Optimum Bi–Impulsive Non Coplanar Elliptic Hohmann Type Transfer

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We optimize the Hohmann type bi–impulsive transfer between inclined elliptic orbits having a common center of attraction, for the four feasible configurations. Our criterion for optimization is the characteristic velocity $\Delta v_T = \Delta v_1 + \Delta v_2$ which is a measure of fuel consumption. We assigned the optimum value of our variable $x$ (ratio between velocity after initial impulse and velocity before initial impulse) by a numerical solution of an algebraic eight degree equation. We have a single plane change angle $\alpha$. We present terse new formulae constituting a new alternative approach for tackling the problem. The derivations of formulae of our treatment are simple, straightforward and exceptionally clear. This is advantageous. By this semi–analytic analysis we avoid many complexities and ambiguity that appear in previous work.

Keywords: Rocket dynamics, transfer orbits, Hohmann transfer, optimization, astrodynamics, aerospace engineering, orbital mechanics

1. Introduction

We assume that the bi–impulsive thrusts concerned with the elliptic Hohmann transfer are provided by a space craft propulsion system. The total characteristic velocity for the transfer maneuver is given by $\Delta v_T = \Delta v_1 + \Delta v_2$. Orbital transfer is utilized in the majority of space crafts placed in orbit around the Earth. Orbit transfer is implemented to acquire a final orbit via a parking one. Periodic corrections should be executed due to perturbations acting on space crafts [1]. The velocity increments at peri–apse and apo–apse are directly proportional to fuel expenditure. Lawden investigated the optimization of a rocket vehicle transfer. He assumed the orbits to be elliptic and coplanar (The Lawden problem) [2]. Gravier et al derived the necessary equations for optimal transfer, in the real case when we take into account the
ellipticity and inclination of the orbits [3]. J. Baker discussed the problem of circular inclined orbits transfer for three cases [4]. L. Rider extended the impulsive thrusts transfer problem to non–coplanar orbits [5], J. Prussing analysed the optimized solution for the coplanar and restricted class of non–coplanar transfer problem [6]. We shall take into consideration the coplanar and non–coplanar Bi–elliptic transfer problem, for the four assumed configurations, strictly after the publication of this work [7].

2. Preliminary Concepts

2.1. Determination of the Feasible Configurations

We have the following four feasible configurations for the elliptic Hohmann type transfer

![Figure 1](image1)

![Figure 2](image2)
1. For Fig. 1, apo – apse of transfer orbit coincides with apo – apse of final orbit (L).

2. For Fig. 2, apo – apse of transfer orbit coincides with peri – apse of final orbit (L).

3. For Fig. 3, apo – apse of transfer orbit coincides with apo – apse of final orbit (R).

4. For Fig. 4, apo – apse of transfer orbit coincides with peri – apse of final orbit (R).

where (L) means that primary is at left focus and (R) means that primary at right focus.

The most convenient way is to regard the problem of orbit transfer as a problem in change of energy [8].
For Fig. 1:
\[ a_1 (1 - e_1) = a_T (1 - e_T) \quad a_T (1 + e_T) = a_2 (1 + e_2) \] (1)

i.e.
\[ 2a_T = a_1 (1 - e_1) + a_T (1 + e_T) \quad 2a_T = a_2 (1 + e_2) + a_T (1 - e_T) \] (2)

We get from Eq. (2) after little reduction:
\[ a_T = \frac{a_1 (1 - e_1) + a_2 (1 + e_2)}{2} \] (3)
\[ e_T = \frac{-a_1 (1 - e_1) + a_2 (1 + e_2)}{a_1 (1 - e_1) + a_2 (1 + e_2)} \]

For the transfer ellipse, we have the following equations for the total and kinetic energy:
\[ C_T = \frac{v_T^2}{2} - \frac{\mu}{r_T} \] (4)
\[ v_T^2 = \mu \left( \frac{2}{r_T} - \frac{1}{a_T} \right) \] (5)

Which yield
\[ C_T = -\frac{\mu}{2a_T} \] (6)

By Eq. (3), we get
\[ C_T = \frac{-\mu}{a_1 (1 - e_1) + a_2 (1 + e_2)} \] (7)

Also we have for the energy increment at point A
\[ \Delta C_A = C_T - C_1 \] (8)

But
\[ C_1 = -\frac{\mu}{2a_1} \] (9)

So, we acquire
\[ \Delta C_A = \frac{\mu}{2a_1} \left[ \frac{a_2 (1 + e_2) - a_1 (1 + e_1)}{a_2 (1 + e_2) + a_1 (1 - e_1)} \right] \] (10)

Similarly
\[ \Delta C_B = C_2 - C_T \] (11)
\[ C_2 = -\frac{\mu}{2a_2} \] (12)

Which gives
\[ \Delta C_B = \frac{\mu}{2a_2} \left[ \frac{a_2 (1 - e_2) - a_1 (1 - e_1)}{a_2 (1 + e_2) + a_1 (1 - e_1)} \right] \] (13)
For the change in kinetic energies at the two points A and B we have the following equalities:

\[
\Delta C_A = \frac{1}{2} \left\{ (v_A + \Delta v_A)^2 - v_A^2 \right\} \\
\Delta C_B = \frac{1}{2} \left\{ -(v_B - \Delta v_B)^2 + v_B^2 \right\}
\] (14) (15)

Where

\[
(v_A)_{Per.} = \left\{ \frac{\mu (1 + e_1)}{a_1 (1 - e_1)} \right\}^{1/2} \\
(v_B)_{Apo.} = \left\{ \frac{\mu (1 - e_2)}{a_2 (1 + e_2)} \right\}^{1/2}
\] (16) (17)

We now find the expressions for \(\Delta v_A\) and \(\Delta v_B\) which are the necessary changes in velocity at A and B respectively to execute the transfer by solving two second degree equations in \(\Delta v_A\) and \(\Delta v_B\), namely

\[
\Delta v_A^2 + 2v_A \Delta v_A - 2\Delta C_A = 0 \\
\Delta v_B^2 - 2v_B \Delta v_B + 2\Delta C_B = 0
\] (18) (19)

i.e.

\[
\Delta v_A = -v_A \pm (v_A^2 + 2\Delta C_A)^{1/2} \\
\Delta v_B = v_B \pm (v_B^2 - 2\Delta C_B)^{1/2}
\]

We get from the solution of (18), (19) and by Eqs (10), (13), (16) and (17):

\[
(\Delta v_A)_{Per.} = \left( \frac{\mu}{a_1} \right)^{1/2} \left[ \pm \left\{ 1 + e_1 + \frac{a_2 (1 + e_2) - a_1 (1 + e_1)}{a_2 (1 + e_2) + a_1 (1 - e_1)} \right\}^{1/2} \\
- \left\{ 1 + e_1 \right\}^{1/2} \right]
\] (20)

\[
(\Delta v_B)_{Apo.} = \left( \frac{\mu}{a_2} \right)^{1/2} \left[ \pm \left\{ 1 - e_2 + \frac{a_2 (1 - e_2) - a_1 (1 - e_1)}{a_2 (1 + e_2) + a_1 (1 - e_1)} \right\}^{1/2} \\
+ \left\{ 1 - e_2 \right\}^{1/2} \right]
\] (21)

