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Optimum Bi–Impulsive Non Coplanar Elliptic Hohmann Type Transfer

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We optimize the Hohmann type bi–impulsive transfer between inclined elliptic orbits having a common center of attraction, for the four feasible configurations. Our criterion for optimization is the characteristic velocity $\Delta v_T = \Delta v_1 + \Delta v_2$ which is a measure of fuel consumption. We assigned the optimum value of our variable x (ratio between velocity after initial impulse and velocity before initial impulse) by a numerical solution of an algebraic eight degree equation. We have a single plane change angle α. We present terse new formulae constituting a new alternative approach for tackling the problem. The derivations of formulae of our treatment are simple, straightforward and exceptionally clear. This is advantageous. By this semi–analytic analysis we avoid many complexities and ambiguity that appear in previous work.

Keywords: Rocket dynamics, transfer orbits, Hohmann transfer, optimization, astrodynamics, aerospace engineering, orbital mechanics

1. Introduction

We assume that the bi–impulsive thrusts concerned with the elliptic Hohmann transfer are provided by a space craft propulsion system. The total characteristic velocity for the transfer maneuver is given by $\Delta v_T = \Delta v_1 + \Delta v_2$. Orbital transfer is utilized in the majority of space crafts placed in orbit around the Earth. Orbit transfer is implemented to acquire a final orbit via a parking one. Periodic corrections should be executed due to perturbations acting on space crafts [1]. The velocity increments at peri–apse and apo–apse are directly proportional to fuel expenditure. Lawden investigated the optimization of a rocket vehicle transfer. He assumed the orbits to be elliptic and coplanar (The Lawden problem) [2]. Gravier et al derived the necessary equations for optimal transfer, in the real case when we take into account the ellipticity and inclination of the orbits [3]. J. Baker discussed the problem of circular inclined orbits transfer for three cases [4]. L. Rider extended the impulsive thrusts transfer problem to non–coplanar orbits [5], J. Prussing analysed the optimized solution for the coplanar and restricted class of non–coplanar transfer problem [6]. We shall take into consideration the coplanar and non–coplanar Bi–elliptic transfer problem, for the four assumed configurations, strictly after the publication of this work [7].

2. Preliminary Concepts

2.1. *Determination of the Feasible Configurations*

We have the following four feasible configurations for the elliptic Hohmann type transfer

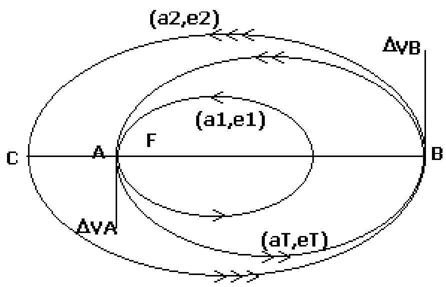

Figure 1

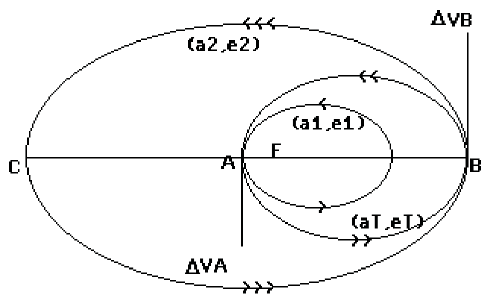

Figure 2

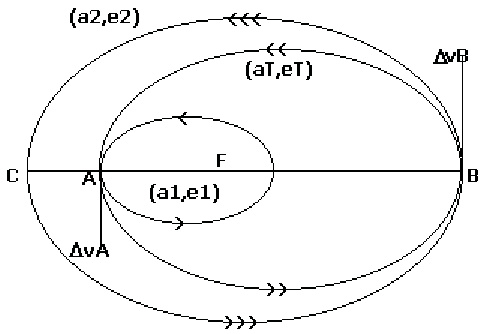

Figure 3

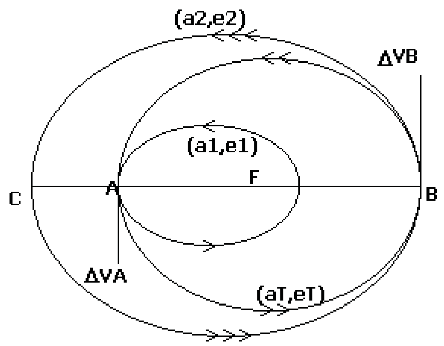

Figure 4

1. For Fig. 1, apo – apse of transfer orbit coincides with apo – apse of final orbit (L).

2. For Fig. 2, apo – apse of transfer orbit coincides with peri – apse of final orbit (L).

3. For Fig. 3, apo – apse of transfer orbit coincides with apo – apse of final orbit (R).

4. For Fig. 4, apo – apse of transfer orbit coincides with peri – apse of final orbit (R).

where (L) means that primary is at left focus and (R) means that primary at right focus.

The most convenient way is to regard the problem of orbit transfer as a problem in change of energy [8].

For Fig. 1:

$$a_1(1-e_1) = a_T(1-e_T) \qquad a_T(1+e_T) = a_2(1+e_2) \tag{1}$$

i.e.

$$2a_T = a_1(1-e_1) + a_T(1+e_T) \qquad 2a_T = a_2(1+e_2) + a_T(1-e_T) \tag{2}$$

We get from Eq. (2) after little reduction :

$$\begin{aligned} a_T &= \frac{a_1(1-e_1) + a_2(1+e_2)}{2} \\ e_T &= \frac{-a_1(1-e_1) + a_2(1+e_2)}{a_1(1-e_1) + a_2(1+e_2)} \end{aligned} \tag{3}$$

For the transfer ellipse, we have the following equations for the total and kinetic energy:

$$C_T = \frac{v_T^2}{2} - \frac{\mu}{r_T} \tag{4}$$

$$v_T^2 = \mu\left(\frac{2}{r_T} - \frac{1}{a_T}\right) \tag{5}$$

Which yield

$$C_T = \frac{-\mu}{2a_T} \tag{6}$$

By Eq. (3), we get

$$C_T = \frac{-\mu}{a_1(1-e_1) + a_2(1+e_2)} \tag{7}$$

Also we have for the energy increment at point A

$$\Delta C_A = C_T - C_1 \tag{8}$$

But

$$C_1 = \frac{-\mu}{2a_1} \tag{9}$$

So, we acquire

$$\Delta C_A = \frac{\mu}{2a_1}\left[\frac{a_2(1+e_2) - a_1(1+e_1)}{a_2(1+e_2) + a_1(1-e_1)}\right] \tag{10}$$

Similarly

$$\Delta C_B = C_2 - C_T \tag{11}$$

$$C_2 = \frac{-\mu}{2a_2} \tag{12}$$

Which gives

$$\Delta C_B = \frac{\mu}{2a_2}\left[\frac{a_2(1-e_2) - a_1(1-e_1)}{a_2(1+e_2) + a_1(1-e_1)}\right] \tag{13}$$

For the change in kinetic energies at the two points A and B we have the following equalities:

$$\Delta C_A = \frac{1}{2}\left\{(v_A + \Delta v_A)^2 - v_A^2\right\} \tag{14}$$

$$\Delta C_B = \frac{1}{2}\left\{-(v_B - \Delta v_B)^2 + v_B^2\right\} \tag{15}$$

Where

$$(v_A)_{Per.} = \left\{\frac{\mu(1+e_1)}{a_1(1-e_1)}\right\}^{\frac{1}{2}} \tag{16}$$

$$(v_B)_{Apo..} = \left\{\frac{\mu(1-e_2)}{a_2(1+e_2)}\right\}^{\frac{1}{2}} \tag{17}$$

