

Nonlinear normal modes of three degree of freedom mechanical oscillator

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In this paper we show the method of calculation of nonlinear normal modes and its application to mechanical coupled systems. We present bifurcation diagram of nonlinear normal modes in three degree of freedom system. We show the appearance of internal resonances and their important role in dynamics of nonlinear coupled oscillators.

Keywords: Nonlinear normal modes, coupled systems, continuation of periodic solutions

1. Introduction

Exploring dynamics of nonlinear systems cause a lot of problems, still there is no universal technique to predict their behavior [1, 2]. In this paper we focus our attention on the conservative coupled systems. It is well known fact that analysis of such systems can give a view on the underlying dynamics of slightly dissipative oscillators, i.e., large class of mechanical and structural systems [3].

To explore dynamics of conservative systems one can calculate the nonlinear normal modes (NNMs) [3–7]. The idea of NNMs comes directly from the theory of the linear vibration, where the linear normal modes (LNMs) are taken into account. The motion of the linear system is governed by eigenfrequencies, for N coupled oscillators one can calculate N eigenfrequencies and corresponding eigenvectors, for this frequencies coupled oscillators behave like decoupled, independent systems. Frequency of their motion is given by eigenfrequency and initial configuration for each eigenfrequency by eigenvectors. The NNMs are extension of this idea for conservative nonlinear systems (for low total energy in the system, where nonlinearity usually have small influence on the dynamics and NNMs overlap with LNMs).

There are two main definitions of NNMs, we present them in the chronological order. Rosenberg [8] defines NNMs as a vibration in union, which mean that all

coupled systems reach extreme amplitudes (maximums or minimums) and cross zero in the same time. This is quite natural continuation of LNMs, newer less the motion of nonlinear system is much more complex then the linear's one, thus this definition do not take into account its all properties (like resonances between oscillators). The next definition was given by Shaw and Pierre [3,], the NNMs where defined as an invariant manifolds in the phase space. Such approach let us calculate NNMs, but invariant manifolds have complicated structure and in most cases it is hard to compute them. There are a lot other analytical techniques to solve NNMs problem, but the main assumptions is the small nonlinearity and few DoFs [3–8]. There are large number of constants of motion for each DoF in the analytical calculation and it is a problem to cope with dozens of them. That is why numerical techniques were applied. The most natural software is AUTO [9] where one can continue periodic solutions (PSs) in parameter space, finding also all bifurcations. The other approach is presented in [10] and can be used also to compute NNMs for systems modeled by finite elements, i.e., discretized beams, shells ect.

The paper is organized as follows. In Sec. 2 we give a brief description how to calculate NNMs in AUTO. Section 3 describes the considered model. The numerical results are discussed in Sec. 4. Finally, our results are summarized in Sec. 5.

2. How to compute NNM in AUTO

In this section we show how the NNMs can be calculated in AUTO. First we describe the algorithm of calculation of PSs which is used in AUTO, than we show how practically compute NNMs.

2.1. Algorithm

The algorithm of PSs calculation in AUTO is very sophisticated, here we describe just the main idea. Assume the system of equations is given in the following form:

$$\dot{\mathbf{x}} = F(\mathbf{x}, \lambda) \quad (1)$$

where $\mathbf{x} = (x_1, \dots, x_N)^T$, $F(x)$ is a matrix of the right hand side and λ is a continuation parameter. Finding PSs in such systems lead to solution of N boundary value problem with $N + 1$ parameters. Assuming, we have the proper starting point $(\mathbf{x}_0, \lambda_0)$ for continuation (how to find it is described later) AUTO is using the pseudo arc-length continuation [11] (see Fig. 1) to calculate next points along the branch of PSs. The pseudo arc-length continuation has two steps: predictor and corrector. The predictor step let us calculate, basing on knowledge of starting point $\mathbf{z}_0 = (\mathbf{x}_0, \lambda_0)$, the next point $\mathbf{z}_1 = (\mathbf{x}_1, \lambda_1)$ which lay on the branch of PSs. The initial approximations of the tangent vector \mathbf{z}_1^* is given by following formula $\mathbf{z}_1^* = \mathbf{z}_0 + \dot{\mathbf{z}}_0 \Delta s$, where $\dot{\mathbf{z}}_0$ is a unit tangent to z_0 vector, Δs is a step size which can modified during computation. The next step is corrector, where by Newton–Raphson algorithm the initial guess is corrected to the proper solution. The correction is made on perpendicular direction to vector to . Finally, after some iterations, one can obtain point which is a PS. To calculate next point along the branch the predictor – corrector procedure is repeated. The detailed description of the pseudo arc-length continuation can be found in [9,11].