For Fig. 2, we have the equalities

\[
a_T = \frac{a_1 (1 - e_1) + a_2 (1 - e_2)}{2} \\
e_T = \frac{a_2 (1 - e_2) - a_1 (1 - e_1)}{a_1 (1 - e_1) + a_2 (1 - e_2)}
\] (22) (23)
For Fig. 3, we have
\[ a_T = \frac{a_1 (1 + e_1) + a_2 (1 + e_2)}{2} \] (24)
\[ e_T = \frac{a_2 (1 + e_2) - a_1 (1 + e_1)}{a_1 (1 + e_1) + a_2 (1 + e_2)} \] (25)

For Fig. 4, we find
\[ a_T = \frac{a_1 (1 + e_1) + a_2 (1 - e_2)}{2} \] (26)
\[ e_T = \frac{a_2 (1 - e_2) - a_1 (1 + e_1)}{a_1 (1 + e_1) + a_2 (1 - e_2)} \] (27)

2.2. Change of vehicle’s energy

The analysis for case (I) has been discussed. For cases (II), (III), and (IV), we have the following relationships:

Case (II):
\[ \Delta C_A = \frac{\mu}{2a_1} \left\{ \frac{a_2 (1 - e_2) - a_1 (1 + e_1)}{a_2 (1 - e_2) + a_1 (1 - e_1)} \right\} \] (28)
\[ \Delta C_B = \frac{\mu}{2a_2} \left\{ \frac{a_2 (1 + e_2) - a_1 (1 - e_1)}{a_2 (1 - e_2) + a_1 (1 - e_1)} \right\} \] (29)
\[ (v_A)_{per.} = \left( \frac{\mu}{a_1} \right)^{1/2} \left\{ \frac{1 + e_1}{1 - e_1} \right\} \] (30)
\[ (\Delta v_A)_{per.} = \left( \frac{\mu}{a_1} \right)^{1/2} \left[ \frac{1 + e_1}{1 - e_1} \right] \] (31)
\[ (v_B)_{per.} = \left( \frac{\mu}{a_2} \right)^{1/2} \left\{ \frac{1 + e_2}{1 - e_2} \right\} \] (32)
\[ (\Delta v_B)_{per.} = \left( \frac{\mu}{a_2} \right)^{1/2} \left[ \frac{1 + e_2}{1 - e_2} \right] \] (33)

Case (III):
\[ \Delta C_A = \frac{\mu}{2a_1} \left\{ \frac{a_2 (1 + e_2) - a_1 (1 - e_1)}{a_2 (1 + e_2) + a_1 (1 + e_1)} \right\} \] (34)
\[ \Delta C_B = \frac{\mu}{2a_2} \left\{ \frac{a_2 (1 - e_2) - a_1 (1 + e_1)}{a_2 (1 - e_2) + a_1 (1 + e_1)} \right\} \] (35)
\[ (v_A)_{apo.} = \left( \frac{\mu}{a_1} \right)^{1/2} \left\{ \frac{1 - e_1}{1 + e_1} \right\} \] (36)
\[ (\Delta v_A)_{Apo.} = \left( \frac{\mu}{a_1} \right)^{\frac{1}{2}} \left[ -\left\{ 1 - e_1 \right\}^{\frac{1}{2}} \right] \]
\[ \pm \left\{ 1 - e_1 + \frac{a_2 (1 + e_2) - a_1 (1 - e_1)}{a_2 (1 + e_2) + a_1 (1 + e_1)} \right\}^{\frac{1}{2}} \]  
(37)

\[ (v_B)_{Apo.} = \left( \frac{\mu}{a_2} \right)^{\frac{1}{2}} \left\{ \frac{1 - e_2}{1 + e_2} \right\}^{\frac{1}{2}} \]  
(38)

\[ (\Delta v_B)_{Apo.} = \left( \frac{\mu}{a_2} \right)^{1/2} \left[ \left\{ \frac{1 - e_2}{1 + e_2} \right\}^{\frac{1}{2}} \right] \]
\[ \pm \left\{ 1 - e_2 + \frac{a_2 (1 - e_2) - a_1 (1 + e_1)}{a_2 (1 - e_2) + a_1 (1 + e_1)} \right\}^{\frac{1}{2}} \]  
(39)

Case (IV):

\[ \Delta C_A = \frac{\mu}{2a_1} \left\{ \frac{a_2 (1 - e_2) - a_1 (1 - e_1)}{a_2 (1 - e_2) + a_1 (1 + e_1)} \right\} \]  
(40)

\[ \Delta C_B = \frac{\mu}{2a_2} \left\{ \frac{a_2 (1 + e_2) - a_1 (1 + e_1)}{a_2 (1 - e_2) + a_1 (1 + e_1)} \right\} \]  
(41)

\[ (v_A)_{Apo.} = \left( \frac{\mu}{a_1} \right)^{\frac{1}{2}} \left\{ \frac{1 - e_1}{1 + e_1} \right\}^{\frac{1}{2}} \]  
(42)

\[ (\Delta v_A)_{Apo.} = \left( \frac{\mu}{a_1} \right)^{\frac{1}{2}} \left[ -\left\{ 1 - e_1 \right\}^{\frac{1}{2}} \right] \]
\[ \pm \left\{ 1 - e_1 + \frac{a_2 (1 - e_2) - a_1 (1 - e_1)}{a_2 (1 - e_2) + a_1 (1 + e_1)} \right\}^{\frac{1}{2}} \]  
(43)

\[ (v_B)_{Per.} = \left( \frac{\mu}{a_2} \right)^{1/2} \left\{ \frac{1 + e_2}{1 - e_2} \right\}^{\frac{1}{2}} \]  
(44)

\[ (\Delta v_B)_{Per.} = \left( \frac{\mu}{a_2} \right)^{1/2} \left[ \left\{ \frac{1 + e_2}{1 - e_2} \right\}^{\frac{1}{2}} \right] \]
\[ \pm \left\{ 1 + e_2 + \frac{a_2 (1 + e_2) - a_1 (1 + e_1)}{a_2 (1 - e_2) + a_1 (1 + e_1)} \right\}^{\frac{1}{2}} \]  
(45)

Evidently $\Delta v_A$ and $\Delta v_B$ are functions of the parameters $a_1$, $a_2$, $e_1$, and $e_2$.

We place here Fig. 5, showing the variation of $\alpha = a_2/a_1$ versus the total transfer impulse $\Delta v_A + \Delta v_B$ (classical characteristic velocity).

A bundle of curves appear corresponding to $e_1 = 0.1$ ; $e_2 = 0.1 - 0.9$.

In Fig. 5, the total impulse increase drastically for the range $\alpha \approx 0.0 - 10.0$, then increase slowly, indicating that the consumption of fuel is greatest for the above range.
3. Presentation of the Method of Optimization

3.1. For the first configuration

Let \( v_A \) be the velocity of the satellite in initial ellipse \((a_1, e_1)\), and \( x_{vA} \) be the velocity in transfer orbit \((a_T, e_T)\) after first impulse \( V_1 \).

