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On The Generalized Hohmann Transfer with Plane Change Using Energy Concepts Part I

Osman M. Kamel
Astronomy and Space Science Dept.
Faculty of Sciences, Cairo University, Egypt

Adel S. Soliman
Theoretical Physics Dept.
National Research Center, Egypt
assoliman16@hotmail.com

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We investigate in this article the optimized orbit transfer of a space vehicle, revolving initially around the primary, in a similar orbit to that of the Earth around the Sun, in an elliptic trajectory, to another similar elliptic orbit of an adequate outer planet. We assume the elements of the initial orbit to be that of the Earth, and the elements of the final orbit to be that of an outer adequate planet, Mars for instance. We assume the elements of the two impulse Hohmann generalized configuration (the case of elliptic, non coplanar orbits) to be a_1, e_1, a_2, e_2, a_T, e_T. From the very beginning, we should assign $\theta = \alpha_1 + \alpha_2$, the total plane change required. α_1 is the plane change at the first instantaneous impulse at peri–apse, which will be minimized, and α_2 the plane change at the second instantaneous thrust at apo–apse.

Keywords: Orbital mechanics, elliptic Hohmann transfer with plane change, optimization

1. Introduction

The Hohmann transfer is the minimum two impulse transfer between coplanar circular and elliptic orbits [1]. As for the derivations of the velocity change requirements ΔV_1, ΔV_2 and transfer time, we can draw a graph which illustrates total energy/satellite mass as a function of orbit period $P = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$ that means a plot of $\frac{-\mu}{2a}$ versus $\left(\frac{2\pi}{\sqrt{\mu}}\right) a^{3/2}$. Plotted results are extensively established [1], [2]. For classical Hohmann transfer if $\frac{r_2}{r_1}\langle 15.58,\ r_2 \rangle r_1$, is not satisfied, then the Hohmann transfer is no longer optimal. For these conditions Bi–elliptic transfers are always more economical in propellant than Hohmann transfer configurations [1]. The Hohmann transfer is a relatively simple maneuver, especially the classical model. It may be simplified or complicated easily, and we may encounter very difficult situations. This can be easily seen from the literature of orbit transfer [3]. There exist four feasible Hohmann configurations according to the coincidence of peri–apse and apo-apse of the three ellipses. We consider the first of them [4]. Radius or major axis change, in the process of orbit transfer may be coupled by a plane change for the circular or elliptic orbit transfer. This is an important practical procedure. The optimal two impulse transfer that satisfy these conditions is the Hohmann transfer, with split plane change. The first ΔV_1 thrust not only produces a transfer ellipse but also induce a rotation of the orbital plane. At the second impulse, a second tilt is induced as well as the production of the final elliptic orbit. An engine firing in the out–of plane direction is required for the change of plane. The point of firing becomes a point in the new orbit, and the burn point becomes the intersection of the current orbit and the desired orbit. Definitely, we should perform plane change in the smartest way, since it is fuel expensive, anyway you do them. Even without the examination of the specific equations, planning a space mission, reduces to a problem of geometry, timing, mechanics of orbital motion, and a lot of common sense.

2. Method and Results

In this article we investigate the generalized Hohmann orbit transfer with split − plane change. We take into account, the first configuration, where the apo–apse of the transfer orbit coincides with the apo–apse of the final orbit, and the peri–apse of the initial and the transfer orbit are coincident [5], Fig. 1. ΔV_1 produces at peri–apse of initial orbit, a transfer ellipse as well as a plane change α_1. Similarly at apo–apse, ΔV_2 rotates the orbit plane through an angle $\alpha_2 = \theta - \alpha_1$, and designs the final elliptic orbit as shown in Fig. 2, Fig. 3 and Fig. 4, represent the velocity vector triangular addition ΔV_1 , ΔV_2 respectively.