We now find the expressions for Δv_A and Δv_B which are the necessary changes in velocity at A and B respectively to execute the transfer by solving two second degree equations in Δv_A and Δv_B, namely

$$\Delta v_A^2 + 2v_A \Delta v_A - 2\Delta C_A = 0 \tag{18}$$

$$\Delta v_B^2 - 2v_B \Delta v_B + 2\Delta C_B = 0 \tag{19}$$

i.e.

$$\Delta v_A = -v_A \pm \left(v_A^2 + 2\Delta C_A\right)^{\frac{1}{2}}$$

$$\Delta v_B = v_B \pm \left(v_B^2 - 2\Delta C_B\right)^{\frac{1}{2}}$$

We get from the solution of (18), (19) and by Eqs (10), (13), (16) and (17):

$$(\Delta v_A)_{Per.} = \left(\frac{\mu}{a_1}\right)^{1/2}\left[\pm\left\{\frac{1+e_1}{1-e_1} + \frac{a_2(1+e_2) - a_1(1+e_1)}{a_2(1+e_2) + a_1(1-e_1)}\right\}^{\frac{1}{2}} - \left\{\frac{1+e_1}{1-e_1}\right\}^{\frac{1}{2}}\right] \tag{20}$$

$$(\Delta v_B)_{Apo.} = \left(\frac{\mu}{a_2}\right)^{1/2}\left[\pm\left\{\frac{1-e_2}{1+e_2} - \frac{a_2(1-e_2) - a_1(1-e_1)}{a_2(1+e_2) + a_1(1-e_1)}\right\}^{\frac{1}{2}} + \left\{\frac{1-e_2}{1+e_2}\right\}^{\frac{1}{2}}\right] \tag{21}$$

For Fig. 2, we have the equalities

$$a_T = \frac{a_1(1-e_1) + a_2(1-e_2)}{2} \tag{22}$$

$$e_T = \frac{a_2(1-e_2) - a_1(1-e_1)}{a_1(1-e_1) + a_2(1-e_2)} \tag{23}$$

For Fig. 3, we have

$$a_T = \frac{a_1(1+e_1) + a_2(1+e_2)}{2} \tag{24}$$

$$e_T = \frac{a_2(1+e_2) - a_1(1+e_1)}{a_1(1+e_1) + a_2(1+e_2)} \tag{25}$$

For Fig. 4, we find

$$a_T = \frac{a_1(1+e_1) + a_2(1-e_2)}{2} \tag{26}$$

$$e_T = \frac{a_2(1-e_2) - a_1(1+e_1)}{a_1(1+e_1) + a_2(1-e_2)} \tag{27}$$

2.2. *Change of vehicle's energy*

The analysis for case (I) has been discussed. For casses (II), (III), and (IV), we have the following relationships:

Case (II):

$$\Delta C_A = \frac{\mu}{2a_1}\left\{\frac{a_2(1-e_2) - a_1(1+e_1)}{a_2(1-e_2) + a_1(1-e_1)}\right\} \tag{28}$$

$$\Delta C_B = \frac{\mu}{2a_2}\left\{\frac{a_2(1+e_2) - a_1(1-e_1)}{a_2(1-e_2) + a_1(1-e_1)}\right\} \tag{29}$$

$$(v_A)_{Per.} = \left(\frac{\mu}{a_1}\right)^{\frac{1}{2}}\left\{\frac{1+e_1}{1-e_1}\right\}^{\frac{1}{2}} \tag{30}$$

$$(\Delta v_A)_{Per.} = \left(\frac{\mu}{a_1}\right)^{\frac{1}{2}}\left[-\left\{\frac{1+e_1}{1-e_1}\right\}^{\frac{1}{2}} \pm\left\{\frac{1+e_1}{1-e_1} + \frac{a_2(1-e_2) - a_1(1+e_1)}{a_2(1-e_2) + a_1(1-e_1)}\right\}^{\frac{1}{2}}\right] \tag{31}$$

$$(v_B)_{Per.} = \left(\frac{\mu}{a_2}\right)^{1/2}\left\{\frac{1+e_2}{1-e_2}\right\}^{1/2} \tag{32}$$

$$(\Delta v_B)_{Per.} = \left(\frac{\mu}{a_2}\right)^{\frac{1}{2}}\left[\left\{\frac{1+e_2}{1-e_2}\right\}^{\frac{1}{2}} \pm\left\{\frac{1+e_2}{1-e_2} - \frac{a_2(1+e_2) - a_1(1-e_1)}{a_2(1-e_2) + a_1(1-e_1)}\right\}^{\frac{1}{2}}\right] \tag{33}$$

Case (III):

$$\Delta C_A = \frac{\mu}{2a_1}\left\{\frac{a_2(1+e_2) - a_1(1-e_1)}{a_2(1+e_2) + a_1(1+e_1)}\right\} \tag{34}$$

$$\Delta C_B = \frac{\mu}{2a_2}\left\{\frac{a_2(1-e_2) - a_1(1+e_1)}{a_2(1+e_2) + a_1(1+e_1)}\right\} \tag{35}$$

$$(v_A)_{Apo.} = \left(\frac{\mu}{a_1}\right)^{\frac{1}{2}}\left\{\frac{1-e_1}{1+e_1}\right\}^{\frac{1}{2}} \tag{36}$$

$$(\Delta v_A)_{Apo.} = \left(\frac{\mu}{a_1}\right)^{\frac{1}{2}} \left[-\left\{\frac{1-e_1}{1+e_1}\right\}^{\frac{1}{2}} \right.$$

$$\left. \pm \left\{ \frac{1-e_1}{1+e_1} + \frac{a_2(1+e_2) - a_1(1-e_1)}{a_2(1+e_2) + a_1(1+e_1)} \right\}^{\frac{1}{2}} \right] \tag{37}$$

$$(v_B)_{Apo.} = \left(\frac{\mu}{a_2}\right)^{\frac{1}{2}} \left\{\frac{1-e_2}{1+e_2}\right\}^{\frac{1}{2}} \tag{38}$$

$$(\Delta v_B)_{Apo.} = \left(\frac{\mu}{a_2}\right)^{1/2} \left[\left\{\frac{1-e_2}{1+e_2}\right\}^{\frac{1}{2}} \right.$$

$$\left. \pm \left\{ \frac{1-e_2}{1+e_2} - \frac{a_2(1-e_2) - a_1(1+e_1)}{a_2(1+e_2) + a_1(1+e_1)} \right\}^{\frac{1}{2}} \right] \tag{39}$$

Case (IV):

$$\Delta C_A = \frac{\mu}{2a_1} \left\{ \frac{a_2(1-e_2) - a_1(1-e_1)}{a_2(1-e_2) + a_1(1+e_1)} \right\} \tag{40}$$

$$\Delta C_B = \frac{\mu}{2a_2} \left\{ \frac{a_2(1+e_2) - a_1(1+e_1)}{a_2(1-e_2) + a_1(1+e_1)} \right\} \tag{41}$$

$$(v_A)_{Apo.} = \left(\frac{\mu}{a_1}\right)^{\frac{1}{2}} \left\{\frac{1-e_1}{1+e_1}\right\}^{\frac{1}{2}} \tag{42}$$

$$(\Delta v_A)_{Apo.} = \left(\frac{\mu}{a_1}\right)^{\frac{1}{2}} \left[-\left\{\frac{1-e_1}{1+e_1}\right\}^{\frac{1}{2}} \right.$$

$$\left. \pm \left\{ \frac{1-e_1}{1+e_1} + \frac{a_2(1-e_2) - a_1(1-e_1)}{a_2(1-e_2) + a_1(1+e_1)} \right\}^{\frac{1}{2}} \right] \tag{43}$$

$$(v_B)_{Per.} = \left(\frac{\mu}{a_2}\right)^{1/2} \left\{\frac{1+e_2}{1-e_2}\right\}^{\frac{1}{2}} \tag{44}$$

$$(\Delta v_B)_{Per.} = \left(\frac{\mu}{a_2}\right)^{\frac{1}{2}} \left[\left\{\frac{1+e_2}{1-e_2}\right\}^{\frac{1}{2}} \right.$$

$$\left. \pm \left\{ \frac{1+e_2}{1-e_2} - \frac{a_2(1+e_2) - a_1(1+e_1)}{a_2(1-e_2) + a_1(1+e_1)} \right\}^{\frac{1}{2}} \right] \tag{45}$$

Evidently Δv_A and Δv_B are functions of the parameters a_1, a_2, e_1, and e_2.