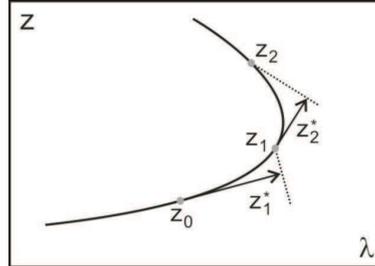


Figure 1 Schematic plot of pseudo arc length continuation

2.2. Practical implementation

Continuation package AUTO is dedicated to calculation of PS in many class of systems, however in case of conservative systems the straight forward calculations are impossible, the equation must be in the special form [13]. Let us take under consideration the dynamical system given by the following equation:

$$\mathbf{x}'(t) = f(\mathbf{x}(t), \lambda) \quad (2)$$

where $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ is a continuation parameter. After rescaling Eq. (2) and adding boundary values condition one have to solve the following set of equation:

$$\begin{aligned} \mathbf{x}(\tau)_1' &= T_1 f(\mathbf{x}(\tau)_1) + \lambda_1 \partial E / \partial \mathbf{x}(\tau)_1 \\ \mathbf{x}(0)_1 &= \mathbf{x}(1)_1, \\ \int_0^1 \mathbf{x}(\tau)_1^T \mathbf{x}(\tau)_0' d\tau &= 0, \end{aligned} \quad (3)$$

where $T_1 \in \mathbb{R}$ is a unknown period of PS, $\tau \in (0, 1)$ is a rescaled time, boundary condition in second line are periodic condition (values of amplitudes of vector are the same for $\tau = 0$ and for $\tau = 1$) and integral, in the last row of Eq. (3), ensure the uniqueness of the solution. In Eq. (3) there are two natural continuation parameter period T_1 and λ_1 , both of are used to calculate NNM. Contrary, to scheme presented in previous subsection now one has $(N + 2)$ unknowns, six boundary condition and one integral, primary continuation parameter is T_1 and secondary λ_1 .

2.3. Stability of periodic solution and bifurcations in AUTO

The stability of PSs can be easily calculated using Floquet theory [12]. Eigenvalues of monodromy matrix govern the stability of PSs. This eigenvalues are complex numbers and called Floquet multipliers (FMs). If all FMs are enclosed in an unit circle the PS is stable and if one of them is outside the PS is unstable, for PS there is always one trivial FM at +1. The typical bifurcation (changes of stability of PS) are saddle-node (when one FM crosses the unit circle by +1), torus bifurcation (when pair of complex conjugate FMs cross the unit circle) and period doubling bifurcation (when one FM crosses the unit circle by -1). In AUTO one can observe also so called branching point (BP) which indicate the appearance of new branches of PS.

