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The Optimality of the Impulsive Hohmann Coplanar Elliptic Transfer Using Energy Change Concepts Aggregation of New Useful Relationships Part II

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In this part we investigate all the four feasible configurations for the generalized Hohmann type transfer. We assign the minimized characteristic velocity ($\Delta v_1 + \Delta v_2)_{Min}$ by the application of ordinary infinitesimal calculus optimum conditions. By some algebraic manipulations, we determine the independent variables $(x)_{Min}$. In addition, we considered the analysis relevant to the two parameters x, y relevant to the two impulses at points A, B. It is demonstrated that the elliptic Hohmann type transfer is the most economic one by this new representation.

Keywords: Orbital mechanics, Hohmann transfer,optimization

1. Introduction

We discuss in detail the concept of optimization in terms of energy concepts, such as the vis viva integral. Astronomical problems will represent geocentric motion of the early stages of orbits that penetrate farther into space.

Heliocentric ellipses will be useful in a later stage of the same orbits. In transition stages, we may employ perturbation techniques.

Recently and especially in optimization and correction theory, the orbit is defined as the state vector as well as being a set including six elements – in our analysis only two elements – or constants of integration. Differential formulae enter into many of the astrodynamics problems: optimization, correction, guidance and error analysis.

We limit ourselves at first to one parameter p, called an element, or state variable, consequently to one function $\Psi = \Psi(\mathrm{p})$, that can be measured (in our investigation this parameter is x or y). Orbit transfer is a major subject with regard to placing a spacecraft in an orbit around the Earth. The velocity increments are directly proportional to motor system thrusts of the rocket / space vehicle. Consequently it is proportional to propellant fuel consumption. It is most convenient to regard the transfer problem as a problem of change of energy [1]. The criterion for optimality is the minimization of the characteristic velocity for the maneuver, [2], [3]. The literature dealing with the optimal transfer is extensive, we may recall the works by Prussing [3], Palmore [4], Edelbaum [5], Barrar [6], Marec [7], Lawden [8], Hiller [9] and Chobotov [10].

2. Method and Results

2.1. *First Configuration*

2.1.1. *Assumption of one single parameter x*

We have the following relationships, for the first configuration, when the apo – apse of transfer orbit coincides with the apo – apse of final orbit of the space vehicle.

We rewrite,

$$I_1 = \Delta v_1 = v_{A2} - v_{A1} = xv_{A1} - v_{A1} = (x-1)\, v_{A1} \tag{1}$$

$$I_2 = \Delta v_2 = v_{B2} - v_{B1} \tag{2}$$

with

$$v_{A2} = \sqrt{\frac{\mu\,(1+e_T)}{a_T\,(1-e_T)}} \qquad v_{A1} = \sqrt{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}}$$
$$v_{B2} = \sqrt{\frac{\mu\,(1-e_2)}{a_2\,(1+e_2)}} \qquad v_{B1} = \sqrt{\frac{\mu\,(1-e_T)}{a_T\,(1+e_T)}} \tag{3}$$

Where

$$x = \frac{xv_{A1}}{v_{A1}} = \frac{\text{velocity after peri - apse initial impulse}}{\text{velocity before peri - apse initial impulse}} > 1 \tag{4}$$

From the geometrical properties of Fig. 1, we get

$$a_T\,(1+e_T) = a_2\,(1+e_2)$$
$$a_T\,(1-e_T) = a_1\,(1-e_1) \tag{5}$$

From the above equations, we acquire

$$x = \sqrt{\frac{1+e_T}{1+e_1}} \quad \text{i.e.} \quad e_T = x^2\,(1+e_1) - 1 \tag{6}$$

