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Influence of a Distributed Delay on Stabilization of Structure Vibration

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The purpose of this theoretical work is to present a stabilization problem of beam with a distributed model of feedback delay. A displacement feedback and particular polarization profiles of piezoelectric sensors and actuators are introduced. The structure is described by integro–partial differential equations with time–dependent coefficient. The uniform stochastic stability criteria of the beam equilibrium are derived using the Liapunov direct method. As the axial force is described by the wide–band gaussian process the dynamic equation has to be written as Itô evolution equation with white–noise coefficient and the Itô differential rule is applied in order to calculate the differential of Liapunov functional. The influence of the time–deley parameter, stiffness and intensity of axial force on dynamic stability regions is shown.

Keywords: Stochastic parametric vibration, structure stabilization, distributed time–delay, Liapunov method, uniform stochastic stability.

1. Introduction

Problem of stabilization of motion for mechanical distributed in space systems was several times solved in the literature. The vibration were actively damped by means of piezoelectric patches glued to the structure surfaces. Despite the fact of large dimensions the presence of time delays was neglected. In the paper [1], [2] theoretical fundamentals of stabilization of beam with shear deformations and rotary inertia effect was presented. The piezoelectric layers were glued to the both sides to the beam compressed by time–dependent axial forces. A velocity feedback and particular polarization profiles of piezoelectric sensors and actuators were introduced. The structure was described by partial differential equations including transverse and rotaryinertia terms, general deformation state with interlaminar shear strains. A viscous model of external damping with the constant proportionality coefficient was assumed to describe a dissipation energy both in the transverse and rotary motion. The beam motion was described by the transverse displacement. In orded to find an exponential one–side estimation the calculus of variations is used. The associated Euler equations in the form of system of differental equations were solved analytically and the stabilization problem was reduced to transcendental algebraic inequality with respect to the exponent of estimation. The effect of spatially uniform time–dependent compressive forces on a stability of circular plates was examined [5]. The purpose of this analytical work was to generalize the previous result relating to the axially symmetric motion and to investigate a uniform stability of plate compressed by time–dependent forces. The stability of parametric vibrations of circular plate subjected to in–plane forces was analysed by the Liapunov method. The method was applied without the earlier finite dimensional or modal approximations. The energy–like functional was proposed; its positiveness was equivalent to the condition in which static buckling does not occur. Taking into account that a plate is compressed radially by time–dependent and uniformly distributed along its edge forces, a dynamic stability of an undeflected state of isotropic elastic circular plate was analysed. Assuming that the compressing force are broad–band normal processes the plate dynamics was described by stochastic Itô equations. The critical damping coefficient has been expressed by the intensity and the mean value of compressing force. Stochastic stability of distributed control [3] and fuzzy control [4] systems with delays is studied.

2. Basic Assumptions, Definitions

Consider the beam of length ℓ, width b, and thickness h_b, loaded by axial–time dependent force with piezoelectric layers mounted on each of two opposite sides. The beam is simply supported on both ends. The piezoelectric layers are bonded on the beam surfaces and the mechanical properties of the bonding material are represented by the effective damping coefficient calculated from the rule of mixtures. Th edamping coefficient is a linear function of both the beam and bonding layer damping coefficients. It is assumed that the transverse motion dominates the axial vibration. The thickness of the actuator and the sensor is denoted by h_a and h_s, respectively. Neglecting the stiffness of piezolayer in comparison with that of the beam and assuming the changing width $b_s(x)$ of the sensor and the changing the width $b_a(x)$ of the actuator the influence of the piezoelectric actuator on on the beam can be reduced to bending moment M^e distributed along the actuator. The position dependent active widths of piezoelements can be realized by introducing suitable shape of metalized electrodes glued to piezoelements. It is the easiest way to perform the position dependent active elements of feedback system. The functions $b_s(x)$ and $b_a(x)$ are called the sensor characteristic function and the actuator characteristic functions, respectively.

2.1. *Sensor and Actuator Equations*

Sensor electric displacement in direction perpendicular to the beam surface is given by

$$D_3 = -e_{31}\epsilon_1 \tag{1}$$

where e_{31} is the piezoelectric stress/charge coefficient, and ϵ_1 sensor strain.