We have

\[
x_{vA} - v_A = V_1
\]

where

\[
x = \frac{\text{the velocity after first impulse}}{\text{the velocity before first impulse}} = \sqrt{\frac{\mu(1+e_T)}{a_T(1-e_T)}}
\]

From Fig. 1,

\[
a_1 (1-e_1) = a_T (1-e_T)
\]

i.e.

\[
x = \sqrt{\frac{1+e_T}{1+e_1}}
\]
\[ V_1 = \sqrt{\frac{\mu (1 + e_1)}{a_1 (1 - e_1)}} \left\{ \frac{1 + e_T}{1 + e_1} - 1 \right\} \]
\[ V_1 = \sqrt{\frac{\mu}{a_1 (1 - e_1)}} \left\{ x\sqrt{1 + e_1} - \sqrt{1 + e_1} \right\} \]
\[ V_1 = \sqrt{\frac{\mu (1 + e_1)}{b_1}} (x - 1) \]

where \( b_1 = a_1 (1 - e_1) \)

we get
\[ \frac{dV_1}{dx} = \sqrt{\frac{\mu (1 + e_1)}{b_1}} \] (46)

Now, let \( v_{B1} \) be the velocity at B before the second impulse \( V_2 \), and \( v_{B2} \) be the velocity at B after the second impulse \( V_2 \). We have
\[ v_{B1} = \sqrt{\frac{\mu (1 - e_T)}{a_T (1 + e_T)}} \]
\[ v_{B2} = \sqrt{\frac{\mu (1 - e_2)}{a_2 (1 + e_2)}} \]

Denote
\[ \eta^2 = \frac{v_{B2}^2}{v_{B1}^2} = \frac{1 - e_2}{1 - e_T} \]

Where, from Fig. 1,
\[ a_T (1 + e_T) = a_2 (1 + e_2) \]

We get from the definition of \( x \), that
\[ 1 - e_T = 2 - x^2 (1 + e_1) \]

Then we write,
\[ \eta^2 = \frac{1 - e_2}{2 - x^2 (1 + e_1)} = \frac{2\mu}{2\mu + x^2 (1 - \mu) (1 + e_1)} \]

Where \( \mu \) – ratio of periapses distances of terminal orbits
\[ \mu = a_2 (1 - e_2) / a_1 (1 - e_1) \]

Then,
\[ \frac{d\eta}{dx} = \frac{x \sqrt{1 - e_2 (1 + e_1)}}{\left\{ 2 - x^2 (1 + e_1) \right\}^{\frac{3}{2}}} \]

If \( \alpha \) be the angle according to a plane change, we get
\[ V_2 = v_{B1} \left( 1 + \eta^2 - 2\eta \cos \alpha \right)^{\frac{1}{2}} \] (47)
We have,

\[ v_{B1} = \sqrt{\frac{\mu}{b_4}} \sqrt{2 - x^2(1 + \epsilon_1)} \quad \text{where} \quad b_4 = a_2(1 + \epsilon_2) \]

Then,

\[ \frac{dv_{B1}}{dx} = \sqrt{\frac{\mu}{b_4}} \left[ \frac{-x(1 + \epsilon_1)}{(2 - x^2(1 + \epsilon_1))^{\frac{3}{2}}} \right] \]

Therefore,

\[ \frac{dV_2}{dx} = v_{B1} \left\{ \frac{\eta \frac{d\eta}{dx} - \cos \alpha \frac{d\eta}{dx}}{(1 + \eta^2 - 2\eta \cos \alpha)^{1/2}} \right\} + \frac{dv_{B1}}{dx} \left(1 + \eta^2 - 2\eta \cos \alpha\right)^{\frac{3}{2}} \]

and after some reductions,

\[ \frac{dV_2}{dx} = \sqrt{\frac{\mu}{b_4}} G \quad (48) \]

where

\[
G = \left[ \frac{x(1+\epsilon_1)\left\{\sqrt{1-\epsilon_2(\eta-\cos \alpha)} - \sqrt{2-x^2(1+\epsilon_1)(1+\eta^2-2\eta \cos \alpha)}\right\}}{(2-x^2(1+\epsilon_1))(1+\eta^2-2\eta \cos \alpha)} \right] \]

The optimum condition is the following:

\[ \frac{dV_T}{dx} = \frac{dV_1}{dx} + \frac{dV_2}{dx} = 0 \]

From Eqs (46) and (48) we get,

\[ \frac{1}{\sqrt{b_1}} + \sqrt{\frac{1 + \epsilon_1}{b_4}} G_1 = 0 \]

where

\[
G_1 = \left[ \frac{x\left\{\sqrt{1-\epsilon_2(\eta-\cos \alpha)} - \sqrt{2-x^2(1+\epsilon_1)(1+\eta^2-2\eta \cos \alpha)}\right\}}{(2-x^2(1+\epsilon_1))(1+\eta^2-2\eta \cos \alpha)} \right] \]

and after some algebraic calculus computations:

\[ \frac{1}{\sqrt{b_1}} + \sqrt{\frac{1 + \epsilon_1}{b_4}} G_2 = 0 \]

where

\[
G_2 = \left[ \frac{x\left\{\sqrt{1-\epsilon_2(\eta-\cos \alpha)} - \sqrt{2-x^2(1+\epsilon_1)}\right\}}{(2-x^2(1+\epsilon_1))(1+\epsilon_2)x^2(1+\epsilon_1))} \right] \]

Then,

\[ \frac{1}{b_1} = \frac{1 + \epsilon_1}{b_4} G_3 \]

where

\[
G_3 = \left[ \frac{x^2\left\{(1-\epsilon_2)e^\alpha - 2\sqrt{(1-\epsilon_2)(2-x^2(1+\epsilon_1))} \right\} \cos \alpha + 2-x^2(1+\epsilon_1)}{(2-x^2(1+\epsilon_1))(1+\epsilon_2)x^2(1+\epsilon_1))} \right] \]
Let
\[-(1 + e_1) (b_1 + b_4) = C_1 \quad -(1 + e_1) (b_1 \cos^2 \alpha + b_4) = C_2\]

Whence
\[
2 \sqrt{1 - e_2} \{2 - x^2 (1 + e_1)\} \cos \alpha \left( x^2 C_1 + 2b_4 \right)
- \{2 - x^2 (1 + e_1)\} \left( x^2 C_1 + 2b_4 \right)
+ (1 - e_2) \{2 - x^2 (1 + e_1)\} \left( x^2 C_2 + 2b_4 \right)
\]