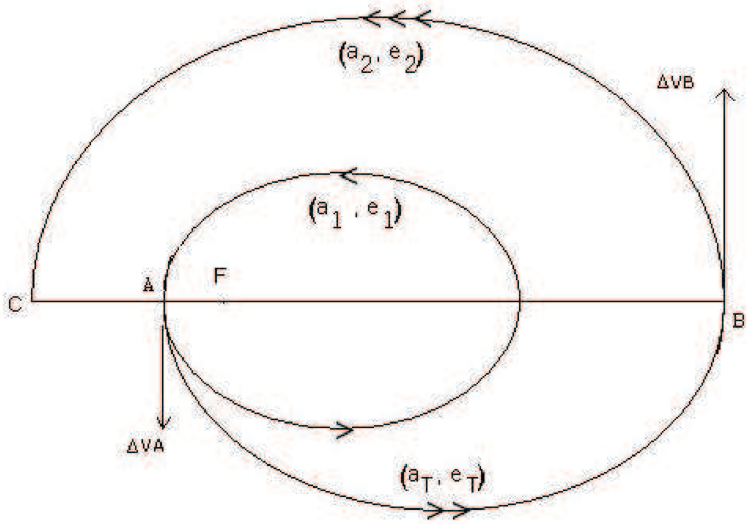

Figure 1

Writing

$$b_1 = a_1 (1 - e_1) \quad b_2 = a_1 (1 + e_1)$$
$$b_3 = a_2 (1 - e_2) \quad b_4 = a_2 (1 + e_2) \tag{7}$$

whence

$$a_T = \frac{b_1}{2 - x^2 (1 + e_1)} = \frac{b_4}{x^2 (1 + e_1)} \tag{8}$$

We can easily derive

$$\Delta v_1 = \sqrt{\frac{\mu (1 + e_1)}{b_1}} (x - 1) \tag{9}$$

$$\Delta v_2 = \sqrt{\frac{\mu (1 - e_2)}{b_4}} - \sqrt{\frac{\mu \{2 - x^2 (1 + e_1)\}}{b_4}} \tag{10}$$

By application of minimization condition

$$\frac{d}{dx}(\Delta v_T) = \frac{d}{dx}(\Delta v_1) + \frac{d}{dx}(\Delta v_2) \tag{11}$$

Whence by differentiation w.r.t. the variable x, we find

$$\sqrt{\frac{\mu(1+e_1)}{b_1}} + \sqrt{\frac{\mu}{b_4}}\frac{x(1+e_1)}{\sqrt{2-x^2(1+e_1)}} = 0 \tag{12}$$

After some reductions and rearrangements, we acquire the explicit form of $(x)_{\mathrm{Mn}}$ in terms of the two elements a,e.

$$(x)_{\mathrm{Min}} = \pm\sqrt{\frac{2b_4}{(1+e_1)(b_1+b_4)}} = \mathrm{const} \tag{13}$$

By substitution, we obtain the unique values of $(a_T)_{\mathrm{Min}}$ & $(e_T)_{\mathrm{Min}}$, namely

$$(a_T)_{\mathrm{Min}} = \frac{b_1+b_4}{2} \tag{14}$$

$$(e_T)_{\mathrm{Min}} = \frac{b_4-b_1}{b_4+b_1} \tag{15}$$

The above immediate two equations demonstrate that the Hohmann type elliptic transfer is an optimal one. We evaluate the minimum total characteristic velocity

$$(\Delta v_T)_{\mathrm{Min}} = (\Delta v_1 + \ \Delta v_2)_{\mathrm{Min}}$$

This is implemented by the substitution for $x = (x)_{\mathrm{Min}}$, and writing the values of b_1, b_2, b_3, b_4 explicity, we find that:

$$\begin{aligned}(\Delta v_T)_{\mathrm{Min}} &= \sqrt{\frac{2\mu b_4}{b_1(b_1+b_4)}} - \sqrt{\frac{\mu(1+e_1)}{b_1}} \\ &\quad + \sqrt{\frac{\mu(1-e_2)}{b_4}} - \sqrt{\frac{2\mu b_1}{b_4(b_1+b_4)}} = \mathrm{const}\end{aligned} \tag{16}$$

