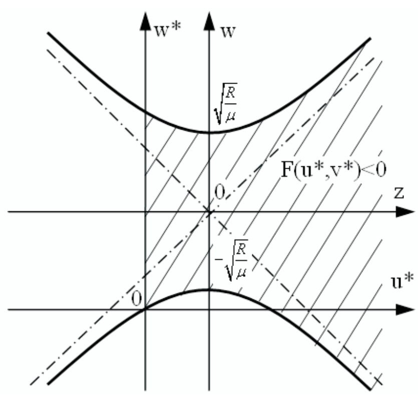

Figure 18

The energy analysis results show that if the inequality (19) is not fulfilled it is possible for the self–induced vibrations to occur depending on the sign μ and R in the field lined in Fig. 16, Fig. 17, Fig. 18.

If the inequalities (17) and (18) are not fulfilled, then, depending on the signγ_1 i γ_2, the increasing vibrations occur in the filed lined in fig.5.7, fig.5.8 and fig.5.9 even though the condition (19) of positive energy dissipation will be fulfilled.

In the case when the inequalities (17) and (18) are fulfilled simultaneously in the field lined in Fig. 13, Fig. 14 and Fig. 15, the fading vibrations will occur with the inequality (19) fulfilled at the same time.

6. Qualitative research

The qualitative research presented here can be employed in mechanical systems design and operation and provides hints for the mechanical characteristics selection to avoid self–induced vibrations dangerous for machines or appliances.

In practice it is also interesting what the amplitude and the limit regime vibration frequency are dependent on and how they change. To determine the amplitude and the frequency the asymptotic Krylov–Bogolubov method was used. It was assumed that attenuation and non–linearity of the elasticity forces are small. The calculations were limited to the first approximation. The stationary value of the amplitude and the frequency obtained from the equations (5) amounts to:

$$\left.\begin{aligned} \left(A_1^{(1)}\right)^2 &= \frac{4}{3\omega_1^2}\left[\frac{\alpha-l(1-\lambda)^2}{\beta}\right] \\ \frac{d\Psi}{dt} &= \omega_1 \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} \left(A_1^{(1)}\right)^2 &= \frac{4}{3\omega_1^2}\left[\frac{\alpha-l(1-\lambda)^2}{\beta+\mu(1-\lambda)^4}\right] \\ \frac{d\Psi}{dt} &= \omega_1 + \frac{3\left(A_1^{(1)}\right)^2}{8\omega_1(m_1+m_2\lambda^2)}\left[\gamma_1+\gamma_2\left(1-\lambda\right)^4\right] \end{aligned}\right\} \tag{22}$$

where:

$$\left.\begin{aligned} \omega_1^2 &= \frac{1}{2}\left[\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)-\sqrt{\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-4\frac{k_1k_2}{m_1m_2}}\right] \\ \lambda &= \frac{A_2^{(1)}}{A_1^{(1)}} = \frac{k_2}{k_2-\omega_1^2m_2} = \frac{(k_1+k_2)-\omega_1^2m_1}{k_2} \end{aligned}\right\} \tag{23}$$

On the basis of the formulas (21) and (22) it can be concluded that in the first approximation the amplitude is influenced, above all, by attenuation while the vibration frequency by the elasticity forces.

The amplitude of the limit regime vibrations depends on the characteristic of the positive and negative silencer. The amplitude can be decreased through the application of the positive silencer of hard characteristic and through the negative silencer non-linearity increase.

The application of the positive silencer of soft characteristic causes the increase of the vibration amplitude. The attenuation coefficient l and μ represents in practice the attenuation material coefficient applied e.g. through a layer of rubber or cork between masses m_l and m_2.

If the attenuation material is selected in such a way that the counter in the formula (22) is small, the received vibrations amplitude will be very little. The conclusion is that the use of attenuation materials cannot be arbitrary. The frequency of the system vibrations, as it can be concluded from (23), depends above all on the non–linearity of the elasticity forces. The bigger the non–linearity, the smaller the vibrations frequency. It can be easily claimed that with the soft characteristics of the springs the frequency of the system vibrations is smaller than with the hard characteristics.

7. Conclusions

The work presented the energy method to research the mechanical systems self–induced vibrations with one and many degrees of freedom as well as the conditions needed for them to occur. In conclusion, the above mentioned dependences should be applied in mechanical systems design and operation to avoid self–induced vibrations which are dangerous for constructions. An inappropriate selection of the construction and operation parameters may result in a failure and in consequence lead to considerable financial outlays and stoppages. The qualitative research provides suggestions for the selection of optimal characteristics selection in designing such systems.

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The work is a part of the research project No. 4 T07A 026 30