By squaring we get
\[
4 (1 - e_2) \{2 - x^2 (1 + e_1)\} \cos^2 \alpha \left( x^4 C_1^2 + 4x^2 C_1 b_4 + 4b_4^2 \right)
= \left\{ 4 - 4x^2 (1 + e_1) + x^4 (1 + e_1)^2 \right\} \left( x^4 C_1^2 + 4x^2 C_1 b_4 + 4b_4^2 \right)
+ 2 (1 - e_2) \{2 - x^2 (1 + e_1)\} \left( x^2 C_1 + 2b_4 \right) \left( x^2 C_2 + 2b_4 \right)
+ (1 - e_2)^2 \left( x^4 C_2^2 + 4x^2 C_2 b_4 + 4b_4^2 \right)
\]
i.e.
\[
-q_1^2 x^8 + x^6 \left\{ 4 q_1 C_1 - 4 q_1 (1 - e_2) C_1 \cos^2 \alpha - 4 q_1 (1 + e_1) b_4 + 2 q_1 (1 - e_2) C_2 \right\}
+ x^4 \left\{ 8 (1 - e_2) C_1^2 \cos^2 \alpha - 16 (1 - e_2) q_1 b_4 \cos^2 \alpha - 4 C_1^2 b_4^2 \right\}
+ x^2 \left\{ 32 (1 - e_2) C_1 b_4 \cos^2 \alpha - 16 (1 - e_2) (1 + e_1) b_4^2 \cos^2 \alpha - 16 C_1 b_4^2 \right\}
\]
where
\[q_1 = C_1 (1 + e_1)\]

whence, we obtain the algebraic equation of the eighth degree, namely:
\[
x^8 + u_1 x^6 + u_2 x^4 + u_3 x^2 + u_4 = 0 \quad (49)
\]
where

\[ u_1 = - \frac{4C_1 - 4(1 - e_2)tC_1 \cos^2 \alpha - 4(1 + e_1)b_4 + 2(1 - e_2C_2)}{q_1} \]

\[ u_2 = - \frac{\left\{ 8(1 - e_2)C_1^2 \cos^2 \alpha - 16(1 - e_2)q_1b_4 \cos^2 \alpha - 4C_1^2 + 16q_1b_4 \right\}}{q_1^2} \]

\[ u_3 = - \frac{\left\{ 32(1 - e_2)C_1b_4 \cos^2 \alpha - 16(1 - e_2)(1 + e_1)b_2^2 \cos^2 \alpha - 16C_1b_4 \right\}}{q_1^2} \]

\[ u_4 = - \frac{\left\{ 32(1 - e_2)b_2^2 \cos^2 \alpha - 16b_4^2 - 16(1 - e_2)b_4^2 - 4(1 - e_2)^2b_4 \right\}}{q_1^2} \]

### 3.2. For the second configuration

We have the additional equations:

\[ a_T(1 + e_T) = a_2(1 - e_2) \]

\[ v_{B1} = \sqrt{\frac{\mu(1 - e_T)}{a_T(1 + e_T)}} \]

\[ v_{B2} = \sqrt{\frac{\mu(1 + e_2)}{a_2(1 - e_2)}} \]

and

\[ \eta^2 = \frac{v_{B2}^2}{v_{B1}^2} = \frac{1 + e_2}{1 - e_T} \]

but

\[ 1 - e_T = 2 - x^2(1 + e_1) \]

then

\[ \eta^2 = \frac{1 + e_2}{2 - x^2(1 + e_1)} \]

By differentiation w.r.t. \( x \), we get

\[ \frac{d\eta}{dx} = \frac{x(1 + e_1)\sqrt{1 + e_2}}{\left\{2 - x^2(1 + e_1)\right\}^2} \]

The second impulse

\[ V_2 = v_{B1}\sqrt{1 + \eta^2 - 2\eta \cos \alpha} \]

with

\[ v_{B1} = \sqrt{\frac{\mu\left\{2 - x^2(1 + e_1)\right\}}{b_3}} \quad \text{where} \quad b_3 = a_2(1 - e_2) \]
Then
\[
\frac{dv_{B1}}{dx} = \sqrt{\frac{\mu}{b_3}} \left[ -x(1+e_1) \right] \]
whence
\[
\frac{dv_2}{dx} = \frac{dv_{B1}}{dx} \sqrt{1 + \eta^2 - 2\eta \cos \alpha} + \nu_{B1} \left[ \frac{d\nu}{dx} \right] (\eta - \cos \alpha) \left[ \sqrt{1 + \eta^2 - 2\eta \cos \alpha} \right]
\]

\[
\frac{dv_2}{dx} = \sqrt{\frac{\mu}{b_3}} \frac{x(1+e_1)}{\sqrt{2 - x^2(1+e_1)}} \left[ \sqrt{1 + e_2 \cos \alpha} - \sqrt{2 - x^2(1+e_1)} \right]
\]

\[
= \left[ \frac{2 - x^2(1+e_1)}{x(1+e_1)} \right] + (1 + e_2) - 2\sqrt{1 + e_2 \cos \alpha} \sqrt{2 - x^2(1+e_1)} \]

(50)
The optimum condition:
\[
\frac{dV_1}{dx} + \frac{dv_2}{dx} = 0
\]
Then
\[
-\frac{1}{\sqrt{b_1}} = \frac{1 + e_1}{b_3} \frac{x}{\sqrt{2 - x^2(1+e_1)}} \left[ \sqrt{1 + e_2 \cos \alpha} - \sqrt{2 - x^2(1+e_1)} \right]
\]

\[
= \left[ \frac{2 - x^2(1+e_1)}{x(1+e_1)} \right] + (1 + e_2) - 2\sqrt{1 + e_2 \cos \alpha} \sqrt{2 - x^2(1+e_1)} \]

Therefore
\[
x^2b_1(1+e_1)(1+e_2)\cos^2\alpha - 2x^2b_1(1+e_1)\sqrt{1 + e_2} \left\{ 2 - x^2(1+e_1) \right\} \frac{1}{2} \cos \alpha
\]
\[
+ x^2b_1(1+e_1) \left\{ 2 - x^2(1+e_1) \right\} = b_3 \left\{ 2 - x^2(1+e_1) \right\}^2
\]
\[
+ b_3(1+e_2) \left\{ 2 - x^2(1+e_1) \right\} - 2b_3\sqrt{1 + e_2 \cos \alpha} \left\{ 2 - x^2(1+e_1) \right\} \frac{1}{2} \cos \alpha
\]

Let
\[
C_3 = b_1(1+e_1)(1+e_2)\cos^2\alpha \quad C_4 = -2b_1(1+e_1)\sqrt{1 + e_2 \cos \alpha}
\]
\[
C_5 = b_3(1+e_1) \quad C_6 = b_3(1+e_2) \quad C_7 = -2b_3\sqrt{1 + e_2 \cos \alpha}
\]
i.e.
\[
x^2 \left\{ C_3 + 2C_5 + 4b_3(1-e_1) + C_6(1+e_1) \right\} + x^4 \left\{ -C_5(1+e_1) - b_3(1+e_1)^2 \right\}
\]
\[
-4b_3 - 2C_6 = C_7 \left\{ 2 - x^2(1+e_1) \right\} \frac{1}{2} - x^2C_4 \left\{ 2 - x^2(1+e_1) \right\} \frac{1}{2}
\]