3. Model of the system

As a model we considered 3-DoF mechanical oscillator presented in Fig. 2. In our calculation we assumed, for simplicity, dimensionless parameters. Transformation of coordinates to this form is easy and was shown for example in Perlikowski et. al. [14]. Also to present fundamental properties the dimensionless parameter are most suitable. Equations of system shown in Fig. 2, after rescaling of time, are given by

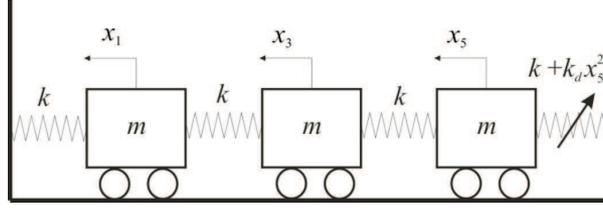


Figure 2 General scheme of the system

six first ordinary differential equations:

$$\begin{aligned}
 \dot{x}(\tau)_1 &= Tx(\tau)_2 \\
 \dot{x}(\tau)_2 &= T(-kx(\tau)_1 - k(x(\tau)_1 - x(\tau)_3))/m \\
 \dot{x}(\tau)_3 &= Tx(\tau)_4 \\
 \dot{x}(\tau)_4 &= T(-k(x(\tau)_3 - x(\tau)_1) - k(x(\tau)_3 - x(\tau)_5))/m \\
 \dot{x}(\tau)_5 &= Tx(\tau)_6 \\
 \dot{x}(\tau)_6 &= T(-kx(\tau)_5 - k_d x(\tau)_5^3 - k(x(\tau)_5 - x(\tau)_3))/m
 \end{aligned} \tag{4}$$

where m is a dimensionless mass of the each system, k is a dimensionless stiffness of linear spring, k_d is a dimensionless stiffness of nonlinear part of spring, T is a period of PS and $\tau \in (0, 1)$ is rescaled time. Motion of oscillators is frictionless. In our calculation we assumed $k = 1$, $m = 1$ and $k_d = 0.25$.

As it was mentioned before to compute NNM in AUTO it is necessary to transform system in to special form. The total energy E of system is given by following equation:

$$\begin{aligned}
 E &= 0.5(x(\tau)_2^2 + x(\tau)_4^2 + x(\tau)_6^2) + 0.5kx(\tau)_1^2 + 0.5k(x(\tau)_1 - x(\tau)_3)^2 + \\
 &\quad + 0.5k(x(\tau)_3 - x(\tau)_5)^2 + 0.5kx(\tau)_5^2 + 0.5k_d x(\tau)_5^4
 \end{aligned} \tag{5}$$

After taking in to consideration Eqs (3, 4, 5) we reach final form:

$$\begin{aligned}
 \dot{x}(\tau)_1 &= Tx(\tau)_2 + \lambda(2x(\tau)_1 - x(\tau)_3) \\
 \dot{x}(\tau)_2 &= T(-2x(\tau)_1 + x(\tau)_3) + \lambda x(\tau)_2 \\
 \dot{x}(\tau)_3 &= Tx(\tau)_4 + \lambda(2x(\tau)_3 - x(\tau)_1 - x(\tau)_5) \\
 \dot{x}(\tau)_4 &= T(-2x(\tau)_3 + x(\tau)_1 + x(\tau)_5) + \lambda x(\tau)_4 \\
 \dot{x}(\tau)_5 &= Tx(\tau)_6 + \lambda(2x(\tau)_5 - x(\tau)_3 - 0.5x(\tau)_5^3) \\
 \dot{x}(\tau)_6 &= T(-2x(\tau)_5 - 0.5x(\tau)_5^3 + x(\tau)_3) + \lambda x(\tau)_6
 \end{aligned} \tag{6}$$

where λ is controlling parameter which should be zero during the whole calculation. The boundary conditions and integral condition correspond to presented in Eq. (3).

4. Numerical examples

The first step in the calculation of NNMs is determination of starting points – beginning of branches of NNMs because for zero energy E NNMs overlap with LNMs one can do this by solving eigenproblem $\det|A - \lambda_e I| = 0$ where A is a linearized right hand side matrix of system (4), λ_e is a vector of eigenvalues of matrix A and I is an identity matrix. For considered system there are three solution: $T = [8.209, 4.443, 3.400]$ where $T = 2\pi/\sqrt{(\lambda_e)}$. The frequency of oscillation is given by $\omega = 1/T$, so the starting values are: $\omega = [0.122, 0.225, 0.294]$.