We place here Fig. 5, showing the variation of $\alpha = a_2/a_1$ versus the total transfer impulse $\Delta v_A + \Delta v_B$ (classical characteristic velocity).

A bundle of curves appear corresponding to $e_1 = 0.1$; $e_2 = 0.1 - 0.9$.

In Fig. 5, the total impulse increase drastically for the range $\alpha \approx 0.0 - 10.0$, then increase slowly, indicating that the consumption of fuel is greatest for the above range.

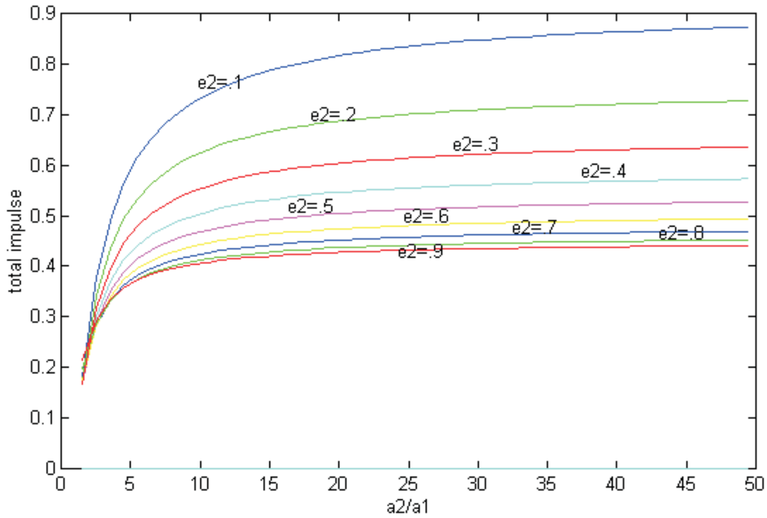

Figure 5

3. Presentation of the Method of Optimization

3.1. *For the first configuration*

Let v_A be the velocity of the satellite in initial ellipse $(a_1,\ e_1)$, and xv_A be the velocity in transfer orbit $(a_T,\ e_T)$ after first impulse V_1.

We have

$$xv_A - v_A = V_1$$

where

$$x = \frac{the\, velocity\, after\, first\, impluse}{the\, velocity\, before\, first\, impulse} = \sqrt{\frac{\frac{\mu(1+e_T)}{a_T(1-e_T)}}{\frac{\mu(1+e_1)}{a_1(1-e_1)}}}$$

From Fig. 1,

$$a_1\,(1-e_1) = a_T\,(1-e_T)$$

i.e.

$$x = \sqrt{\frac{1+e_T}{1+e_1}}$$

$$V_1 = \sqrt{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}}\left\{\sqrt{\frac{1+e_T}{1+e_1}}-1\right\}$$

$$V_1 = \sqrt{\frac{\mu}{a_1\,(1-e_1)}}\left\{x\sqrt{1+e_1}-\sqrt{1+e_1}\right\}$$

$$V_1 = \sqrt{\frac{\mu\,(1+e_1)}{b_1}}\,(x-1)$$

where $\quad b_1 = a_1\,(1-e_1)$

we get

$$\frac{dV_1}{dx} = \sqrt{\frac{\mu\,(1+e_1)}{b_1}} \tag{46}$$

Now, let v_{B1} be the velocity at B before the second impulse V_2 , and V_{B2} be the velocity at B after the second impulse V_2. We have

$$v_{B1} = \sqrt{\frac{\mu\,(1-e_T)}{a_T\,(1+e_T)}}$$

$$v_{B2} = \sqrt{\frac{\mu\,(1-e_2)}{a_2\,(1+e_2)}}$$

Denote

$$\eta^2 = \frac{v_{B2}^2}{v_{B1}^2} = \frac{1-e_2}{1-e_T}$$

Where, from Fig. 1,

$$a_T\,(1+e_T) = a_2\,(1+e_2)$$

We get from the definition of x, that

$$1-e_T = 2-x^2\,(1+e_1)$$

Then we write,

$$\eta^2 = \frac{1-e_2}{2-x^2\,(1+e_1)} = \frac{2\mu}{2\mu+x^2\,(1-\mu)\,(1+e_1)}$$

Where μ – ratio of periapses distances of terminal orbits

$$\mu = a_2(1-e_2)/a_1(1-e_1)$$

Then,

$$\frac{d\eta}{dx} = \frac{x\sqrt{1-e_2}\,(1+e_1)}{\{2-x^2\,(1+e_1)\}^{\frac{3}{2}}}$$

If α be the angle according to a plane change, we get

$$V_2 = v_{B1}\left(1+\eta^2-2\eta\cos\alpha\right)^{\frac{1}{2}} \tag{47}$$

We have,

$$v_{B1} = \sqrt{\frac{\mu}{b_4}}\sqrt{2 - x^2\,(1 + e_1)} \qquad \text{where} \quad b_4 = a_2\,(1 + e_2)$$

Then,

$$\frac{dv_{B1}}{dx} = \sqrt{\frac{\mu}{b_4}}\left[\frac{-x\,(1 + e_1)}{\{2 - x^2\,(1 + e_1)\}^{\frac{1}{2}}}\right]$$

Therefore,

$$\frac{dV_2}{dx} = v_{B1}\left\{\frac{\eta\frac{d\eta}{dx} - \cos\alpha\frac{d\eta}{dx}}{(1 + \eta^2 - 2\eta\cos\alpha)^{1/2}}\right\} + \frac{dv_{B1}}{dx}\left(1 + \eta^2 - 2\eta\cos\alpha\right)^{\frac{1}{2}}$$

and after some reductions,

$$\frac{dV_2}{dx} = \sqrt{\frac{\mu}{b_4}}G \tag{48}$$

where
$G = \left[\frac{x(1+e_1)\left\{\sqrt{1-e_2}(\eta-\cos\alpha)-\sqrt{2-x^2(1+e_1)}\left(1+\eta^2-2\eta\cos\alpha\right)\right\}}{\{2-x^2(1+e_1)\}\sqrt{(1+\eta^2-2\eta\cos\alpha)}}\right]$
The optimum condition is the following:

$$\frac{dV_T}{dx} = \frac{dV_1}{dx} + \frac{dV_2}{dx} = 0$$

From Eqs (46) and (48) we get,

$$\frac{1}{\sqrt{b_1}} + \sqrt{\frac{1 + e_1}{b_4}}G_1 = 0$$

where
$G_1 = \left[\frac{x\left\{\sqrt{1-e_2}(\eta-\cos\alpha)-\sqrt{2-x^2(1+e_1)}\left(1+\eta^2-2\eta\cos\alpha\right)\right\}}{\{2-x^2(1+e_1)\}\sqrt{(1+\eta^2-2\eta\cos\alpha)}}\right]$
and after some algebraic calculus computations:

$$\frac{1}{\sqrt{b_1}} + \sqrt{\frac{1 + e_1}{b_4}}G_2 = 0$$

where
$G_2 = \left[\frac{x\left\{\sqrt{1-e_2}\cos\alpha-\sqrt{2-x^2(1+e_1)}\right\}}{\sqrt{2-x^2(1+e_1)}\left\{3-e_2-x^2(1+e_1)-2\sqrt{(1-e_2)\{2-x^2(1+e_1)\}}\cos\alpha\right\}^{\frac{1}{2}}}\right]$
Then,

$$\frac{1}{b_1} = \frac{1 + e_1}{b_4}G_3$$

where
$G_3 = \left[\frac{x^2\left\{(1-e_2)\cos^2\alpha-2\sqrt{(1-e_2)\{2-x^2(1+e_1)\}}\cos\alpha+2-x^2(1+e_1)\right\}}{\{2-x^2(1+e_1)\}\left\{3-e_2-x^2(1+e_1)-2\sqrt{(1-e_2)\{2-x^2(1+e_1)\}}\cos\alpha\right\}}\right]$

i.e.