Let
\[
C_8 = C_3 + 2C_5 + 4b_3(1-e_1) + C_6(1+e_1)
\]
\[
C_9 = -C_5(1+e_1) - b_3(1+e_1)^2 \quad C_{10} = -4b_3 - 2C_6
By squaring, we get
\[C_9^2 x^8 + x^6 \left\{2C_8 C_9 + C_7^2 (1 + e_1)^3 + 2C_4 C_7 (1 + e_1)^2 + C_4^2 (1 + e_1)\right\} +
+ x^4 \left\{C_8^2 + 2C_9 C_{10} - 6C_7^2 (1 + e_1)^2 - 8C_4 C_7 (1 + e_1) - 2C_4^2\right\} +
+ x^2 \left\{2C_8 C_{10} + 12C_7^2 (1 + e_1) + 8C_4 C_7 \right\} + C_{10}^2 - 8C_7^2 = 0\]

Put,
\[
D_1 = \frac{\left\{2C_8 C_9 + C_7^2 (1 + e_1)^3 + 2C_4 C_7 (1 + e_1)^2 + C_4^2 (1 + e_1)\right\}}{C_9^2}
\]
\[
D_2 = \frac{\left\{C_8^2 + 2C_9 C_{10} - 6C_7^2 (1 + e_1)^2 - 8C_4 C_7 (1 + e_1) - 2C_4^2\right\}}{C_9^2}
\]
\[
D_3 = \frac{\left\{2C_8 C_{10} + 12C_7^2 (1 + e_1) + 8C_4 C_7 \right\}}{C_9^2}
\]
\[
D_4 = \frac{C_{10}^2 - 8C_7^2}{C_9^2}
\]

Therefore, we get an algebraic equation of the eight degree in \(x\),
\[x^8 + D_1 x^6 + D_2 x^4 + D_3 x^2 + D_4 = 0 \quad (51)\]

### 3.3. For the third configuration

The initial impulse is at apo–apse.

We have the following relationships:

\[x = \frac{\text{velocity after initial impulse}}{\text{velocity before initial impulse}} = \sqrt{\frac{\mu (1 + e_T)}{a_T (1 - e_T)}} \sqrt{\frac{\mu (1 - e_1)}{a_1 (1 + e_1)}}\]

For Fig. 3,
\[a_T (1 - e_T) = a_1 (1 + e_1)\]

Then,
\[x = \sqrt{\frac{1 + e_T}{1 - e_1}} \quad 1 + e_T = x^2 (1 - e_1) \quad 1 - e_T = 2 - x^2 (1 - e_1)\]

But, we have
\[V_1 = v_A (x - 1) = (x - 1) \sqrt{\frac{\mu (1 - e_1)}{a_1 (1 + e_1)}}\]

i.e.
\[V_1 = (x - 1) \sqrt{\frac{\mu (1 - e_1)}{b_2}} \quad \text{where} \quad b_2 = a_1 (1 + e_1)\]
Therefore,

\[
\frac{dV_1}{dx} = \sqrt{\frac{\mu (1 - e_1)}{b_2}}
\]  

(52)

We have, from Fig. 3.

\[a_T (1 + e_T) = a_2 (1 + e_2)\]

and

\[\eta^2 = \frac{v_{B2}^2}{v_{B1}^2} = \frac{1 - e_2}{1 - e_T}\]

\[v_{B1} = \sqrt{\frac{\mu (1 - e_T)}{a_T (1 + e_T)}}\]

\[v_{B2} = \sqrt{\frac{\mu (1 - e_2)}{a_2 (1 + e_2)}}\]

whence

\[\eta^2 = \frac{1 - e_2}{2 - x^2 (1 - e_1)}\]

\[\eta = \sqrt{\frac{1 - e_2}{2 - x^2 (1 - e_1)}}\]

\[\frac{d\eta}{dx} = \frac{x (1 - e_1) \sqrt{1 - e_2}}{(2 - x^2 (1 - e_1))^\frac{3}{2}}\]

\[v_{B1} = \sqrt{\frac{\mu \{2 - x^2 (1 - e_1)\}}{b_4}}\]  with  \[b_4 = a_2 (1 + e_2)\]

\[\frac{dv_{B1}}{dx} = \sqrt{\frac{\mu}{b_4}} \left[ -\frac{x (1 - e_1)}{(2 - x^2 (1 - e_1))^\frac{3}{2}} \right]\]

The second impulse,

\[V_2 = v_{B1} \sqrt{1 + \eta^2 - 2\eta \cos \alpha}\]

\[\frac{dV_2}{dx} = \frac{dv_{B1}}{dx} \sqrt{1 + \eta^2 - 2\eta \cos \alpha} + v_{B1} \left[ \frac{(\eta - \cos \alpha) \frac{dn}{dx}}{\sqrt{1 + \eta^2 - 2\eta \cos \alpha}} \right]\]

Substituting for \(v_{B1}, dv_{B1}/dx, \eta, d\eta/dx\) we may write

\[\frac{dV_2}{dx} = \sqrt{\frac{\mu}{b_4}} \frac{x (1 - e_1)}{(2 - x^2 (1 - e_1))^{\frac{3}{2}}} \]

\[\left[ \frac{\sqrt{1 - e_2 \cos \alpha - \{2 - x^2 (1 - e_1)\}^{\frac{1}{2}}}{(2 - x^2 (1 - e_1)) + (1 - e_2) - 2 \sqrt{(1 - e_2) \{2 - x^2 (1 - e_1)\} \cos \alpha}} \right] \]

(53)
But we have for optimum condition

\[ \frac{dV_1}{dx} + \frac{dV_2}{dx} = 0 \]

From Eqs (52) and (53), we get

\[
\frac{1}{\sqrt{b_2}} = \frac{-x\sqrt{1 - e_1}}{\sqrt{b_4 \{2 - x^2(1 - e_1)\}}} \\
\left[ \sqrt{1 - e_2} \cos \alpha - \left\{ 2 - x^2(1 - e_1) \right\}^{1/2} \right] \\
\left[ \{2 - x^2(1 - e_1)\} + (1 - e_2) - 2\sqrt{(1 - e_2) \{2 - x^2(1 - e_1)\}} \cos \alpha \right]^{1/2}
\]

\[
x^2b_2(1 - e_1)(1 - e_2)\cos^2 \alpha - 2x^2b_2(1 - e_1)\sqrt{1 - e_2}\{2 - x^2(1 - e_1)\}^{1/2} \cos \alpha \\
+ x^2b_2(1 - e_1)\{2 - x^2(1 - e_1)\} = b_4\{2 - x^2(1 - e_1)\} \left\{ \{2 - x^2(1 - e_1)\} \right\}^{1/2} \cos \alpha \\
+ (1 - e_2) - 2\sqrt{1 - e_2}\{2 - x^2(1 - e_1)\}^{1/2} \cos \alpha
\]

Let

\[
g_1 = b_2(1 - e_1)(1 - e_2)\cos^2 \alpha \quad g_2 = -2b_2(1 - e_1)\sqrt{1 - e_2} \cos \alpha \\
g_3 = b_2(1 - e_1) \quad g_4 = b_4(1 - e_2) \quad g_5 = -2b_4\sqrt{1 - e_2} \cos \alpha \\
d\eta = \frac{x(1 - e_1)\sqrt{1 - e_2}}{(2 - x^2(1 - e_1))^{1/2}}
\]

whence

\[
x^2\{g_1 + 2g_3 + 4b_4(1 - e_1) + g_4(1 - e_1)\} + x^4\left\{ -g_3(1 - e_1) - b_4(1 - e_1)^2 \right\} \\
- 4b_4 - 2g_4 = g_5\{2 - x^2(1 - e_1)\}^{1/2} - x^2g_2\{2 - x^2(1 - e_1)\}^{1/2}
\]

put

\[
g_6 = g_1 + 2g_3 + 4b_4(1 - e_1) + g_4(1 - e_1) \\
g_7 = -g_3(1 - e_1) - b_4(1 - e_1)^2 \\
g_8 = -4b_4 - 2g_4
\]