For the classical circular Hohmann transfer $e_1 = 0$and $e_2 = 0$, whence we obtain the quite symmetric formula

$$\begin{aligned}(\Delta v_T)_{\mathrm{Min}} &= \sqrt{\frac{\mu}{a_1}}\left\{\sqrt{\frac{2a_2}{a_1+a_2}} - 1\right\} \\ &\quad + \sqrt{\frac{\mu}{a_2}}\left\{1 - \sqrt{\frac{2a_1}{a_1+a_2}}\right\} = \mathrm{const}\end{aligned} \tag{17}$$

2.1.2. *Consideration of two parameters x, y*

Let

$$y = \frac{v_{B2}}{v_{B1}} = \frac{\text{velocity after impulse at point B}}{\text{velocity before impulse at point B}} > 1$$

whence

$$y = \sqrt{\frac{1 - e_2}{1 - e_T}}$$

and

$$\Delta v_T = \Delta v_1 + \Delta v_2 = \sqrt{\frac{\mu\left(1 + e_T\right)}{a_T\left(1 - e_T\right)}} - \sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}} + \sqrt{\frac{\mu\left(1 - e_2\right)}{b_4}} - \sqrt{\frac{\mu\left(1 - e_T\right)}{a_T\left(1 + e_T\right)}}$$

i.e.

$$\Delta v_T = \sqrt{\mu}\left[\frac{\sqrt{1 + e_T} - \sqrt{1 + e_1}}{\sqrt{b_1}} + \frac{\sqrt{1 - e_2} - \sqrt{1 - e_T}}{\sqrt{b_4}}\right]$$

We may write

$$\Delta v_T = \sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}}\left(x - 1\right) + \sqrt{\frac{\mu\left(1 - e_2\right)}{b_4}}\left(1 - \frac{1}{y}\right) = f\left(x, y\right) \tag{18}$$

But, we have from rules of partial differentiation of two variables

$$\frac{\partial}{\partial x} f\left(x, y\right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} \tag{19}$$

From optimum condition

$$\frac{\partial \Delta v_T}{\partial x} = \sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}} + \sqrt{\frac{\mu\left(1 - e_2\right)}{b_4}}\frac{1}{y^2}\frac{\partial y}{\partial x} = 0 \tag{20}$$

where

$$y^2 = \frac{1 - e_2}{2 - x^2\left(1 + e_1\right)}$$

But

$$\frac{1}{y^2}\frac{\partial y}{\partial x} = \frac{x\left(1 + e_1\right)}{\sqrt{\left(1 - e_2\right)\left\{2 - x^2\left(1 + e_1\right)\right\}}}$$

After some rearrangements and reductions, we get

$$\left(x^2\right)_{Min} = \frac{2b_4}{\left(1 + e_1\right)\left\{b_1 + b_4\right\}} = constant \tag{21}$$

Which is the same as Eq. (12).

Also, we may write

$$\frac{\partial}{\partial y} f\left(x, y\right) = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} \quad \ldots \tag{22}$$

By analogy, we find

$$\left(y^2\right)_{Min} = \left(\frac{1-e_2}{2}\right)\left\{1+\frac{b_4}{b_1}\right\} \quad \ldots \tag{23}$$

2.2. *Second Configuration*

2.2.1. *Assumption of one single parameter x*

Now we investigate the second feasible configuration

$$I_1 = \Delta v_1 = v_{A2} - v_{A1} = x v_{A1} - v_{A1} = (x-1)\, v_{A1} \tag{24}$$

$$I_2 = \Delta v_2 = v_{B2} - v_{B1} \tag{25}$$

with

$$\begin{aligned} v_{A2} &= \sqrt{\frac{\mu\,(1+e_T)}{b_1}} \qquad v_{A1} = \sqrt{\frac{\mu\,(1+e_1)}{b_1}} \\ v_{B2} &= \sqrt{\frac{\mu\,(1+e_2)}{b_3}} \qquad v_{B1} = \sqrt{\frac{\mu\,(1-e_T)}{b_3}} \end{aligned} \tag{26}$$