$$2\sqrt{(1-e_2)\{2-x^2(1+e_1)\}}\cos\alpha\left[-x^2(1+e_1)(b_1+b_4)+2b_4\right]$$
$$=\left\{2-x^2(1+e_1)\right\}\left\{-x^2(1+e_1)(b_1+b_4)+2b_4\right\}$$
$$+(1-e_2)\left\{-x^2(1+e_1)\left(b_1\cos^2\alpha+b_4\right)+2b_4\right\}$$

Let

$$-(1+e_1)(b_1+b_4)=C_1 \qquad -(1+e_1)\left(b_1\cos^2\alpha+b_4\right)=C_2$$

Whence

$$2\sqrt{(1-e_2)\{2-x^2(1+e_1)\}}\cos\alpha\left(x^2C_1+2b_4\right)$$
$$-\left\{2-x^2(1+e_1)\right\}\left(x^2C_1+2b_4\right)+(1-e_2)\left(x^2C_2+2b_4\right)$$

By squaring we get

$$4(1-e_2)\left\{2-x^2(1+e_1)\right\}\cos^2\alpha\left(x^4C_1^2+4x^2C_1b_4+4b_4^2\right)$$
$$=\left\{4-4x^2(1+e_1)+x^4(1+e_1)^2\right\}\left(x^4C_1^2+4x^2C_1b_4+4b_4^2\right)$$
$$+2(1-e_2)\left\{2-x^2(1+e_1)\right\}\left(x^2C_1+2b_4\right)\left(x^2C_2+2b_4\right)$$
$$+(1-e_2)^2\left(x^4C_2^2+4x^2C_2b_4+4b_4^2\right)$$

i.e.

$$-q_1^2x^8+x^6\left\{4q_1C_1-4q_1(1-e_2)C_1\cos^2\alpha-4q_1(1+e_1)b_4+2q_1(1-e_2)C_2\right\}$$
$$+x^4\left\{\begin{array}{l}8(1-e_2)C_1^2\cos^2\alpha-16(1-e_2)q_1b_4\cos^2\alpha-4C_1^2+16q_1b_4\\-4(1+e_1)^2b_4^2--4(1-e_2)C_1C_2+4q_1(1-e_2)b_4\\+4(1-e_2)(1+e_1)C_2b_4-(1-e_2)^2C_2^2\end{array}\right\}$$
$$+x^2\left\{\begin{array}{l}32(1-e_2)C_1b_4\cos^2\alpha-16(1-e_2)(1+e_1)b_4^2\cos^2\alpha-16C_1b_4\\+16(1+e_1)b_4^2-8(1-e_2)C_1b_4-8(1-e_2)C_2b_4\\+8(1-e_2)(1+e_1)b_4^2-4(1-e_2)^2C_2b_4\end{array}\right\}$$
$$+32(1-e_2)b_4^2\cos^2\alpha-16b_4^2-16(1-e_2)b_4^2-4(1-e_2)^2b_4^2=0$$

where
$q_1=C_1(1+e_1)$
whence, we obtain the algebraic equation of the eigth degree, namely:

$$x^8+u_1x^6+u_2x^4+u_3x^2+u_4=0 \qquad (49)$$

where

$$u_1 = -\frac{\left\{4C_1 - 4(1-e_2t)C_1\cos^2\alpha - 4(1+e_1)b_4 + 2(1-e_2C_2\right\}}{q_1}$$

$$u_2 = -\frac{\left\{\begin{array}{l} 8\left(1-e_2\right)C_1^2\cos^2\alpha - 16\left(1-e_2\right)q_1b_4\cos^2\alpha - 4C_1^2 + 16q_1b_4 \\ -4\left(1+e_1\right)^2b_4^2 - 4\left(1-e_2\right)C_1C_2 + 4q_1\left(1-e_2\right)b_4 \\ +4\left(1-e_2\right)\left(1+e_1\right)C_2b_4 - \left(1-e_2\right)^2C_2^2 \end{array}\right\}}{q_1^2}$$

$$u_3 = -\frac{\left\{\begin{array}{l} 32\left(1-e_2\right)C_1b_4\cos^2\alpha - 16\left(1-e_2\right)\left(1+e_1\right)b_4^2\cos^2\alpha - 16C_1b_4 \\ +16\left(1+e_1\right)b_4^2 - 8\left(1-e_2\right)C_1b_4 - 8\left(1-e_2\right)C_2b_4 \\ +8\left(1-e_2\right)\left(1+e_1\right)b_4^2 - 4\left(1-e_2\right)^2C_2b_4 \end{array}\right\}}{q_1^2}$$

$$u_4 = -\frac{\left\{32\left(1-e_2\right)b_4^2\cos^2\alpha - 16b_4^2 - 16\left(1-e_2\right)b_4^2 - 4\left(1-e_2\right)^2b_4^2\right\}}{q_1^2}$$

3.2. *For the second configuration*

We have the additional equations:

$$a_T\left(1+e_T\right) = a_2\left(1-e_2\right)$$

$$v_{B1} = \sqrt{\frac{\mu\left(1-e_T\right)}{a_T\left(1+e_T\right)}}$$

$$v_{B2} = \sqrt{\frac{\mu\left(1+e_2\right)}{a_2\left(1-e_2\right)}}$$

and

$$\eta^2 = \frac{v_{B2}^2}{v_{B1}^2} = \frac{1+e_2}{1-e_T}$$

but

$$1-e_T = 2-x^2\left(1+e_1\right)$$

then

$$\eta^2 = \frac{1+e_2}{2-x^2\left(1+e_1\right)}$$

By differentiation w.r.t. x, we get

$$\frac{d\eta}{dx} = \frac{x\left(1+e_1\right)\sqrt{1+e_2}}{\left\{2-x^2\left(1+e_1\right)\right\}^{\frac{3}{2}}}$$

The second impulse

$$V_2 = v_{B1}\sqrt{1+\eta^2-2\eta\cos\alpha}$$

with

$$v_{B1} = \sqrt{\frac{\mu\left\{2-x^2\left(1+e_1\right)\right\}}{b_3}} \qquad \text{where} \qquad b_3 = a_2\left(1-e_2\right)$$