After squaring, we get

\[
g_7^2x^8 + x^6\left\{ 2g_6g_7 + g_5^2(1 - e_1)^3 + 2g_2g_5(1 - e_1)^2 + g_2^2(1 - e_1) \right\} + \\
x^4\left\{ g_6^2 + 2g_7g_8 - 6g_5^2(1 - e_1)^2 - 8g_2g_5(1 - e_1) - 2g_2^2 \right\} + \\
x^2\left\{ 2g_6g_8 + 12g_5^2(1 - e_1) + 8g_2g_5 \right\} + g_8^2 - 8g_5^2 = 0
\]

Let

\[
g_9 = \left\{ 2g_6g_7 + g_5^2(1 - e_1)^3 + 2g_2g_5(1 - e_1)^2 + g_2^2(1 - e_1) \right\} / g_7^2 \\
g_{10} = \left\{ g_6^2 + 2g_7g_8 - 6g_5^2(1 - e_1)^2 - 8g_2g_5(1 - e_1) - 2g_2^2 \right\} / g_7^2 \\
g_{11} = \left\{ 2g_6g_8 + 12g_5^2(1 - e_1) + 8g_2g_5 \right\} / g_7^2 \\
g_{12} = \left\{ g_8^2 - 8g_5^2 \right\} / g_7^2
\]

Therefore, we get an equation of degree eight in \( x \) in the form

\[
x^8 + g_9x^6 + g_{10}x^4 + g_{11}x^2 + g_{12} = 0 \quad (54)
\]
3.4. For the fourth configuration

The initial impulse is at apo-apse.

And we have

\[ v_{B1} = \sqrt{\frac{\mu (1 - e_T)}{a_T (1 + e_T)}} \quad v_{B2} = \sqrt{\frac{\mu (1 + e_2)}{a_2 (1 - e_2)}} \]

From Fig. 4, we notice that

\[ a_T (1 + e_T) = a_2 (1 - e_2) \]

Therefore,

\[ v_{B1} = \sqrt{\frac{\mu \{2 - x^2 (1 - e_1)\}}{b_3}} \quad v_{B2} = \sqrt{\frac{\mu (1 + e_2)}{b_3}} \]

where \( b_3 = a_2 (1 - e_2) \)

\[ \frac{dv_{B1}}{dx} = \frac{\mu}{b_3} \left[ \frac{-x (1 - e_1)}{(2 - x^2 (1 - e_1))^2} \right] \]

Let

\[ \eta = \frac{v_{B2}}{v_{B1}} = \sqrt{\frac{1 + e_2}{2 - x^2 (1 - e_1)}} \]

\[ \frac{d\eta}{dx} = \frac{x (1 - e_1) \sqrt{1 + e_2}}{(2 - x^2 (1 - e_1))^{\frac{3}{2}}} \]

\[ V_2 = v_{B1} \sqrt{1 + \eta^2 - 2\eta \cos \alpha} \]

\[ \frac{dV_2}{dx} = \frac{dv_{B1}}{dx} \left(1 + \eta^2 - 2\eta \cos \alpha\right)^{\frac{1}{2}} + v_{B1} \left[ \frac{(\eta - \cos \alpha)}{(1 + \eta^2 - 2\eta \cos \alpha)^{\frac{1}{2}}} \right] \]

\[ \frac{dV_2}{dx} = \sqrt{\frac{\mu}{b_3 \{2 - x^2 (1 - e_1)\}}} \left\{ x (1 - e_1) \right\} \]

\[
\begin{bmatrix}
\sqrt{1 + e_2 \cos \alpha - \left\{ 2 - x^2 (1 - e_1) \right\}^{\frac{1}{2}}}
\left\{ 2 - x^2 (1 - e_1) \right\} + (1 + e_2) - 2\sqrt{(1 + e_2) \left\{ 2 - x^2 (1 - e_1) \right\} \cos \alpha}^{\frac{1}{2}}
\end{bmatrix}
\]

But, from optimum condition

\[ \frac{d}{dx} (V_1 + V_2) = 0 \]

Using Eqs (52) and (55), we get

\[ -\frac{1}{\sqrt{b_2}} = \frac{x \sqrt{1 - e_1}}{\sqrt{b_3 \{2 - x^2 (1 - e_1)\}}} \]

\[
\begin{bmatrix}
\sqrt{1 + e_2 \cos \alpha - \left\{ 2 - x^2 (1 - e_1) \right\}^{\frac{1}{2}}}
\left\{ 2 - x^2 (1 - e_1) \right\} + (1 + e_2) - 2\sqrt{(1 + e_2) \left\{ 2 - x^2 (1 - e_1) \right\} \cos \alpha}^{\frac{1}{2}}
\end{bmatrix}
\]
By squaring, we get

\[x^2 b_2 \left(1 - e_1\right) \left(1 + e_2\right) \cos^2 \alpha - 2x^2 b_2 \left(1 - e_1\right) \sqrt{1 + e_2} \left\{2 - x^2 \left(1 - e_1\right)\right\}^{\frac{3}{2}} \cos \alpha + x^2 b_2 \left(1 - e_1\right) \left\{2 - x^2 \left(1 - e_1\right)\right\} = b_3 \left\{2 - x^2 \left(1 - e_1\right)\right\}^2 + b_3 \left(1 + e_2\right) \left\{2 - x^2 \left(1 - e_1\right)\right\} - 2b_3 \sqrt{1 + e_2} \left\{2 - x^2 \left(1 - e_1\right)\right\}^{\frac{3}{2}} \cos \alpha\]

Let

\[E_1 = b_2 \left(1 - e_1\right) \left(1 + e_2\right) \cos^2 \alpha \quad E_2 = -2b_2 \left(1 - e_1\right) \sqrt{1 + e_2} \cos \alpha \]

\[E_3 = b_2 \left(1 - e_1\right) \quad E_4 = b_3 \left(1 + e_2\right) \quad E_5 = -2b_3 \sqrt{1 + e_2} \cos \alpha\]

Then

\[x^2 \left[E_1 + E_2 \left\{2 - x^2 \left(1 - e_1\right)\right\}^{\frac{3}{2}} + E_3 \left\{2 - x^2 \left(1 - e_1\right)\right\}\right] = \left\{2 - x^2 \left(1 - e_1\right)\right\} \left[b_3 \left\{2 - x^2 \left(1 - e_1\right)\right\} + E_4 + E_5 \left\{2 - x^2 \left(1 - e_1\right)\right\}^{\frac{3}{2}}\right]\]

i.e.