Expressing strains be the beam curvature and the distance from the neutral axis we integrate the electric displacement over the sensor area

$$D_3 = -d_s \frac{(h_s + h_b)E_s}{2} \int_0^{\ell} b_s(x) w_{,xx} dx \tag{2}$$

where d_s is the piezoelectric strain/charge coefficient of sensor. Finally, the sensor voltage is calculated using the formula for a flat capacitor

$$V_s = -d_s \frac{(h_s + h_b)E_s h_s}{2\epsilon_{33} A_s} \int_0^{\ell} b_s(x) w_{,xx} dx \tag{3}$$

where A_s is the effective sensor area and ϵ_{33} is the permittivity coefficient. Using the displacement feedback control the voltage applied to the actuator is

$$V_a = K_a V_s \tag{4}$$

The control bending moment can be expressed by the actuator stresses σ_a, moment arm $h_a + h_b$ and the cross–section area $h_a b_a(x)$ of the actuator in the following way

$$M^e = D b_a(x) \tag{5}$$

where $D = d_{a31} V_a h_a \frac{h_b + h_a}{2}$.

2.2. *Distributed Model of Delayed Feedback*

Due to a delay in the displacement feedback system the control voltage V_a and the control bending moment are disturbed, that is decreasing the stabilization effect [7]. The delay is not fully determined and the delayed moment M^e is modelled by means of convolution in the form

$$M^e = D b_a(x) \int_0^t \int_0^{\ell} b_s(x) w_{,xx}(x, t-\tau) \exp(-\lambda\tau) \; d\, x \; d\, \tau \tag{6}$$

Eq. (6) represents decreasing influence of distributed time-delay from the range $\{0,\, t\}$ on the feedback signal V_a. Constant λ describes a level of signal decreasing. It should be also remembered that due to the position dependent widths $b_a(x)$ the feedback bending moment is also space dependent $M^e = M^e(x,\, t)$.

2.3. *Stability Definition*

The main purpose of the paper is to examine a uniform stability of the equilibrium state. To estimate a perturbed solution of beam dynamics equation it is necessary to introduce a measure of distance $\|.\|$ of the solution of dynamics equation with nontrivial initial conditions from the trivial one. In order to make the transition to the Liapunov stability in probability (uniform stochastic stability) we merely have to estimate the probability of perturbed solutions for all $t > 0$. More precisely, the equilibrium state of dynamics equation is said to be uniformly stochastically stable, if the following logic sentence is true [8]

$$\bigwedge_{\epsilon \geq 0} \bigwedge_{\delta \geq 0} \bigvee_{r \geq 0} \|w(.,0)\| \leq r \Rightarrow P(\sup_{t \geq 0} \|w(.,t)\| \geq \epsilon) \leq \delta \tag{7}$$

In the present paper the direct Liapunov method is proposed to establish criteria for the uniform stochastic stability of the unperturbed (trivial) solution of the beam–like plate compressed by the uniformly distributed time-dependent forces.

3. Dynamics Equation

Consider the beam loaded axially by a time–dependent force $F_o + F(t)$ with piezoelectric layers mounted on each of the two opposite sides of the beam of length ℓ. The piezoelectric layers are assumed to be perfectly bonded on the beam surfaces. The sensing and actuating effects of piezoelectric layers are used to stabilize parametric vibrations excited by the oscillating axial force. Assuming the negligible stiffness of the sensor in comparison with that of the beam the influence of the piezoelectric actuator on the beam is reduced to a bending moment M^e distributed along the beam. The Kirchhoff hypothesis on deformable normal element to the middle line is used and the rotary and coupling inertial are neglected. The mass density is denoted by ρ, the cross-section area of the beam by A_o and the bending stiffness by EJ. The transverse displacement w of the beam is measured from the equilibrium state and is governed by the following dynamics equation

$$\rho A w_{,tt} + EJ w_{,xxxx} + (F_o + F(t)) w_{,xx} + M^e_{,xx} = 0 \tag{8}$$

Dividing by ρA and substituting Eq. (6)

$$\begin{aligned} & w_{,tt} + \frac{EJ}{\rho A} w_{,xxxx} + \frac{f_o + f(t)}{\rho A} w_{,xx} + \\ & + \frac{D}{\rho A} b_{a,xx} \int_0^t \exp(-\lambda \tau) \int_0^\ell b_s(x) w_{,xx}(t-\tau) d\,x\,d\,\tau = 0 \end{aligned} \tag{9}$$