Then

$$\frac{dv_{B1}}{dx} = \sqrt{\frac{\mu}{b_3}}\left[\frac{-x\left(1+e_1\right)}{\left\{2-x^2\left(1+e_1\right)\right\}^{\frac{1}{2}}}\right]$$

whence

$$\frac{dV_2}{dx} = \frac{dv_{B1}}{dx}\sqrt{1+\eta^2-2\eta\cos\alpha} + v_{B1}\left[\frac{\frac{d\eta}{dx}\left(\eta-\cos\alpha\right)}{\sqrt{1+\eta^2-2\eta\cos\alpha}}\right]$$

$$\frac{dV_2}{dx} = \sqrt{\frac{\mu}{b_3}}\frac{x\left(1+e_1\right)}{\sqrt{2-x^2\left(1+e_1\right)}}$$

$$\left[\frac{\sqrt{1+e_2}\cos\alpha-\sqrt{2-x^2\left(1+e_1\right)}}{\left[\left\{2-x^2\left(1+e_1\right)\right\}+\left(1+e_2\right)-2\sqrt{1+e_2}\cos\alpha\sqrt{2-x^2\left(1+e_1\right)}\right]^{\frac{1}{2}}}\right] \quad (50)$$

The optimum condition:

$$\frac{dV_1}{dx}+\frac{dV_2}{dx}=0$$

Then

$$\frac{-1}{\sqrt{b_1}} = \sqrt{\frac{1+e_1}{b_3}}\frac{x}{\sqrt{2-x^2\left(1+e_1\right)}}$$

$$\left[\frac{\sqrt{1+e_2}\cos\alpha-\sqrt{2-x^2\left(1+e_1\right)}}{\left[\left\{2-x^2\left(1+e_1\right)\right\}+\left(1+e_2\right)-2\sqrt{1+e_2}\cos\alpha\sqrt{2-x^2\left(1+e_1\right)}\right]^{\frac{1}{2}}}\right]$$

Therefore

$$x^2b_1\left(1+e_1\right)\left(1+e_2\right)\cos^2\alpha-2x^2b_1\left(1+e_1\right)\sqrt{1+e_2}\left\{2-x^2\left(1+e_1\right)\right\}^{\frac{1}{2}}\cos\alpha$$
$$+x^2b_1\left(1+e_1\right)\left\{2-x^2\left(1+e_1\right)\right\} = b_3\left\{2-x^2\left(1+e_1\right)\right\}^2$$
$$+b_3\left(1+e_2\right)\left\{2-x^2\left(1+e_1\right)\right\}-2b_3\sqrt{1+e_2}\left\{2-x^2\left(1+e_1\right)\right\}^{\frac{3}{2}}\cos\alpha$$

Let

$$C_3=b_1\left(1+e_1\right)\left(1+e_2\right)\cos^2\alpha \qquad C_4=-2b_1\left(1+e_1\right)\sqrt{1+e_2}\cos\alpha$$
$$C_5=b_1\left(1+e_1\right) \qquad C_6=b_3\left(1+e_2\right) \qquad C_7=-2b_3\sqrt{1+e_2}\cos\alpha$$

i.e.

$$x^2\left\{C_3+2C_5+4b_3\left(1-e_1\right)+C_6\left(1+e_1\right)\right\}+x^4\left\{-C_5\left(1+e_1\right)-b_3\left(1+e_1\right)^2\right\}$$
$$-4b_3-2C_6=C_7\left\{2-x^2\left(1+e_1\right)\right\}^{\frac{3}{2}}-x^2C_4\left\{2-x^2\left(1+e_1\right)\right\}^{\frac{1}{2}}$$

Let

$$C_8=C_3+2C_5+4b_3\left(1-e_1\right)+C_6\left(1+e_1\right)$$
$$C_9=-C_5\left(1+e_1\right)-b_3\left(1+e_1\right)^2 \qquad C_{10}=-4b_3-2C_6$$

By squaring, we get

$$C_9^2 x^8 + x^6 \left\{2C_8C_9 + C_7^2 (1+e_1)^3 + 2C_4C_7 (1+e_1)^2 + C_4^2 (1+e_1)\right\} +$$
$$+x^4 \left\{C_8^2 + 2C_9C_{10} - 6C_7^2 (1+e_1)^2 - 8C_4C_7 (1+e_1) - 2C_4^2\right\} +$$
$$+x^2 \left\{2C_8C_{10} + 12C_7^2 (1+e_1) + 8C_4C_7\right\} + C_{10}^2 - 8C_7^2 = 0$$

Put,

$$D_1 = \frac{\left\{2C_8C_9 + C_7^2 (1+e_1)^3 + 2C_4C_7 (1+e_1)^2 + C_4^2 (1+e_1)\right\}}{C_9^2}$$
$$D_2 = \frac{\left\{C_8^2 + 2C_9C_{10} - 6C_7^2 (1+e_1)^2 - 8C_4C_7 (1+e_1) - 2C_4^2\right\}}{C_9^2}$$
$$D_3 = \frac{\left\{2C_8C_{10} + 12C_7^2 (1+e_1) + 8C_4C_7\right\}}{C_9^2}$$
$$D_4 = \frac{C_{10}^2 - 8C_7^2}{C_9^2}$$

Therefore, we get an algebraic equation of the eight degree in x,

$$x^8 + D_1x^6 + D_2x^4 + D_3x^2 + D_4 = 0 \tag{51}$$

3.3. *For the third configuration*

The initial impulse is at apo–apse.

We have the following relationships:

$$x = \frac{velocity\,after\,initial\,impulse}{velocity\,before\,initial\,impulse}$$
$$= \sqrt{\frac{\frac{\mu(1+e_T)}{a_T(1-e_T)}}{\frac{\mu(1-e_1)}{a_1(1+e_1)}}}$$

For Fig. 3,

$$a_T (1-e_T) = a_1 (1+e_1)$$

Then,

$$x = \sqrt{\frac{1+e_T}{1-e_1}} \qquad 1+e_T = x^2 (1-e_1) \qquad 1-e_T = 2 - x^2 (1-e_1)$$

But, we have

$$V_1 = v_A (x-1) = (x-1) \sqrt{\frac{\mu (1-e_1)}{a_1 (1+e_1)}}$$

i.e.

$$V_1 = (x-1) \sqrt{\frac{\mu (1-e_1)}{b_2}} \qquad \text{where} \qquad b_2 = a_1 (1+e_1)$$

Therefore,

$$\frac{dV_1}{dx} = \sqrt{\frac{\mu\,(1-e_1)}{b_2}} \tag{52}$$

We have, from Fig. 3.

$$a_T\,(1+e_T) = a_2\,(1+e_2)$$

and

$$\eta^2 = \frac{v_{B2}^2}{v_{B1}^2} = \frac{1-e_2}{1-e_T}$$
$$v_{B1} = \sqrt{\frac{\mu\,(1-e_T)}{a_T\,(1+e_T)}}$$
$$v_{B2} = \sqrt{\frac{\mu\,(1-e_2)}{a_2\,(1+e_2)}}$$

whence

$$\eta^2 = \frac{1-e_2}{2-x^2\,(1-e_1)}$$
$$\eta = \sqrt{\frac{1-e_2}{2-x^2\,(1-e_1)}}$$
$$\frac{d\eta}{dx} = \frac{x\,(1-e_1)\,\sqrt{1-e_2}}{\{2-x^2\,(1-e_1)\}^{\frac{3}{2}}}$$
$$v_{B1} = \sqrt{\frac{\mu\,\{2-x^2\,(1-e_1)\}}{b_4}} \qquad \text{with} \qquad b_4 = a_2\,(1+e_2)$$
$$\frac{dv_{B1}}{dx} = \sqrt{\frac{\mu}{b_4}}\left[\frac{-x\,(1-e_1)}{\{2-x^2\,(1-e_1)\}^{\frac{1}{2}}}\right]$$

The second impulse,

$$V_2 = v_{B1}\sqrt{1+\eta^2-2\eta\cos\alpha}$$
$$\frac{dV_2}{dx} = \frac{dv_{B1}}{dx}\sqrt{1+\eta^2-2\eta\cos\alpha} + v_{B1}\left[\frac{(\eta-\cos\alpha)\,\frac{d\eta}{dx}}{\sqrt{1+\eta^2-2\eta\cos\alpha}}\right]$$