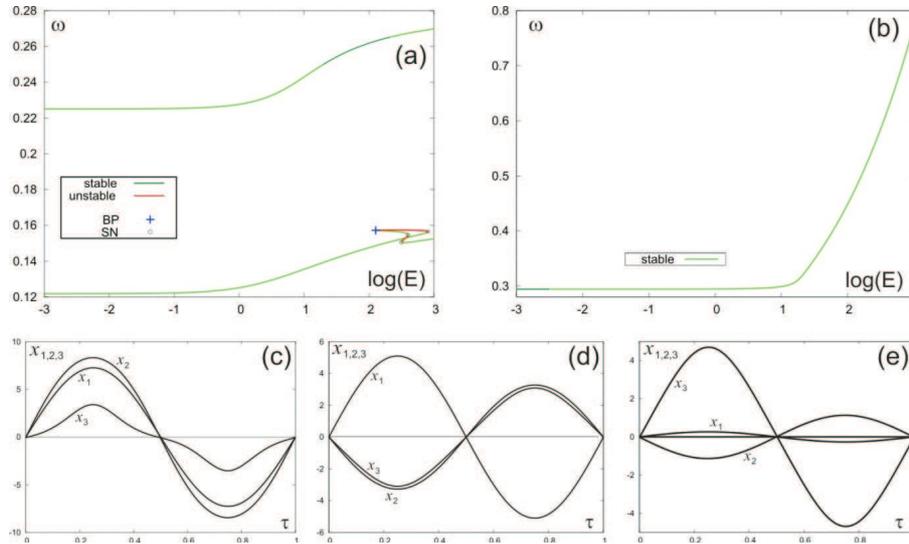


Figure 3 NNMs in energy–frequency plots: (a) two branches of NNMs, (b) third branch of NNMs, green and red color indicate the stable and unstable periodic solutions. The loop observed in first branch of NNMs in (a) show occurrence of inner resonances (for details see Fig. 4). In (c–e) the time traces of each system for fixed values of energy ($\log E = 1.75$) and frequency $\omega = 0.144$ (c), $\omega = 0.259$ (d), $\omega = 0.39$ (e). In (c) all oscillators are in–phase oscillation, in (d) second and third are in anti–phase motion to first one and in (e) first and third are in–phase and second in anti–phase to them.

The typical way of presenting NNM is an energy–frequency plot, where the logarithm of total system energy E (see Eq. (5)) is plotted versus frequency of oscillation ω . In numerical calculation we used RAUTO [15]. In Fig. 3a,b we show branches of NNMs, for better visibility we present them on two plots. One can easily see that till $\log E = 0$ the frequency is constant, so the nonlinearity has neglecting influence on the motion of the system. For larger energy levels values of frequencies start to increasing, especially strongly for third branch. We limit our calculation to upper bound $\log E = 3$, which is large energy and in practice hardly ever met. In Fig. 3a there is a loop on the first branch which indicates the internal resonance, the detailed description of this phenomena is shown in Fig. 4. Now, let us show the shape of PSS on different branches for fixed values of energy ($\log E = 1.75$). The

corresponding frequencies are: in Fig. 3c $\omega = 0.144$, in Fig. 3d $\omega = 0.259$, in Fig. 3e $\omega = 0.39$. In Fig. 3c all oscillators are in-phase oscillations, in Fig. 3d second and third are in anti-phase motion to first one and in Fig. 3e first and third are in-phase and second in anti-phase to them. The influence of nonlinearity on shape of periodic solution is visible especially in Fig. 3c where the trace of third oscillator is strongly deformed.

In Fig. 4 we show the detailed investigations of internal resonances, which are crucial points for nonlinear systems. In Fig. 4a one can see the zoom of the resonance loop for the first branch (frequency of first branch resonate with frequency of third branch), circles correspond to saddle-node bifurcations (SN) and plus to branching point (BP). The BP in Fig. 4a indicate touching with branch which have the initial period three times higher than first branch (if there exist the branch with initial period T there are also branches with initial period $2T$, $3T$ and so on), we do not plot this branches here. The SN cause change of stability along the branch of PS.