\[x^2 \left[E_1 + 2E_3 + 4b_3 \left(1 - e_1\right) + E_4 \left(1 - e_1\right)\right] + x^4 \left\{-E_3 \left(1 - e_1\right) - b_3 \left(1 - e_1\right)^2\right\} - 4b_3 - 2E_4 = E_5 \left\{2 - x^2 \left(1 - e_1\right)\right\}^{\frac{3}{2}} - x^2 E_2 \left\{2 - x^2 \left(1 - e_1\right)\right\}^{\frac{7}{2}}\]

Put

\[E_6 = E_1 + 2E_3 + 4b_3 \left(1 - e_1\right) + E_4 \left(1 - e_1\right) \quad E_7 = -E_3 \left(1 - e_1\right) - b_3 \left(1 - e_1\right)^2 \quad E_8 = -4b_3 - 2E_4\]

By squaring, we get

\[(x^2 E_6 + x^4 E_7 + E_8)^2 = E_6^2 \left\{2 - x^2 \left(1 - e_1\right)\right\}^3 - 2E_2 E_5 x^2 \left\{2 - x^2 \left(1 - e_1\right)\right\}^2 + E_8^2 x^4 \left\{2 - x^2 \left(1 - e_1\right)\right\}\]

i.e.

\[E_7 x^8 + x^6 \left\{2E_6 E_7 + E_6^2 \left(1 - e_1\right)^3 + 2E_2 E_5 \left(1 - e_1\right)^2 + E_2 \left(1 - e_1\right)\right\} + x^4 \left\{E_6^2 + 2E_7 E_8 - 6E_6 \left(1 - e_1\right)^2 - 8E_2 E_5 \left(1 - e_1\right) - 2E_2^2\right\} + x^2 \left\{2E_6 E_8 + 12E_6 \left(1 - e_1\right) + 8E_2 E_5\right\} + E_8^2 - 8E_6^2 = 0\]

Let

\[E_9 = \left\{2E_6 E_7 + E_6^2 \left(1 - e_1\right)^3 + 2E_2 E_5 \left(1 - e_1\right)^2 + E_2 \left(1 - e_1\right)\right\} / E_7^2\]

\[E_{10} = \left\{E_6^2 + 2E_7 E_8 - 6E_6 \left(1 - e_1\right)^2 - 8E_2 E_5 \left(1 - e_1\right) - 2E_2^2\right\} / E_7^2\]

\[E_{11} = \left\{2E_6 E_8 + 12E_6 \left(1 - e_1\right) + 8E_2 E_5\right\} / E_7^2\]

\[E_{12} = \left\{E_8^2 - 8E_6^2\right\} / E_7^2\]

Finally, we get an equation of degree eight in \(x\)

\[x^8 + E_9 x^6 + E_{10} x^4 + E_{11} x^2 + E_{12} = 0 \quad (56)\]
4. Reduction to coplanar classical Hohmann system

4.1.
Evidently when we set $\alpha = 0$, we reduce to the coplanar case. For the first configuration, we have

$$\frac{dV_1}{dx} = \sqrt{\frac{\mu (1 + e_1)}{b_1}}$$

$$v_{B1} = \sqrt{\frac{\mu \{2 - x^2 (1 + e_1)\}}{b_4}}$$

$$\frac{dv_{B1}}{dx} = \{-x (1 + e_1)\} \sqrt{\frac{\mu}{b_4 \{2 - x^2 (1 + e_1)\}}}$$

Set $\alpha = 0$ in Eq. (47), then

$$V_2 = v_{B1} (1 - \eta)$$

$$\frac{dV_2}{dx} = v_{B1} (-\frac{d\eta}{dx}) + \frac{dv_{B1}}{dx} (1 - \eta)$$

After some reductions, the first derivative is given by

$$\frac{dV_2}{dx} = \{-x (1 + e_1)\} \sqrt{\frac{\mu}{b_4 \{2 - x^2 (1 + e_1)\}}}$$

The optimum condition is

$$\frac{d}{dx} (V_1 + V_2) = 0$$

i.e.

$$\sqrt{\frac{\mu (1 + e_1)}{b_1}} + \{-x (1 + e_1)\} \sqrt{\frac{\mu}{b_4 \{2 - x^2 (1 + e_1)\}}} = 0$$

From which

$$x = \pm \left\{ \frac{2b_4}{(1 + e_1) (b_1 + b_4)} \right\}^{\frac{1}{2}}$$

(57)

4.2.
Similarly for the second configuration, we have

$$\frac{dV_1}{dx} = \sqrt{\frac{\mu (1 + e_1)}{b_1}}$$

$$V_2 = v_{B1} (1 + \eta^2 - 2\eta)^{\frac{1}{2}} = v_{B1} (1 - \eta)$$

$$\frac{dV_2}{dx} = v_{B1} \frac{d}{dx} (1 - \eta) + \frac{dv_{B1}}{dx} (1 - \eta)$$
By a similar analysis, we get

\[
\frac{dV_2}{dx} = \{-x(1+e_1)\} \sqrt{\frac{\mu}{b_3 \{2-x^2(1+e_1)\}}}
\]

From the optimum condition

\[
\frac{d}{dx} (V_1 + V_2) = 0
\]

We get

\[
\frac{1}{\sqrt{b_1}} = x \sqrt{\frac{1+e_1}{b_3 \{2-x^2(1+e_1)\}}}
\]

i.e.

\[
x = \pm \left\{ \frac{2b_3}{(1+e_1)(b_1 + b_3)} \right\}^{\frac{1}{2}}
\]

**4.3.**

Similarly for the third configuration, when \( \alpha = 0 \), we have

\[
\frac{dV_1}{dx} = \sqrt{\frac{\mu (1-e_1)}{b_2}} \quad \text{and} \quad V_2 = v_B (1-\eta)
\]

By substitution

\[
V_2 = \frac{\mu}{b_4} \left\{2 - x^2(1-e_1)\right\}^{1/2} - \sqrt{1-e_2}
\]

\[
\frac{dV_2}{dx} = \{-x(1-e_1)\} \sqrt{\frac{\mu}{b_4 \{2-x^2(1-e_1)\}}}
\]

For optimum, we have

\[
\frac{d}{dx} (V_1 + V_2) = 0
\]

i.e.

\[
\sqrt{\frac{\mu (1-e_1)}{b_2}} + \sqrt{\frac{\mu (1-e_1)}{b_4}} \left[ \frac{-x \sqrt{1-e_1}}{\{2-x^2(1-e_1)\}^{\frac{1}{2}}} \right] = 0
\]

Finally

\[
x = \pm \left\{ \frac{2b_4}{(1+e_1)(b_2 + b_4)} \right\}^{\frac{1}{2}}
\]

**4.4.**

For the fourth configuration, when \( \alpha = 0 \), we have

\[
\frac{dV_1}{dx} = \sqrt{\frac{\mu (1-e_1)}{b_2}} \quad \text{and} \quad V_2 = v_B (1-\eta)
\]
By substitution

\[ V_2 = \sqrt{\frac{\mu}{b_3}} \left( \{2 - x^2(1 - e_1)\}^{\frac{3}{2}} - \sqrt{1 + e_2} \right) \]

\[ \frac{dV_2}{dx} = \left\{ -x(1 - e_1) \right\} \sqrt{\frac{\mu}{b_3}} \frac{1}{(2 - x^2(1 - e_1))^{\frac{3}{2}}} \]

Applying optimum condition

\[ \frac{d}{dx} (V_1 + V_2) = 0 \]

Then

\[ \sqrt{\frac{\mu}{b_2}} (1 - e_1) + \sqrt{\frac{\mu}{b_3}} \left\{ \frac{-x\sqrt{1 - e_1}}{(2 - x^2(1 - e_1))^{\frac{3}{2}}} \right\} = 0 \]

i.e.

\[ x = \pm \left\{ \frac{2b_3}{(1 - e_1)(b_2 + b_3)} \right\} \]

Moreover, we may set \( e_1 = e_2 = 0 \) (the original Hohmann classical circular orbit transfer) and find the corresponding equalities.