Substituting for v_{B1}, dv_{B1}/dx, η, $d\eta/dx$ we may write

$$\frac{dV_2}{dx} = \sqrt{\frac{\mu}{b_4}}\frac{x\,(1-e_1)}{\{2-x^2\,(1-e_1)\}^{1/2}}$$
$$\left[\frac{\sqrt{1-e_2}\cos\alpha - \{2-x^2\,(1-e_1)\}^{1/2}}{\left[\{2-x^2\,(1-e_1)\} + (1-e_2) - 2\sqrt{(1-e_2)\,\{2-x^2\,(1-e_1)\}}\cos\alpha\right]^{\frac{1}{2}}}\right] \tag{53}$$

But we have for optimum condition

$$\frac{dV_1}{dx} + \frac{dV_2}{dx} = 0$$

From Eqs (52) and (53), we get

$$\frac{1}{\sqrt{b_2}} = \frac{-x\sqrt{1-e_1}}{\sqrt{b_4\left\{2-x^2\left(1-e_1\right)\right\}}}$$

$$\left[\frac{\sqrt{1-e_2}\cos\alpha - \left\{2-x^2\left(1-e_1\right)\right\}^{\frac{1}{2}}}{\left[\left\{2-x^2\left(1-e_1\right)\right\}+\left(1-e_2\right)-2\sqrt{\left(1-e_2\right)\left\{2-x^2\left(1-e_1\right)\right\}}\cos\alpha\right]^{\frac{1}{2}}}\right]$$

$$x^2b_2\left(1-e_1\right)\left(1-e_2\right)\cos^2\alpha - 2x^2b_2\left(1-e_1\right)\sqrt{1-e_2}\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{1}{2}}\cos\alpha$$
$$+x^2b_2\left(1-e_1\right)\left\{2-x^2\left(1-e_1\right)\right\} = b_4\left\{2-x^2\left(1-e_1\right)\right\}\left[\left\{2-x^2\left(1-e_1\right)\right\}\right.$$
$$\left.+\left(1-e_2\right)-2\sqrt{1-e_2}\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{1}{2}}\cos\alpha\right]$$

Let

$$g_1 = b_2\left(1-e_1\right)\left(1-e_2\right)\cos^2\alpha \qquad g_2 = -2b_2\left(1-e_1\right)\sqrt{1-e_2}\cos\alpha$$
$$g_3 = b_2\left(1-e_1\right) \qquad g_4 = b_4\left(1-e_2\right) \qquad g_5 = -2b_4\sqrt{1-e_2}\cos\alpha$$
$$\frac{d\eta}{dx} = \frac{x\left(1-e_1\right)\sqrt{1-e_2}}{\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{3}{2}}}$$

whence

$$x^2\left\{g_1+2g_3+4b_4\left(1-e_1\right)+g_4\left(1-e_1\right)\right\}+x^4\left\{-g_3\left(1-e_1\right)-b_4\left(1-e_1\right)^2\right\}$$
$$-4b_4-2g_4 = g_5\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{3}{2}} - x^2g_2\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{1}{2}}$$

put

$$g_6 = g_1+2g_3+4b_4\left(1-e_1\right)+g_4\left(1-e_1\right)$$
$$g_7 = -g_3\left(1-e_1\right)-b_4\left(1-e_1\right)^2$$
$$g_8 = -4b_4-2g_4$$

After squaring, we get

$$g_7^2x^8+x^6\left\{2g_6g_7+g_5^2\left(1-e_1\right)^3+2g_2g_5\left(1-e_1\right)^2+g_2^2\left(1-e_1\right)\right\}+$$
$$+x^4\left\{g_6^2+2g_7g_8-6g_5^2\left(1-e_1\right)^2-8g_2g_5\left(1-e_1\right)-2g_2^2\right\}+$$
$$+x^2\left\{2g_6g_8+12g_5^2\left(1-e_1\right)+8g_2g_5\right\}+g_8^2-8g_5^2=0$$

Let

$$g_9 = \left\{2g_6g_7+g_5^2\left(1-e_1\right)^3+2g_2g_5\left(1-e_1\right)^2+g_2^2\left(1-e_1\right)\right\}/g_7^2$$
$$g_{10} = \left\{g_6^2+2g_7g_8-6g_5^2\left(1-e_1\right)^2-8g_2g_5\left(1-e_1\right)-2g_2^2\right\}/g_7^2$$
$$g_{11} = \left\{2g_6g_8+12g_5^2\left(1-e_1\right)+8g_2g_5\right\}/g_7^2$$
$$g_{12} = \left\{g_8^2-8g_5^2\right\}/g_7^2$$

Therefore, we get an equation of degree eight in x in the form

$$x^8+g_9x^6+g_{10}x^4+g_{11}x^2+g_{12}=0 \tag{54}$$

3.4. *For the fourth configuration*

The initial impulse is at apo–apse.

And we have

$$v_{B1} = \sqrt{\frac{\mu\left(1-e_T\right)}{a_T\left(1+e_T\right)}} \qquad v_{B2} = \sqrt{\frac{\mu\left(1+e_2\right)}{a_2\left(1-e_2\right)}}$$

From Fig. 4, we notice that

$$a_T\left(1+e_T\right) = a_2\left(1-e_2\right)$$

Therefore,

$$v_{B1} = \sqrt{\frac{\mu\left\{2-x^2\left(1-e_1\right)\right\}}{b_3}} \qquad v_{B2} = \sqrt{\frac{\mu\left(1+e_2\right)}{b_3}}$$

where $b_3 = a_2\left(1-e_2\right)$

$$\frac{dv_{B1}}{dx} = \sqrt{\frac{\mu}{b_3}}\left[\frac{-x\left(1-e_1\right)}{\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{1}{2}}}\right]$$

Let

$$\eta = \frac{v_{B2}}{v_{B1}} = \sqrt{\frac{1+e_2}{2-x^2\left(1-e_1\right)}}$$

$$\frac{d\eta}{dx} = \frac{x\left(1-e_1\right)\sqrt{1+e_2}}{\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{3}{2}}}$$

$$V_2 = v_{B1}\sqrt{1+\eta^2-2\eta\cos\alpha}$$

$$\frac{dV_2}{dx} = \frac{dv_{B1}}{dx}\left(1+\eta^2-2\eta\cos\alpha\right)^{\frac{1}{2}} + v_{B1}\left[\frac{\left(\eta-\cos\alpha\right)\frac{d\eta}{dx}}{\left(1+\eta^2-2\eta\cos\alpha\right)^{\frac{1}{2}}}\right]$$

$$\frac{dV_2}{dx} = \sqrt{\frac{\mu}{b_3\left\{2-x^2\left(1-e_1\right)\right\}}}\left\{x\left(1-e_1\right)\right\} \left[\frac{\sqrt{1+e_2}\cos\alpha-\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{1}{2}}}{\left[\left\{2-x^2\left(1-e_1\right)\right\}+\left(1+e_2\right)-2\sqrt{\left(1+e_2\right)\left\{2-x^2\left(1-e_1\right)\right\}}\cos\alpha\right]^{\frac{1}{2}}}\right] \tag{55}$$

But, from optimum condition

$$\frac{d}{dx}\left(V_1+V_2\right) = 0$$

Using Eqs (52) and (55), we get

$$\frac{-1}{\sqrt{b_2}} = \frac{x\sqrt{1-e_1}}{\sqrt{b_3\left\{2-x^2\left(1-e_1\right)\right\}}} \left[\frac{\sqrt{1+e_2}\cos\alpha-\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{1}{2}}}{\left[\left\{2-x^2\left(1-e_1\right)\right\}+\left(1+e_2\right)-2\sqrt{1+e_2}\left\{2-x^2\left(1-e_1\right)\right\}^{\frac{1}{2}}\cos\alpha\right]^{\frac{1}{2}}}\right]$$

i.e.