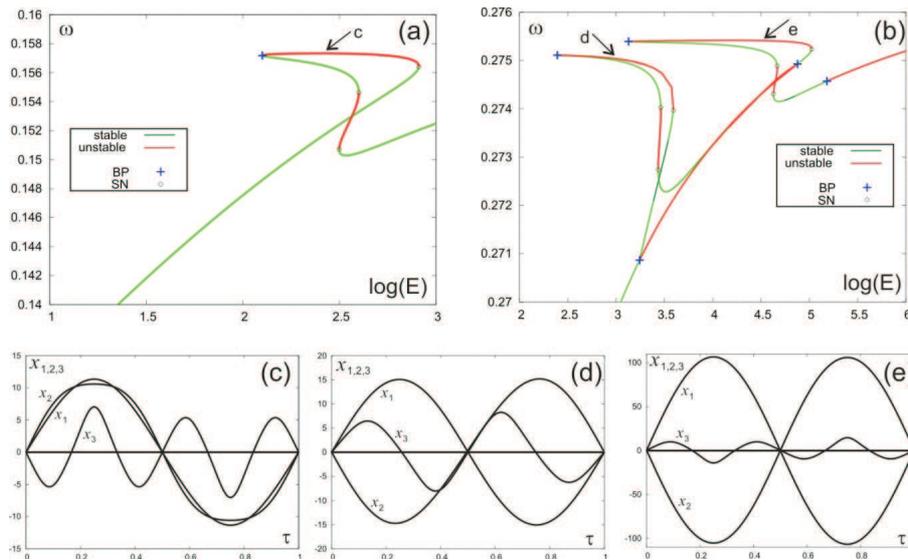


Figure 4 NNMs in energy–frequency plots: (a) first branch of NNMs, (b) second branch of NNMs, green and red color indicate the stable and unstable periodic solutions. The loops observed along the branches of NNMs indicate the occurrence of internal resonances between coupled systems. In (c–f) one can see time traces of oscillators in resonances. In (c) for the first branch and in (e) for second one can see 1:1:3 locking and in (d) for second branch 1:1:2 locking. For all cases first and second oscillators stay in in-phase motion (c) or in anti-phase motion (d, e) while third one show motion with period 2 or 3.

In Fig. 3c one can see time traces for $\log E = 2.39$ and $\log E = 1.75$. It is clearly visible that resonance is 1:1:3, i.e., for one period of first and second oscillators third one has three periods. In Fig. 4b we show internal resonances for second branch (also resonance with third branch), they occur for larger values of energy

but it is interesting to show them. The first branch is a 1:1:2 locking, this branch appear from BP and also finished in BP, at the sharp edge of this branch met with period $2T$ branch. The next branch indicate 1:1:3 resonance. If one follow the main branch further there is a next branching point - the starting point of branch with 1:1:4 locking. In Fig. 4d,e one can find time traces of oscillators for points indicate in Fig. 4b, once again the resonant is the third oscillator, first and second are in anti-phase motion. The stability of NNMs play important role when looking on the dynamics of slightly damped systems, where some structure of oscillation preserve, and jumps between stable PSs can be observed.

5. Conclusions

In this paper we show the NNNs for coupled mechanical systems calculated in AUTO. One can observe the bifurcation diagram with described bifurcations along the branches of PSs. The influence of nonlinearity is visible for larger energy values: $\log E > 0$. From practical point of view that this is a good feature. Internal resonances occur for $\log E > 2$ which mean that for slightly damped mechanical oscillators of this type the locking solutions are far from usually working range.

This approximation contrary to integration of the system's equations can give a overview of dynamics in conservative oscillators. In case of more than two DoF one can obtain the fast information about dynamics contrary to classical Poincare section for fixed energy levels (there are just ineffective).

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