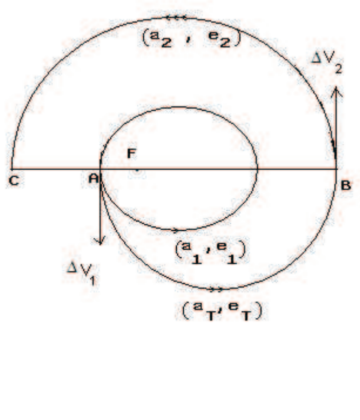

Figure 1 Generalized Hohmann Transfer

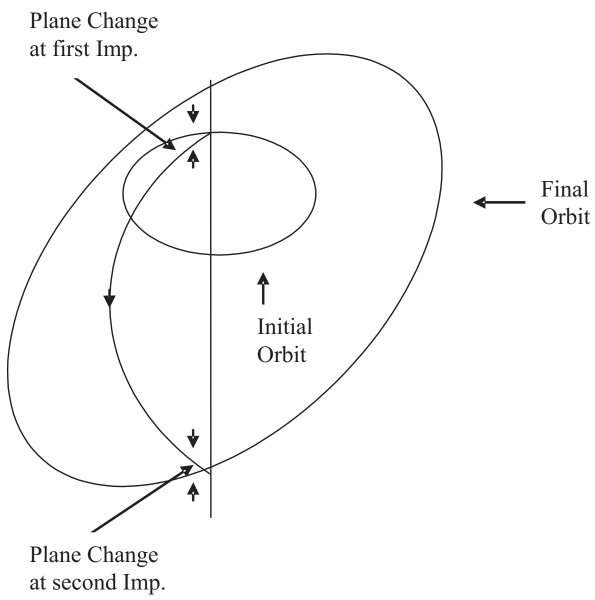

Figure 2

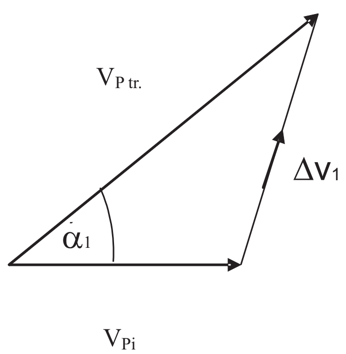

Figure 3

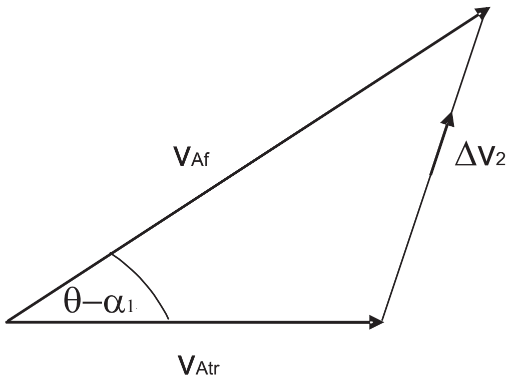

Figure 4

where

$$V_{P_i} = \sqrt{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}}$$

$$V_{P_{tr}} = \sqrt{\frac{\mu\,(1+e_T)}{a_T\,(1-e_T)}}$$

$$V_{A_{tr}} = \sqrt{\frac{\mu\,(1-e_T)}{a_T\,(1+e_T)}}$$

$$V_{A_f} = \sqrt{\frac{\mu\,(1-e_2)}{a_2\,(1+e_2)}}$$

The increments of velocities at peri–apse and apo–apse of the elliptic transfer orbit is given by

$$\begin{aligned} \Delta V_1^2 &= \frac{\mu\,(1+e_1)}{a_1\,(1-e_1)} + \frac{\mu\,(1+e_T)}{a_T\,(1-e_T)} \\ &\quad -2\sqrt{\left\{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}\right\}\left\{\frac{\mu\,(1+e_T)}{a_T\,(1-e_T)}\right\}}\cos\alpha_1 \qquad (1) \\ \Delta V_2^2 &= \frac{\mu\,(1-e_2)}{a_2\,(1+e_2)} + \frac{\mu\,(1-e_T)}{a_T\,(1+e_T)} \\ &\quad -2\sqrt{\left\{\frac{\mu\,(1-e_2)}{a_2\,(1+e_2)}\right\}\left\{\frac{\mu\,(1-e_T)}{a_T\,(1+e_T)}\right\}}\cos(\theta-\alpha_1) \qquad (2) \\ \Delta V_T &= \sqrt{\Delta V_1^2} + \sqrt{\Delta V_2^2} = \Delta V_1 + \Delta V_2 \qquad (3) \end{aligned}$$