The beam is assumed to be simply supported at both ends. If the stochastic parametric excitation can be modelled as the Gaussian white noise with intensity σ^* Eq. (9) can be treated as the Itô integro – partial differential equations

$$\begin{aligned} & dw = v dt \\ & dv = -\Big[\frac{D}{\rho A} w_{,xxxx} + \frac{f_o}{\rho A} w_{,xx} + \\ & + \frac{D}{\rho A} b_{a,xx} \int_0^t \exp(-\lambda \tau) \int_0^\ell b_s(x) w_{,xx}(t-\tau) d\,x\,d\tau\Big]\,dt - \sigma^* w_{,xx} d\,\mathcal{W} \end{aligned} \tag{10}$$

where standard Wiener process is denoted by $\mathcal{W}$. In order to avoid qualitative analysis of integro–partial differential equations we expand the transverse displacement into series of functions satisfying simply supported boundary conditions on both ends

$$w(x,t) = \sum_{i=1,2,..}^{\infty} P_i(t) \sin \frac{i\pi x}{\ell} \tag{11}$$

In order to analyse equilibrium stability it is necessary to fix shapes of piezoelectric sensor and piezoactuator. The width of both elements will be described by sine function in the following way

$$b_s(x) = b_s^o \sin \frac{\pi x}{\ell} \tag{12}$$

$$b_a(x) = b_a^o \sin \frac{\pi x}{\ell} \tag{13}$$

Introducing functions $P(t)$, $Q(t)$ and $R(t)$ we obtain the equation of the first mode corresponding to Eq. (10) in the form of the following system of Itô evolution equations

$$\begin{aligned} dP &= Qd\,t \\ dQ &= -[(k^2 - f_o)P - \gamma R]d\,t + \sigma P d\,\mathcal{W} \\ dR &= (P - \lambda R)d\,t \end{aligned} \tag{14}$$

where

$$\gamma = \frac{D}{\rho A}(\pi/\ell)^4$$
$$k^2 = \frac{EJ}{\rho A}(\pi/\ell)^4$$
$$f_o = \frac{F_o}{\rho A}(\pi/\ell)^2$$
$$\sigma = \frac{\sigma^*}{\rho A}(\pi/\ell)^2$$

Solutions of Eq. (14) are evolutionary processes with known infinitesimal operator $L(.)$.

4. Uniform Stability Analysis

Using Barabashin method [6] the Liapunov function is given by a quadratic form with respect to P, Q, R with unknown coefficients

$$V = w_{11}P^2 + w_{22}Q^2 + w_{33}R^2 + 2w_{12}PQ + 2w_{23}QR + 2w_{13}PR \tag{15}$$

Using the main advantage of the proposed method we impose condition on the time–derivative of function in Eq. (15) along solutions of Eq. (14) with cancelled stochastic component

$$\frac{dV}{dt} = -P^2 \tag{16}$$

According to Barabashin's algorithm [6] the Liapunov function is chosen in the form

$$\begin{aligned} V &= (k^2-\gamma/\lambda-f_o)(k^2f_o-\lambda^2)\lambda P^2+\lambda(k^2-\gamma/\lambda-f_o+\lambda^2)Q^2+\lambda k^4\gamma^2R^2+ \\ &\quad -2k^2\gamma(k^2-\gamma/\lambda-f_o)PR+2\lambda^2k^2\gamma QR \end{aligned} \tag{17}$$

The functional (17) is positive definite if the static criterion of stability is fulfilled

$$f_o < k^2-\gamma/\lambda \tag{18}$$

Infinitesimal operator has the form [8]

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{3} b_i\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} \tag{19}$$

where b_i is a column matrix of right hand side deterministic functions of Eq. (14) and a_{ij} is a matrix containing intensities of Wiener processes in Eq. (14). In the problem we have just one stochastic force present in the second equation of system Eq. (14). Therefore, matrix a has one nonzero element $a_{22}=-\sigma x_2$. Calculating the infinitesimal operator LV along solutions of Eq. (14) we have

$$LV=-\Big[2k^2\gamma\lambda(k^2-\gamma/\lambda-f_o)-\sigma^2(k^2-\gamma/\lambda-f_o+\lambda^2)\Big]P^2 \tag{20}$$

Thus, the trivial solution of equilibrium state is uniformly stable if the intensity coefficient is sufficiently small

$$\sigma^2<\frac{2k^2\gamma\lambda(k^2-\gamma/\lambda-f_o)}{k^2-\gamma/\lambda-f_o+\lambda^2} \tag{21}$$

Obtained formula contains all parameter of the system with active vibration control and the parameter of distrbuted delay λ. The proposed analysis can be extended to more general feedback control using a flexibility of Barabashin algorythm.

5. Conclusions

The stabilization of vibrating beam with distributed piezoelectric sensor, actuator, and delayed displacement feedback has been studied. The stabilization of parametric vibrations needs sufficiently small intensity of axial force σ. Admissible disturbance intensity strongly depends on the feedback gain factor. The assumed model of delay significantly decreases the stabilization effect. Increase of constant component of axial force decreases stability region.

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