From the geometrical properties of Fig. 2, we get

$$\begin{aligned} a_T\,(1+e_T) &= a_2\,(1-e_2) = b_3 \\ a_T\,(1-e_T) &= a_1\,(1-e_1) = b_1 \end{aligned} \tag{27}$$

From the above equations, we acquire

$$x = \sqrt{\frac{1+e_T}{1+e_1}}\text{i.e. } e_T = x^2\,(1+e_1) - 1 \tag{28}$$

whence

$$a_T = \frac{b_1}{2 - x^2\,(1+e_1)} = \frac{b_3}{x^2\,(1+e_1)} \tag{29}$$

We can easily derive

$$\Delta v_1 = \sqrt{\frac{\mu\,(1+e_1)}{b_1}}\,(x-1) \tag{30}$$

$$\Delta v_2 = \sqrt{\frac{\mu\,(1+e_2)}{b_3}} - \sqrt{\frac{\mu\,\{2 - x^2\,(1+e_1)\}}{b_3}} \tag{31}$$

Whence by differentiation w.r.t. variable x, we find

$$\frac{d}{dx}\,(\Delta v_1) + \frac{d}{dx}\,(\Delta v_2) == \sqrt{\frac{\mu\,(1+e_1)}{b_1}} + \sqrt{\frac{\mu}{b_3}}\,\frac{x\,(1+e_1)}{\sqrt{2 - x^2\,(1+e_1)}} = 0 \tag{32}$$

After some reductions and rearrangements, we may write

$$(x)_{\text{Min}} = \pm\sqrt{\frac{2b_3}{(1+e_1)(b_1+b_3)}} \tag{33}$$

By substitution, we obtain the unique values of $(a_T)_{\text{Min}}$ & $(e_T)_{\text{Min}}$, namely

$$(a_T)_{Min} = \frac{b_1+b_3}{2} \tag{34}$$

$$(e_T)_{Min} = \frac{b_3-b_1}{b_3+b_1} \tag{35}$$

We evaluate the minimum total characteristic velocity

$$\begin{aligned}(\Delta v_T)_{\text{Min}} &= (\Delta v_1 + \Delta v_2)_{\text{Min}} \\ &= \sqrt{\frac{2\mu b_3}{b_1(b_1+b_3)}} - \sqrt{\frac{\mu(1+e_1)}{b_1}} + \sqrt{\frac{\mu(1+e_2)}{b_3}} - \sqrt{\frac{2\mu b_1}{b_3(b_1+b_3)}} = \text{const}\end{aligned} \tag{36}$$

2.2.2. *Consideration of two parameters x, y*

Let

$$y = \frac{v_{B2}}{v_{B1}} = \frac{\text{velocity after impulse at point B}}{\text{velocity before impulse at point B}} > 1$$

whence

$$y = \sqrt{\frac{1+e_2}{1-e_T}} \tag{37}$$

and

$$\begin{aligned}\Delta v_T &= \Delta v_1 + \Delta v_2 = \\ &= \sqrt{\frac{\mu(1+e_T)}{b_1}} - \sqrt{\frac{\mu(1+e_1)}{b_1}} + \sqrt{\frac{\mu(1+e_2)}{b_3}} - \sqrt{\frac{\mu(1-e_T)}{b_3}}\end{aligned} \tag{38}$$

i.e.

$$\Delta v_T = \sqrt{\mu}\left[\frac{\sqrt{1+e_T}-\sqrt{1+e_1}}{\sqrt{b_1}} + \frac{\sqrt{1+e_2}-\sqrt{1-e_T}}{\sqrt{b_3}}\right]$$