where θ is arbitrary and given by $\theta = \alpha_1 + \alpha_2$ and e_T , a_T are calculated from the formulae

$$e_T = \frac{[a_2(1+e_2) - a_1(1-e_1)]}{[a_2(1+e_2) + a_1(1-e_1)]} = \frac{b_4 - b_1}{b_4 + b_1} \tag{4}$$

$$a_T = \frac{[a_2(1+e_2) + a_1(1-e_1)]}{2} = \frac{b_4 + b_1}{2} \tag{5}$$

where

$b_1 = a_1(1-e_1)$

$b_4 = a_2(1+e_2)$

α_1 to be optimized by the condition of minimization $\frac{\partial \Delta V_T}{\partial \alpha_1} = 0$.

When checking the second order conditions, we treat the functional $V[y]$ as a function of ε i.e. $V(\varepsilon)$.

We set $\frac{dV}{d\varepsilon} = 0$ and consequently, we acquire the first order necessary condition for an extremal. For distinction between maximization and minimization we calculate the second derivative $\frac{d^2V}{dt^2}$. We find the following second order necessary conditions:

$\frac{d^2V}{dt^2} \le 0$ – for maximize of V,

$\frac{d^2V}{dt^2} \ge 0$ – for minimize of V.

As for second order sufficient conditions:

$\frac{d^2V}{dt^2} < 0$ – for maximize of V,

$\frac{d^2V}{dt^2} > 0$ – for minimize of V,

cf. [5] for definitions.

For the first configuration

Let

$$A_1 = \frac{\mu}{a_1}\left(\frac{1+e_1}{1-e_1}\right) \qquad B_1 = \frac{\mu}{a_T}\left(\frac{1+e_T}{1-e_T}\right)$$
$$C_1 = \frac{\mu}{a_2}\left(\frac{1-e_2}{1+e_2}\right) \qquad D_1 = \frac{\mu}{a_T}\left(\frac{1-e_T}{1+e_T}\right) \tag{6}$$

whence

$$\Delta V_T = \left(A_1 + B_1 - 2\sqrt{A_1 B_1}\cos\alpha_1\right)^{1/2} + \left(C_1 + D_1 - 2\sqrt{C_1 D_1}\cos(\theta - \alpha_1)\right)^{1/2} \tag{7}$$

By partial differentiation with respect to α_1 and equating to zero, and after rearrangements and clearing fractions, we find that

$$\frac{A_1 B_1 \sin^2\alpha_1}{A_1 + B_1 - 2\sqrt{A_1 B_1}\cos\alpha_1} = \frac{C_1 D_1 \sin^2(\theta - \alpha_1)}{C_1 + D_1 - 2\sqrt{C_1 D_1}\cos(\theta - \alpha_1)} \tag{8}$$

From which we can deduce

$$A_1 B_1 \left[C_1 + D_1 - 2\sqrt{C_1 D_1}\left(bx + a\sqrt{1-x^2}\right)\right]\left(1-x^2\right)$$
$$= C_1 D_1 \left[A_1 + B_1 - 2\sqrt{A_1 B_1}x\right]\left[\left(a^2-b^2\right)x^2 + b^2 - 2abx\sqrt{1-x^2}\right] \tag{9}$$

where:

$$x = \cos\alpha_1 \qquad \sqrt{1-x^2} = \sin\alpha_1 \qquad a = \sin\theta \qquad b = \cos\theta \tag{10}$$

After some reductions, we find

$$\begin{aligned}
&A_1 B_1 \left(C_1 + D_1\right) - C_1 D_1 \left(A_1 + B_1\right) b^2 \\
&+ \left(2C_1 D_1 \sqrt{A_1 B_1} b^2 - 2bA_1 B_1 \sqrt{C_1 D_1}\right) x \\
&- \left\{A_1 B_1 \left(C_1 + D_1\right) + C_1 D_1 \left(A_1 + B_1\right)\left(a^2 - b^2\right)\right\} x^2 \\
&+ \left\{2bA_1 B_1 \sqrt{C_1 D_1} + 2C_1 D_1 \sqrt{A_1 B_1}\left(a^2 - b^2\right)\right\} x^3 \\
&= \sqrt{1-x^2}\left\{2aA_1 B_1 \sqrt{C_1 D_1} - 2abC_1 D_1 \left(A_1 + B_1\right) x\right. \\
&\left. + \left(4abC_1 D_1 \sqrt{A_1 B_1} - 2aA_1 B_1 \sqrt{C_1 D_1}\right) x^2\right\}
\end{aligned}$$