$$\begin{aligned}&x^2 b_2 (1-e_1)(1+e_2)\cos^2\alpha - 2x^2 b_2 (1-e_1)\sqrt{1+e_2}\left\{2-x^2(1-e_1)\right\}^{\frac{1}{2}}\cos\alpha\\&+x^2 b_2(1-e_1)\left\{2-x^2(1-e_1)\right\} = b_3\left\{2-x^2(1-e_1)\right\}^2\\&+b_3(1+e_2)\left\{2-x^2(1-e_1)\right\} - 2b_3\sqrt{1+e_2}\left\{2-x^2(1-e_1)\right\}^{\frac{3}{2}}\cos\alpha\end{aligned}$$

Let

$$\begin{aligned}&E_1 = b_2(1-e_1)(1+e_2)\cos^2\alpha \qquad E_2 = -2b_2(1-e_1)\sqrt{1+e_2}\cos\alpha\\&E_3 = b_2(1-e_1) \qquad E_4 = b_3(1+e_2) \qquad E_5 = -2b_3\sqrt{1+e_2}\cos\alpha\end{aligned}$$

Then

$$\begin{aligned}&x^2\left[E_1 + E_2\left\{2-x^2(1-e_1)\right\}^{\frac{1}{2}} + E_3\left\{2-x^2(1-e_1)\right\}\right]\\&=\left\{2-x^2(1-e_1)\right\}\left[b_3\left\{2-x^2(1-e_1)\right\} + E_4 + E_5\left\{2-x^2(1-e_1)\right\}^{\frac{1}{2}}\right]\end{aligned}$$

i.e.

$$\begin{aligned}&x^2\left\{E_1 + 2E_3 + 4b_3(1-e_1) + E_4(1-e_1)\right\} + x^4\left\{-E_3(1-e_1) - b_3(1-e_1)^2\right\}\\&-4b_3 - 2E_4 = E_5\left\{2-x^2(1-e_1)\right\}^{\frac{3}{2}} - x^2 E_2\left\{2-x^2(1-e_1)\right\}^{\frac{1}{2}}\end{aligned}$$

Put

$$\begin{aligned}&E_6 = E_1 + 2E_3 + 4b_3(1-e_1) + E_4(1-e_1)\\&E_7 = -E_3(1-e_1) - b_3(1-e_1)^2 \qquad E_8 = -4b_3 - 2E_4\end{aligned}$$

By squaring, we get

$$\begin{aligned}&\left(x^2E_6 + x^4E_7 + E_8\right)^2 = E_5^2\left\{2-x^2(1-e_1)\right\}^3 - 2E_2E_5x^2\left\{2-x^2(1-e_1)\right\}^2\\&+E_2^2x^4\left\{2-x^2(1-e_1)\right\}\end{aligned}$$

i.e.

$$\begin{aligned}&E_7^2x^8 + x^6\left\{2E_6E_7 + E_5^2(1-e_1)^3 + 2E_2E_5(1-e_1)^2 + E_2^2(1-e_1)\right\} +\\&+x^4\left\{E_6^2 + 2E_7E_8 - 6E_5^2(1-e_1)^2 - 8E_2E_5(1-e_1) - 2E_2^2\right\} +\\&+x^2\left\{2E_6E_8 + 12E_5^2(1-e_1) + 8E_2E_5\right\} + E_8^2 - 8E_5^2 = 0\end{aligned}$$

Let

$$\begin{aligned}&E_9 = \left\{2E_6E_7 + E_5^2(1-e_1)^3 + 2E_2E_5(1-e_1)^2 + E_2^2(1-e_1)\right\}/E_7^2\\&E_{10} = \left\{E_6^2 + 2E_7E_8 - 6E_5^2(1-e_1)^2 - 8E_2E_5(1-e_1) - 2E_2^2\right\}/E_7^2\\&E_{11} = \left\{2E_6E_8 + 12E_5^2(1-e_1) + 8E_2E_5\right\}/E_7^2\\&E_{12} = \left\{E_8^2 - 8E_5^2\right\}/E_7^2\end{aligned}$$

Finally, we get an equation of degree eight in x

$$x^8 + E_9x^6 + E_{10}x^4 + E_{11}x^2 + E_{12} = 0 \qquad (56)$$

4. Reduction to coplanar classical Hohmann system

4.1.

Evidently when we set $\alpha = 0$, we reduce to the coplanar case. For the first configuration, we have

$$\frac{dV_1}{dx} = \sqrt{\frac{\mu\,(1+e_1)}{b_1}}$$

$$v_{B1} = \sqrt{\frac{\mu\,\{2 - x^2\,(1+e_1)\}}{b_4}}$$

$$\frac{dv_{B1}}{dx} = \{-x\,(1+e_1)\}\sqrt{\frac{\mu}{b_4\,\{2 - x^2\,(1+e_1)\}}}$$

Set $\alpha = 0$ in Eq. (47), then

$$V_2 = v_{B1}\,(1-\eta)$$

$$\frac{dV_2}{dx} = v_{B1}\left(-\frac{d\eta}{dx}\right) + \frac{dv_{B1}}{dx}\,(1-\eta)$$

After some reductions, the first derivative is given by

$$\frac{dV_2}{dx} = \{-x\,(1+e_1)\}\sqrt{\frac{\mu}{b_4\,\{2 - x^2\,(1+e_1)\}}}$$

The optimum condition is

$$\frac{d}{dx}\,(V_1 + V_2) = 0$$

i.e.

$$\sqrt{\frac{\mu\,(1+e_1)}{b_1}} + \{-x\,(1+e_1)\}\sqrt{\frac{\mu}{b_4\,\{2 - x^2\,(1+e_1)\}}} = 0$$

From which

$$x = \pm\left\{\frac{2b_4}{(1+e_1)\,(b_1+b_4)}\right\}^{\frac{1}{2}} \tag{57}$$

4.2.

Similarly for the second configuration, we have

$$\frac{dV_1}{dx} = \sqrt{\frac{\mu\,(1+e_1)}{b_1}}$$

$$V_2 = v_{B1}\left(1+\eta^2-2\eta\right)^{\frac{1}{2}} = v_{B1}\,(1-\eta)$$

$$\frac{dV_2}{dx} = v_{B1}\frac{d}{dx}\,(1-\eta) + \frac{dv_{B1}}{dx}\,(1-\eta)$$

By a similar analysis, we get

$$\frac{dV_2}{dx} = \{-x(1+e_1)\}\sqrt{\frac{\mu}{b_3\{2-x^2(1+e_1)\}}}$$

From the optimum condition

$$\frac{d}{dx}(V_1+V_2) = 0$$

We get

$$\frac{1}{\sqrt{b_1}} = x\sqrt{\frac{1+e_1}{b_3\{2-x^2(1+e_1)\}}}$$

i.e.

$$x = \pm\left\{\frac{2b_3}{(1+e_1)(b_1+b_3)}\right\}^{\frac{1}{2}} \tag{58}$$

4.3.