We may write

$$\Delta v_T = \sqrt{\frac{\mu(1+e_1)}{b_1}}(x-1) + \sqrt{\frac{\mu(1+e_2)}{b_3}}\left(1-\frac{1}{y}\right) = f(x,y) \tag{39}$$

For optimization condition and after differentiation, we get

$$\frac{\partial \Delta v_T}{\partial x} = \sqrt{\frac{\mu(1+e_1)}{b_1}} + \sqrt{\frac{\mu(1+e_2)}{b_3}}\,\frac{1}{y^2}\frac{\partial y}{\partial x} = 0 \tag{40}$$

where

$$y^2 = \frac{1+e_2}{2-x^2\left(1+e_1\right)}, i.e. y = \sqrt{\frac{1+e_2}{2-x^2\left(1+e_1\right)}} \tag{41}$$

But

$$\frac{1}{y^2}\frac{\partial y}{\partial x} = \frac{x\left(1+e_1\right)}{\sqrt{\left(1+e_2\right)\left\{2-x^2\left(1+e_1\right)\right\}}}$$

After some rearrangements and reductions, we get

$$\left(x^2\right)_{Min} = \frac{2b_3}{\left(1+e_1\right)\left\{b_1+b_3\right\}} = \text{constant} \tag{42}$$

which is the same as Eq. (33). Also, we may write

$$\frac{\partial}{\partial y} f\left(x, y\right) = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} \tag{43}$$

By analogy, we find

$$\left(y^2\right)_{Min} = \left(\frac{1+e_2}{2}\right)\left\{1+\frac{b_3}{b_1}\right\} \tag{44}$$

2.3. *Third Configuration*

2.3.1. *Assumption of one single parameter x*

For Fig. 3, we deduce the following equalities:

$$\begin{aligned} a_1\left(1+e_1\right) &= a_T\left(1+e_T\right) = b_2 \\ a_2\left(1-e_2\right) &= a_T\left(1-e_T\right) = b_3 \end{aligned} \tag{45}$$

with

$$\begin{aligned} v_{A2} &= \sqrt{\frac{\mu\left(1-e_T\right)}{b_2}} \qquad v_{A1} = \sqrt{\frac{\mu\left(1-e_1\right)}{b_2}} \\ v_{B2} &= \sqrt{\frac{\mu\left(1+e_2\right)}{b_3}} \qquad v_{B1} = \sqrt{\frac{\mu\left(1+e_T\right)}{b_3}} \end{aligned} \tag{46}$$

$$\begin{aligned} I_1 &= \Delta v_1 = v_{A2} - v_{A1} = xv_{A1} - v_{A1} = \left(x-1\right)v_{A1} \\ I_2 &= \Delta v_2 = v_{B2} - v_{B1} \end{aligned}$$

where

$$x = \frac{xv_{A1}}{v_{A1}} = \sqrt{\frac{1-e_T}{1-e_1}} > 1$$

i.e.

$$e_T = 1 - x^2\left(1-e_1\right) \tag{47}$$

Therefore,

$$a_T = \frac{b_2}{2 - x^2 (1 - e_1)} = \frac{b_3}{x^2 (1 - e_1)} \tag{48}$$

and

$$v_{B1} = \sqrt{\frac{\mu \{2 - x^2 (1 - e_1)\}}{b_3}} \tag{49}$$

Also,weget

$$\Delta v_1 = (x - 1) \sqrt{\frac{\mu (1 - e_1)}{b_2}} \tag{50}$$

$$\Delta v_2 = \frac{\sqrt{\mu} \left\{ \sqrt{1 + e_2} - \sqrt{2 - x^2 (1 - e_1)} \right\}}{\sqrt{b_3}} \tag{51}$$

By differentiation w.r.t. the variable x, and after some reductions, we get

$$x^2 = \frac{2b_3}{(1 - e_1)(b_2 + b_3)} \tag{52}$$

i.e.