Let

$$\begin{aligned}
&A_1 B_1 \left(C_1 + D_1\right) - C_1 D_1 \left(A_1 + B_1\right) b^2 = E_1 \\
&2C_1 D_1 \sqrt{A_1 B_1} b^2 - 2bA_1 B_1 \sqrt{C_1 D_1} = \quad E_2 \\
&-\{A_1 B_1 \left(C_1 + D_1\right) + C_1 D_1 \left(A_1 + B_1\right)\left(a^2 - b^2\right)\} = E_3 \\
&2bA_1 B_1 \sqrt{C_1 D_1} + 2C_1 D_1 \sqrt{A_1 B_1}\left(a^2 - b^2\right) = E_4 \\
&2aA_1 B_1 \sqrt{C_1 D_1} = E_5 \\
&-2abC_1 D_1 \left(A_1 + B_1\right) = E_6 \\
&4abC_1 D_1 \sqrt{A_1 B_1} - 2aA_1 B_1 \sqrt{C_1 D_1} = E_7
\end{aligned}$$

i.e.

$$E_1 + E_2 x + E_3 x^2 + E_4 x^3 = \sqrt{1-x^2}\left(E_5 + E_6 x + E_7 x^2\right) \tag{11}$$

After squaring and some reductions, we may write

$$\begin{aligned}
&\left(E_1^2 - E_5^2\right) + \left(2E_1 E_2 - 2E_5 E_6\right) x \\
&+ \left(2E_1 E_3 + E_2^2 - 2E_5 E_7 - E_6^2 + E_5^2\right) x^2 \\
&+ \left(2E_1 E_4 + 2E_2 E_3 - 2E_6 E_7 + 2E_5 E_6\right) x^3 \\
&+ \left(2E_2 E_4 + E_3^2 - E_7^2 + 2E_5 E_7 + E_6^2\right) x^4 \\
&+ \left(2E_3 E_4 + 2E_6 E_7\right) x^5 + \left(E_4^2 + E_7^2\right) x^6 = 0
\end{aligned} \tag{12}$$

Set

$$\begin{aligned}
&E_1^2 - E_5^2 = \Psi_1 \quad 2E_1E_2 - 2E_5E_6 = \Psi_2 \\
&2E_1E_3 + E_2^2 - 2E_5E_7 - E_6^2 + E_5^2 = \Psi_3 \\
&2E_1E_4 + 2E_2E_3 - 2E_6E_7 + 2E_5E_6 = \Psi_4 \\
&2E_2E_4 + E_3^2 + 2E_5E_7 + E_6^2 - E_7^2 = \Psi_5 \\
&2E_3E_4 + 2E_6E_7 = \Psi_6; E_4^2 + E_7^2 = \Psi_7
\end{aligned}$$

That means after some reductions, we get an equation of degree 6 in $x = \cos\alpha_1$:

$$\psi_1 + \psi_2 x + \psi_3 x^2 + \psi_4 x^3 + \psi_5 x^4 + \psi_6 x^5 + \psi_7 x^6 = 0 \tag{13}$$

where the E's appearing in the ψ's are functions of one parameter α_1 and could be expressed in terms of a_1, a_2, e_1, e_2 [6], which are constants since they are the known elements of the terminal orbits.

3. Discussion

The Hohmann transfer is an optimal two impulse transfer. We suppose that the first increment at peri–apse ΔV_1, not only produces a transfer elliptic orbit, but also rotates the orbital plane by an optimal angle α_1.