5. Numerical Results

We regard the Earth–Mars transfer system, the data available are [10]

\[ a_E = 1.00000011 \quad e_E = 0.01671022 \]
\[ a_M = 1.52366231 \quad e_M = 0.09341233 \]

Let \( \alpha = 25.5^0 = 0.4451 \) rad.

5.1.

Relevant to first configuration, we have

\[ u_1 = -5.3512 \quad u_2 = 11.2976 \quad u_3 = -11.2720 \quad u_4 = 4.541 \]

By solving Eq. (49), using a Mathematica program, we acquire the following 8 roots:

\[ x_{1,2} = \pm 1.30043 \quad x_{3,4} = \pm 1.12807 \quad x_{5,6} = 1.14726 \pm 0.349919I \]
\[ x_{7,8} = -1.14726 \pm 0.349919I \]

5.2.

Similarly relevant to second configuration,

\[ D_1 = -4.6236 \quad D_2 = 8.5415 \quad D_3 = -7.6817 \quad D_4 = 2.8272 \]
\[ x_{1,2} = \pm 1.24936 \quad x_{3,4} = \pm 1.08291 \quad x_{5,6} = -1.0459 \pm 0.385868I \]
\[ x_{7,8} = 1.0459 \pm 0.385868I \]

5.3.

With regard to third configuration, we have

\[ g_9 = -5.3043 \quad g_{10} = 11.0255 \quad g_{11} = -10.7918 \quad g_{12} = 4.1914 \]
\[ x_{1,2} = \pm 1.30264 \quad x_{3,4} = \pm 1.13589 \quad x_{5,6} = -1.12743 \pm 0.335436I \]
\[ x_{7,8} = 1.12743 \pm 0.335436I \]
5.4.

As for fourth configuration,

\[ E_9 = -4.8370 \quad E_{10} = 9.4059 \quad E_{11} = -8.9407 \quad E_{12} = 3.4691 \]

\[ x_{1,2} = \pm 1.09547 \quad x_{3,4} = \pm 1.2786 \quad x_{5,6} = -1.07954 \pm 0.405399 \]

\[ x_{7,8} = 1.07954 \pm 0.405399 \]

Whence the following table

<table>
<thead>
<tr>
<th>Configuration</th>
<th>( x_{Min.} (\alpha \neq 0) )</th>
<th>( x_{Min.} (\alpha = 0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>1.1281</td>
<td>1.1122</td>
</tr>
<tr>
<td>(II)</td>
<td>1.0822</td>
<td>1.0720</td>
</tr>
<tr>
<td>(III)</td>
<td>1.1359</td>
<td>1.1239</td>
</tr>
<tr>
<td>(IV)</td>
<td>1.0955</td>
<td>1.0824</td>
</tr>
</tbody>
</table>

6. Conclusions

We pointed out four feasible configurations for the elliptic Hohmann bi-impulsive transfer. We considered the problem as a change of energy of the vehicle. We computed the increments of energy \( \Delta v_A \), \( \Delta v_B \) at peri–apo–apse due to motor thrusts, for the four feasible configurations Eqs. (20), (21), (31), (33), (37), (39), (43), (45). Our parameter is \( x \) [9] which is a measure of the initial perpendicular impulse at peri–apse and apo–apse. We established the optimum condition

\[
\frac{d}{dx} (V_1 + V_2) = 0
\]

using the cosine law and the application of ordinary infinitesimal calculus. We eliminated after a rather lengthy reductions and rearrangements the power fractions, and we attained an algebraic equation of the eighth degree in \( x \), which could be resolved numerically. We determined the 8 roots for each of these four algebraic equations. The resolution yield real as well as complex roots, positive as well as negative roots. Only one real positive root is adequate and consistent with our assumptions, for the adopted Earth–Mars Hohmann elliptic transfer. A reduction of the problem to the coplanar transfer case occurs by setting \( \alpha = 0 \). A second degree algebraic equation in \( x \) results when we set \( \alpha = 0 \). We have the following four propositions corresponding to \( \alpha \), \( e_1 \), \( e_2 \) is equal to null or not:

(I) \( \alpha \neq 0 \quad e_1, e_2 \neq 0 \)

(II) \( \alpha \neq 0 \quad e_1, e_2 = 0 \)

(III) \( \alpha = 0 \quad e_1, e_2 \neq 0 \)

(IV) \( \alpha = 0 \quad e_1, e_2 = 0 \)

(I) represents the elliptic non-coplanar case whilst, (II) represents the circular non-coplanar case, and (III) relevant to the elliptic coplanar case, whilst (IV) is relevant to the circular coplanar case. It is obvious from table 1 that

\[
[x]_{\alpha \neq 0}^{Min.} = 1.0822
\]
represents the most economic expenditure of fuel since it is the least value of $x$ for $\alpha \neq 0$, for the Earth–Mars transfer system. Similarly for $\alpha = 0$

$$[x]_{\alpha=0}^{\text{Min.}} = 1.0720$$

is the most economic. Both are relevant to the second configuration where apo–apse of transfer orbit coincides with peri–apse of final orbit, the primary is at left focus. We note that

$$[x]_{\alpha \neq 0}^{\text{Min.}} / [x]_{\alpha=0}^{\text{Min.}}$$

which indicates that the energy needed at initial impulse for non–coplanar Hohmann elliptic transfers exceeds that for coplanar ones.

References


Nomenclature:

- $a_1$: semi-major axis of initial elliptic orbit
- $a_2$: semi-major axis of final elliptic orbit
- $a_T$: semi-major axis of transfer elliptic orbit
- $e_1$: eccentricity of initial elliptic orbit
- $e_2$: eccentricity of final elliptic orbit
- $e_T$: eccentricity of transfer elliptic orbit
- $\Delta v_A$: increment of velocity at point A
- $\Delta v_B$: increment of velocity at point B
- $\mu$: constant of gravitation
- $r$: radius vector measured from center of attraction
- $v$: scalar velocity
- $\Delta v_A + \Delta v_B$: classical characteristic velocity
- $x$: single parameter of optimization
- $\alpha$: single plane change angle
- $\eta$: velocity after second impulse / velocity before second impulse
- $V_1$: magnitude of first impulse
- $V_2$: magnitude of second impulse