$$(x)_{Min} = \pm \sqrt{\frac{2b_3}{(1 - e_1)(b_2 + b_3)}} = \text{constant} \tag{53}$$

Easily, we can find

$$(a_T)_{Min} = \frac{b_2 + b_3}{2} = \text{constant} \tag{54}$$

and

$$(e_T)_{Min} = \frac{b_2 - b_3}{b_2 + b_3} \tag{55}$$

Finally, we get

$$\begin{aligned}(\Delta v_T)_{Min} &= \sqrt{\frac{\mu (1 + e_2)}{b_3}} - \sqrt{\frac{\mu (1 - e_1)}{b_2}} \\ &+ \sqrt{\frac{2\mu b_3}{b_2 (b_2 + b_3)}} - \sqrt{\frac{2\mu b_2}{b_3 (b_2 + b_3)}} = \text{constant}\end{aligned} \tag{56}$$

2.3.2. *Consideration of two parameters x, y*

Let

$$y = \frac{v_{B2}}{v_{B1}} = \frac{\text{velocity after impulse at point B}}{\text{velocity before impulse at point B}} > 1 \tag{57}$$

whence

$$y = \sqrt{\frac{1 + e_2}{1 + e_T}} \tag{58}$$

and

$$\begin{aligned}\Delta v_T &= \Delta v_1 + \Delta v_2 \\ &= \sqrt{\frac{\mu(1-e_T)}{b_2}} - \sqrt{\frac{\mu(1-e_1)}{b_2}} + \sqrt{\frac{\mu(1+e_2)}{b_3}} - \sqrt{\frac{\mu(1+e_T)}{b_3}}\end{aligned}$$

i.e.

$$\Delta v_T = \sqrt{\mu}\left[\frac{\sqrt{1-e_T}-\sqrt{1-e_1}}{\sqrt{b_2}} + \frac{\sqrt{1+e_2}-\sqrt{1+e_T}}{\sqrt{b_3}}\right] \tag{59}$$

But,we have

$$1-e_T = x^2(1-e_1) \qquad 1+e_T = \frac{1+e_2}{y^2} \tag{60}$$

Therefore,

$$\Delta v_T = (x-1)\sqrt{\frac{\mu(1-e_1)}{b_2}} + \left(1-\frac{1}{y}\right)\sqrt{\frac{\mu(1+e_2)}{b_3}} = f(x,y) \tag{61}$$

whence,

$$\frac{\partial \Delta v_T}{\partial x} = \sqrt{\frac{\mu(1-e_1)}{b_2}} + \sqrt{\frac{\mu(1+e_2)}{b_3}}\frac{1}{y^2}\frac{\partial y}{\partial x} = 0 \tag{62}$$

But,

$$\frac{\partial y}{\partial x} = \frac{x(1-e_1)\sqrt{1+e_2}}{\{2-x^2(1-e_1)\}^{3/2}}$$

After some reductions, we find

$$(x^2)_{Min} = \frac{2b_3}{(1-e_1)(b_2+b_3)} \tag{63}$$

and

$$(y^2)_{Min} = \frac{(1+e_2)}{2}\left\{1+\frac{b_3}{b_2}\right\} \tag{64}$$

2.4. ***Fourth Configuration***

2.4.1. *Assumption of one single parameterx*

For figure 4, we get the following relationships

$$\begin{aligned} a_1(1+e_1) = a_T(1-e_T) = b_2 \\ a_2(1+e_2) = a_T(1+e_T) = b_4 \end{aligned} \tag{65}$$

with

$$\begin{aligned} v_{A2} = \sqrt{\frac{\mu(1+e_T)}{b_2}} \qquad v_{A1} = \sqrt{\frac{\mu(1-e_1)}{b_2}} \\ v_{B2} = \sqrt{\frac{\mu(1-e_2)}{b_4}} \qquad v_{B1} = \sqrt{\frac{\mu(1-e_T)}{b_4}} \end{aligned} \tag{66}$$

where

$$x = \frac{v_{A2}}{v_{A1}} = \sqrt{\frac{1+e_T}{1-e_1}} > 1$$

i.e.