At apo–apse the second increment of velocity ΔV_2 will produce the trajectory of the final elliptic orbit and rotates the orbit plane by an angle $\alpha_2 = \Theta - \alpha_1$. We have $\Delta V_T = \Delta V_1 + \Delta V_2$. For the minimization of ΔV_T we have the condition $\partial \Delta V_T \ / \ \partial \alpha_1 = 0$. By expansions, rearrangements and clearing fractions, we acquire through a purely analytical method, except for the resolution of the algebraic sixth degree equation, the value of the optimized α_1 i.e. $(\alpha_1)_{opt}$, whence $(\alpha_2)_{opt} = \Theta - (\alpha_1)_{opt}$. By substitution of $(\alpha_1)_{opt}$, we can easily evaluate $(\Delta V_1)_{opt}$ and $(\Delta V_2)_{opt}$ from Eqs 1, 2, and $(\Delta V_T)_{Min} = (\Delta V_1)_{opt} + (\Delta V_2)_{opt}$ [6]. The sixth degree algebraic equation $O(x^6)$ Eq.13 could be easily solved by a mathematica software program.

In this article we laid the foundation of the theory. The approach is purely analytical except the solution of Eq. 13 which is numerical. No numerical illustrations are included, for the time being, for instance for the obtenition of numerical results for the elliptic Hohmann Earth – outer planets, configurations. This will be done in the second part of the paper. We restrict ourselves here to the consideration of the first configuration.

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Appendix

David Eagle [7], exposed an excellent clear topic about the optimization of the classical Hohmann transfer, for coplanar and non–coplanar circular orbits invented in the year 1925 by the German engineer Walter Hohmann. Eagle cited the following remarks concerning this transfer:

1. Two impulses 180^0 apart are required in the direction of motion, collinear with velocity vector at peri–apse and apo–apse of transfer orbit.

2. Velocity and not position of vehicle is changed instantaneously.

3. Both thrusts are posigrade, i.e. in direction of orbital motion.

4. The transfer time from first to second impulse is given by

   $\tau = \pi\sqrt{\frac{a^3}{\mu}} \qquad a = \frac{1}{2}\left(r_i + r_f\right)$

   Script i, f refer to initial circular orbit and final circular orbit respectively; a denotes semi major axis of transfer ellipse.

He derived the following formulae for the purpose of his article:

$$\Delta V_1 = V_{lc}\sqrt{1 + R_1^2 - 2R_1\cos\theta_1}$$
$$\Delta V_2 = V_{lc}\sqrt{R_2^2 + R_1^2R_3^2 - 2R_2^2R_3\cos\theta_2}$$
$$\Delta V_t = \Delta V_1 + \Delta V_2$$
$$\theta_t = \theta_1 + \theta_2$$
$$R_1 = \sqrt{2\frac{r_f}{r_i + r_f}} \qquad R_2 = \sqrt{2\frac{r_i}{r_f}} \qquad R_3 = \sqrt{2\frac{r_i}{r_i + r_f}}$$
$$V_{lc} = \text{local circular velocity} = \sqrt{\frac{\mu}{ri}}$$

θ_1 = plane change associated with first impulse

θ_2 = plane change associated with second impulse

θ_t = total plane change angle between initial and final orbit

The necessary condition for optimization is:

$$\frac{\partial \Delta V_t}{\partial \theta_1} = \frac{R_1\sin\theta_1}{\sqrt{1 + R_1^2 - 2R_1\cos\theta_1}} + \frac{R_2^2R_3\sin\theta_2}{\sqrt{R_2^2 + R_1^2R_3^2 - 2R_2^2R_3\cos\theta_2}} = 0$$

Nomenclature:

ΔV_1	increment of velocity at peri–apse impulse.
ΔV_2	increment of velocity at apo–apse impulse.
$\Delta V_T = \Delta V_1 + \Delta V_2$	
μ	constant of gravitation.
a_1	semi–major axis of initial orbit.
a_2	semi–major axis of final orbit.
a_T	semi–major axis of transfer orbit.
e_1	eccentricity of initial orbit.
e_2	eccentricity of final orbit.
e_T	eccentricity of transfer orbit.
r_1	initial radius (classical Hohmann).
r_2	final radius (classical Hohmann).
α_1	plane change at peri–apse.
α_2	plane change at apo–apse.
$\Theta = \alpha_1 + \alpha_2$	total plane change.