Similarly for the third configuration, when $\alpha = 0$, we have

$$\frac{dV_1}{dx} = \sqrt{\frac{\mu(1-e_1)}{b_2}} \qquad \text{and} \qquad V_2 = v_{B1}(1-\eta)$$

By substitution

$$V_2 = \sqrt{\frac{\mu}{b_4}}\left[\left\{2-x^2(1-e_1)\right\}^{1/2} - \sqrt{1-e_2}\right]$$

$$\frac{dV_2}{dx} = \{-x(1-e_1)\}\sqrt{\frac{\mu}{b_4\{2-x^2(1-e_1)\}}}$$

For optimum, we have

$$\frac{d}{dx}(V_1+V_2) = 0$$

i.e.

$$\sqrt{\frac{\mu(1-e_1)}{b_2}} + \sqrt{\frac{\mu(1-e_1)}{b_4}}\left[\frac{-x\sqrt{1-e_1}}{\{2-x^2(1-e_1)\}^{\frac{1}{2}}}\right] = 0$$

Finally

$$x = \pm\left\{\frac{2b_4}{(1-e_1)(b_2+b_4)}\right\}^{\frac{1}{2}} \tag{59}$$

4.4.

For the fourth configuration, when $\alpha = 0$, we have

$$\frac{dV_1}{dx} = \sqrt{\frac{\mu(1-e_1)}{b_2}} \qquad \text{and} \qquad V_2 = v_{B1}(1-\eta)$$

By substitution

$$V_2 = \sqrt{\frac{\mu}{b_3}}\left[\left\{2 - x^2\left(1 - e_1\right)\right\}^{\frac{1}{2}} - \sqrt{1 + e_2}\right]$$

$$\frac{dV_2}{dx} = \left\{-x\left(1 - e_1\right)\right\}\sqrt{\frac{\mu}{b_3\left\{2 - x^2\left(1 - e_1\right)\right\}}}$$

Applying optimum condition

$$\frac{d}{dx}\left(V_1 + V_2\right) = 0$$

Then

$$\sqrt{\frac{\mu\left(1 - e_1\right)}{b_2}} + \sqrt{\frac{\mu\left(1 - e_1\right)}{b_3}}\left[\frac{-x\sqrt{1 - e_1}}{\left\{2 - x^2\left(1 - e_1\right)\right\}^{\frac{1}{2}}}\right] = 0$$

i.e.

$$x = \pm\left\{\frac{2b_3}{\left(1 - e_1\right)\left(b_2 + b_3\right)}\right\} \tag{60}$$

Moreover, we may set $e_1 = e_2 = 0$ (the original Hohmann classical circular orbit transfer) and find the corresponding equalities.

5. Numerical Results

We regard the Earth–Mars transfer system, the data available are [10]

$a_E = 1.00000011$ $\quad e_E = 0.01671022$

$a_M = 1.52366231$ $\quad e_M = 0.09341233$

Let $\alpha = 25.5^0 = 0.4451\,\text{rad}$.

5.1.

Relevant to first configuration, we have

$u_1 = -5.3512 \quad u_2 = 11.2976 \quad u_3 = -11.2720 \quad u_4 = 4.4541$

By solving Eq. (49), using a Mathematica program, we acquire the following 8 roots:

$x_{1,2} = \pm 1.30043 \quad x_{3,4} = \pm 1.12807 \quad x_{5,6} = 1.14726 \pm 0.349919I$

$x_{7,8} = -1.14726 \pm 0.349919I$.

5.2.

Similarly relevant to second configuration,

$D_1 = -4.6236 \quad D_2 = 8.5415 \quad D_3 = -7.6817 \quad D_4 = 2.8272$

$x_{1,2} = \pm 1.24936 \quad x_{3,4} = \pm 1.08291 \quad x_{5,6} = -1.0459 \pm 0.385868\,I$

$x_{7,8} = 1.0459 \pm 0.385868\,I$

5.3.

With regard to third configuration, we have

$g_9 = -5.3043 \quad g_{10} = 11.0255 \quad g_{11} = -10.7918 \quad g_{12} = 4.1914.$

$x_{1,2} = \pm 1.30264 \quad x_{3,4} = \pm 1.13589 \quad x_{5,6} = -1.12743 \pm 0.335436\,I$

$x_{7,8} = 1.12743 \pm 0.335436\,I$

5.4.

As for fourth configuration,

$E_9 = -4.8370 \quad E_{10} = 9.4059 \quad E_{11} = -8.9407 \quad E_{12} = 3.4691$

$x_{1,2} = \pm 1.09547 \quad x_{3,4} = \pm 1.2786 \quad x_{5,6} = -1.07954 \pm 0.405399\, I$

$x_{7,8} = 1.07954 \pm 0.405399\, I$

Whence the following table

Table 1

Configuration	$x_{Min.}\ (\alpha \neq 0)$	$x_{Min.}(\alpha = 0)$
(I)	1.1281	1.1122
(II)	1.0822	1.0720
(III)	1.1359	1.1239
(IV)	1.0955	1.0824

6. Conclusions

We pointed out four feasible configurations for the elliptic Hohmann bi–impulsive transfer. We considered the problem as a change of energy of the vehicle. We computed the increments of energy Δv_A , Δv_B at peri–apo–apse due to motor thrusts, for the four feasible configurations Eqs. (20), (21), (31), (33), (37), (39), (43), (45). Our parameter is x [9] which is a measure of the initial perpendicular impulse at peri–apse and apo–apse. We established the optimum condition

$$\frac{d}{dx}(V_1 + V_2) = 0$$

using the cosine law and the application of ordinary infinitesimal calculus. We eliminated after a rather lengthy reductions and rearrangements the power fractions, and we attained an algebraic equation of the eigth degree in x, which could be resolved numerically. We determined the 8 roots for each of these four algebraic equations. The resolution yield real as well as complex roots, positive as well as negative roots. Only one real positive root is adequate and consistent with our assumptions, for the adopted Earth–Mars Hohmann elliptic transfer. A reduction of the problem to the coplanar transfer case occurs by setting $\alpha = 0$. A second degree algebraic equation in x results when we set $\alpha = 0$. We have the following four propositions corresponding to α , e_1, e_2 is equal to null or not:

(I) $\alpha \neq 0 \quad e_1, e_2 \neq 0$

(II) $\alpha \neq 0 \quad e_1, e_2 = 0$

(III) $\alpha = 0 \quad e_1, e_2 \neq 0$

(IV) $\alpha = 0 \quad e_1, e_2 = 0$

(I) represents the elliptic non–coplanar case whilst, (II) represents the circular non–coplanar case, and (III) relevant to the elliptic coplanar case, whilst (IV) is relevant to the circular coplanar case. It is obvious from table 1 that

$$[x]^{Min.}_{\alpha \neq 0} = 1.0822$$

represents the most economic expenditure of fuel since it is the least value of x for $\alpha \neq 0$, for the Earth–Mars transfer system. Similarly for $\alpha = 0$

$$[x]^{Min.}_{\alpha=0} = 1.0720$$

is the most economic. Both are relevant to the second configuration where apo–apse of transfer orbit coincides with peri–apse of final orbit, the primary is at left focus. We note that

$$[x]^{Min.}_{\alpha\neq 0} \rangle \, [x]^{Min.}_{\alpha=0}$$

which indicates that the energy needed at initial impulse for non–coplanar Hohmann elliptic transfers exceeds that for coplanar ones.

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Nomenclature:

a_1	semi–major axis of initial elliptic orbit
a_2	semi–major axis of final elliptic orbit
a_T	semi–major axis of transfer elliptic orbit
e_1	eccentricity of initial elliptic orbit
e_2	eccentricity of final elliptic orbit
e_T	eccentricity of transfer elliptic orbit
Δv_A	increment of velocity at point A
Δv_B	increment of velocity at point B
μ	constant of gravitation
r	radius vector measured from center of attraction
v	scalar velocity
$\Delta v_A + \Delta v_B$	classical characteristic velocity
x	single parameter of optimization
α	single plane change angle
η	velocity after second impulse / velocity before second impulse
V_1	magnitude of first impulse
V_2	magnitude of second impulse