$$e_T = x^2 (1 - e_1) - 1 \tag{67}$$

Therefore,

$$a_T = \frac{b_2}{2 - x^2 (1 - e_1)} = \frac{b_4}{x^2 (1 - e_1)} \tag{68}$$

and

$$v_{B1} = \sqrt{\frac{\mu \{2 - x^2 (1 - e_1)\}}{b_4}} \tag{69}$$

Also,we get

$$\Delta v_1 = (x - 1) \sqrt{\frac{\mu (1 - e_1)}{b_2}} \tag{70}$$

$$\Delta v_2 = \frac{\sqrt{\mu} \left\{ \sqrt{1 - e_2} - \sqrt{2 - x^2 (1 - e_1)} \right\}}{\sqrt{b_4}} \tag{71}$$

By differentiation w.r.t. the variable x, and after some reductions, we get

$$x^2 = \frac{2b_4}{(1 - e_1)(b_2 + b_4)} \tag{72}$$

i.e.

$$(x)_{Min} = \pm \sqrt{\frac{2b_4}{(1 - e_1)(b_2 + b_4)}} = \text{constant} \tag{73}$$

we can find, from previous steps, after some reductions

$$(a_T)_{Min} = \frac{b_2 + b_4}{2} = \text{constant} \tag{74}$$

and

$$(e_T)_{Min} = \frac{b_4 - b_2}{b_2 + b_4} \tag{75}$$

Finally, we get

$$\begin{aligned} (\Delta v_T)_{Min} &= \sqrt{\frac{\mu (1 - e_2)}{b_4}} - \sqrt{\frac{\mu (1 - e_1)}{b_2}} \\ &+ \sqrt{\frac{2\mu b_4}{b_2 (b_2 + b_4)}} - \sqrt{\frac{2\mu b_2}{b_4 (b_2 + b_4)}} = \text{constant} \end{aligned} \tag{76}$$

2.4.2. *Consideration of two parameters x, y*

Let

$$y = \frac{v_{B2}}{v_{B1}} = \frac{\text{velocity after impulse at point B}}{\text{velocity before impulse at point B}} > 1$$

whence

$$y = \sqrt{\frac{1-e_2}{1-e_T}}$$

and

$$\Delta v_T = \Delta v_1 + \Delta v_2 = \\ = \sqrt{\frac{\mu(1+e_T)}{b_2}} - \sqrt{\frac{\mu(1-e_1)}{b_2}} + \sqrt{\frac{\mu(1-e_2)}{b_4}} - \sqrt{\frac{\mu(1-e_T)}{b_4}} \tag{77}$$

i.e.

$$\Delta v_T = \sqrt{\mu}\left[\frac{\sqrt{1+e_T}-\sqrt{1-e_1}}{\sqrt{b_2}} + \frac{\sqrt{1-e_2}-\sqrt{1-e_T}}{\sqrt{b_4}}\right] \tag{78}$$

But, we have

$$1+\ e_T = x^2(1-e_1)\,;1-e_T = \frac{1-e_2}{y^2} \tag{79}$$

Therefore,

$$\Delta v_T = (x-1)\sqrt{\frac{\mu(1-e_1)}{b_2}} + \left(1-\frac{1}{y}\right)\sqrt{\frac{\mu(1-e_2)}{b_4}} = f(x,y) \tag{80}$$

whence,

$$\frac{\partial \Delta v_T}{\partial x} = \sqrt{\frac{\mu(1-e_1)}{b_2}} + \sqrt{\frac{\mu(1-e_2)}{b_4}}\frac{1}{y^2}\frac{\partial y}{\partial x} = 0 \tag{81}$$

But,

$$\frac{\partial y}{\partial x} = \frac{x(1-e_1)\sqrt{1-e_2}}{\{2-x^2(1-e_1)\}^{3/2}} \tag{82}$$

After application of optimization condition and some reductions, we find

$$(x^2)_{Min} = \frac{2b_4}{(1-e_1)(b_2+b_4)} \tag{83}$$

and

$$\left(y^2\right)_{Min} = \frac{(1-e_2)}{2}\left\{1+\frac{b_4}{b_2}\right\} \tag{84}$$

3. Concluding Remarks

The choice of x as our independent variable leads to the most simple and exact formulae of the problem of optimization. From the concept of optimum condition, we can determine the unique values of $(e_T)_{\text{Min}}$, $(a_T)_{\text{Min}}$ from Eqs (5) & (7), knowing the given values of a_1, e_1, a_2, e_2 of the initial and final orbit.

The minimum characteristic velocity $(\Delta v_T)_{\text{Min}}$ expressed by Eq. (15) is obviously cited in terms of the initial and final orbital elements a, e; and correspondingly for configurations 2, 3, 4. The optimization procedure is based on formulas stemming from first principles considerations. It is not a special case, arising from the general problem, when we assume the non-coplanar trajectories.

There are four feasible configurations for this transfer problem, namely sub articles 2.1, 2.2, 2.3, 2.4. Two of the four configurations are relevant to the peri-apse perpendicular initial impulse, the other two are relevant to the final perpendicular apo-apse impulse.

We demonstrated that the elliptic Hohmann type transfer is the most economic in the expenditure of fuel, because when we substitute the value of $(x)_{\text{Min}}$ in Eqs (5) and (7), we get the unique values of $(e_T)_{\text{Min}}$ and $(a_T)_{\text{Min}}$ for the elliptic Hohmann type of orbit transfer.

$(\Delta v_T)_{\text{Min}}$ is a measure of the extremum of the characteristic velocity, and consequently expenditure of fuel. The two produced analysis of Arts: 2.1.1, 2.1.2 lead to the same value of $(x)_{\text{Min}}$, when we assume a single parameter x or when we propose the two parameters x, y, asserting the optimality of the Hohmann transfer.

Config.	$(\Delta v_T)_{Min}$	Notes
1	0.184290	most economic
2	0.186960	
3	0.187265	
4	0.185015	

We assign $(\Delta v_T)_{\text{Min}}$ in the case of one parameter x and two parameters x, y by the substitution of $(x)_{\text{Min}}$ and $(y)_{\text{Min}}$ in the adequate equalities. We conclude that the first configuration is the most economic, compared with the other three ones.

In other words the derivations above in Art. 2.1.1 and 2.1.2 demonstrate that the generalized Hohmann transfer is an optimum one when we apply two impulsive thrusts.

The results of computations are implied in the following table, assuming the elliptic orbit transfer from Earth to planet Mars.

We put $\mu = 1$, i.e., we adopted canonical units. The numerical values of the elements of the orbits are the following [1]:

a_1 (Earth) = 1 e_1 (Earth) = 0.016726

a_2(Mars) = 1.523691 e_2(Mars) = 0.093368

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Nomenclature:

x	ratio of velocities after and before initial impulse at point A.
y	ratio of velocities after and before second impulse at point B.
a_1	semi–major axis of initial orbit.
a_2	semi–major axis of final orbit.
e_1	eccentricity of initial orbit.
e_2	eccentricity of final orbit.
a_T	semi–major axis of transfer orbit.
e_T	eccentricity of transfer orbit.
v_{A1}	peri–apse velocity in initial orbit at point A.
v_{A2}	peri–apse velocity of transfer orbit at point A.
v_{B1}	apo–apse velocity of transfer orbit at point B.
v_{B2}	apo–apse velocity in final orbit at point B.
Δv_1	increment of velocity at point A.
Δv_2	increment of velocity at point B.
Δv_T	characteristic velocity.
μ	constant of gravitation.