

## Elastic Distortional Buckling of Thin-Walled Bars of Closed Quadratic Cross-Section

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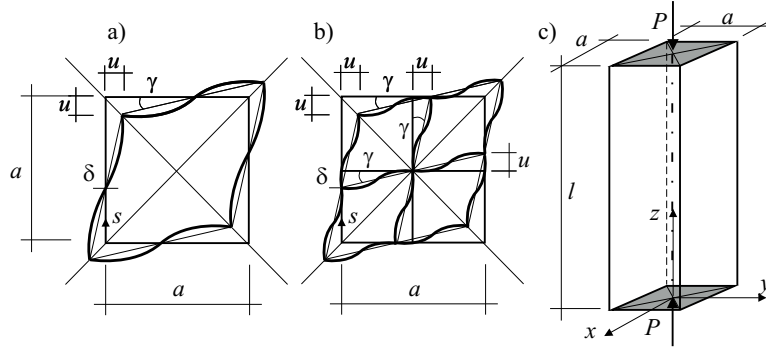
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In this study a thin-walled bar with closed quadratic cross-section is considered. The elastic stability of axially compressed bar related to the cross-section distortion is investigated. The governing differential equation is derived with aid of the principle of stationary total potential energy. The critical load for the simply supported bar is found in analytical form and it is compared with the FEM solution. Sufficient accuracy of the results is worth of noticing.

*Keywords:* Thin-walled bar, elastic stability, energetic approach, FEM.

### 1. Introduction

Both in the classical Euler theory and the theory of restrained torsion of thin-walled bars it is assumed that the bar cross-section is non-deformable. In the context of engineering experience, this implies the need to use diaphragms in small intervals. If there are no diaphragms or the distance between them is large one should take into consideration the cross-section deformation of the bar. Up to date the local stability of walls in the frame of plate buckling analysis and the global stability of the bar has been well developed [1, 2, 4, 3, 5]. However, there are hardly a few papers



**Figure 1** Schematic diagram of the bar and expected deformation of cross-section a) without and b) with internal walls

dealing with the stability of bars including the deformability of the bar cross-section. Particularly noteworthy is the article written by A. Chudzikiewicz [6], in which the possibility of stability loss due to the cross-section deformation is investigated. The main topic of this paper is a detailed stability analysis of a bar with closed deformable quadratic cross-section with or without internal walls and equal the wall thickness (Fig. 1a, b). In this case flexural buckling, torsional buckling and distortional buckling due to the cross-section deformation are independent of each other therefore in this paper only elastic distortional buckling analysis is considered. The one dimensional model of the bar is taken into account and the governing differential equation is derived using the stationary energy theorem. The most important in engineering practice is to describe the geometrical properties of the bar cross-section in which the critical distortional buckling load is less than the flexural or torsional one.

## 2. Elastic energy of distortional buckling

Let us consider an axially compressed bar stiffened by two diaphragms at both ends and with freedom of warping (Fig. 1c). We apply for solving our problem a fundamental concept of minimum total potential energy II. The total potential energy II, is the sum of the elastic strain energy  $V$ , stored in the deformed body and the potential energy  $U$ , of the applied loads.

The potential energy of cross-sectional distortions  $V_p$  can be described as:

$$V_p = \frac{1}{2} 4 \int_0^l \int_0^a \frac{M^2}{EJ} ds dz = \frac{2}{EJ} \int_0^l \int_0^a M^2 ds dz = \frac{24EJ}{a} \int_0^l \gamma^2 dz = \frac{1}{2} K_\gamma^2 dz \quad (1)$$

where bending moments of one wall  $M$  and the factor  $K_\gamma$  Eq. (1) are defined as:

$$M = 2 \frac{6EJ}{a^2} u = 6EJ \frac{\gamma}{a} = 6E \frac{1}{12} \frac{\delta^3}{a} \gamma = \frac{E\gamma\delta^3}{6a} \quad (2)$$

$$K_\gamma \stackrel{def}{=} \frac{48EJ}{a} = \frac{4E\delta^3}{a}$$

The elastic bending energy  $V_g$  of the bar in longitudinal direction is:

$$V_g = \frac{1}{2} 4 \int_0^l \frac{M_z^2}{EJ_g} dz = \frac{1}{2} EJ_g a^2 \int_0^l \gamma''^2 dz = \frac{1}{2} K_g \int_0^l \gamma''^2 dz \quad (3)$$

where the bending moment  $M_z$  and the factor  $K_g$  Eq. (3) are defined as:

$$M_z = -EJ_g u'' = -EJ_g \frac{a}{2} \gamma'' \quad (4)$$

$$K_g \stackrel{def}{=} EJ_g a^2 = \frac{Ea^5 \delta}{12}$$

In addition, the potential energy of torsion of the cross-section walls  $V_s$  is taken into account:

$$V_s = \frac{1}{2} GJ_d 4 \int_0^l \gamma'^2 dz = \frac{1}{2} K_s \int_0^l \gamma'^2 dz \quad (5)$$

where  $GJ_d$  is the free torsional rigidity of walls. The factor  $K_s$  Eq. (5) is defined as:

$$K_s \stackrel{def}{=} 4GJ_d = \frac{4}{3} G\delta^3 a \quad (6)$$

Moreover, the potential energy  $U_p^I$  of the compressive loads  $P$  due to all cross-section walls bending:

$$U_p^I = -\frac{1}{2} 4 \frac{P}{4} \int_0^l u'^2 dz = -\frac{1}{2} P \left(\frac{a}{2}\right)^2 \int_0^l \gamma'^2 dz = -\frac{1}{8} Pa^2 \int_0^l \gamma'^2 dz \quad (7)$$

and walls torsion  $U_p^{II}$  is taken into account:

$$U_p^{II} = -\frac{1}{2} 4 \frac{P}{4} r_0^2 \int_0^l \gamma'^2 dz = -\frac{1}{2} Pr_0^2 \int_0^l \gamma'^2 dz \quad (8)$$

where  $r_0^2$  is a square of polar radius of gyration, which is given by:

$$r_0^2 = \frac{J_0}{A} = \frac{\frac{1}{2}(a^3 \delta + \delta^3 a)}{a\delta} = \frac{1}{12}(a^2 + \delta^2)$$

in which  $a$  it is a height of cross-section and  $\delta$  is a wall thickness.

The total potential energy  $\Pi$  can be written as a sum of previously defined parts of energy:

$$\Pi = V_p + V_g + V_s + U_p^I + U_p^{II} \quad (9)$$

After substitution of the individual components Eqs (1), (3), (5), (7), (8) we get:

$$\Pi = \frac{1}{2} K_g \int_0^l \gamma''^2 dz + \frac{1}{2} \left( K_s - \frac{1}{4} Pa^2 - Pr_0^2 \right) \int_0^l \gamma'^2 dz + \frac{1}{2} K_g \int_0^l \gamma''^2 dz \quad (10)$$

The necessary condition of the stationary total potential energy can be written as:

$$\Pi_{min} \rightarrow \delta \Pi = 0 \quad (11)$$

**Table 1** Values of the coefficient  $\chi$  in case of bars without internal walls (for  $n = 1$  and  $\nu = 0.33$ )

		$l/a$					
		4	8	10	12	16	20
$a/\delta$	20	<b>0.50</b>	2.30	5.04	10.02	30.75	74.42
	50	0.39	<b>0.68</b>	1.12	1.92	5.24	12.23
	100	0.38	0.45	<b>0.56</b>	<b>0.76</b>	1.59	3.34
	150	0.37	0.40	0.45	<b>0.54</b>	<b>0.91</b>	1.69
	200	0.37	0.39	0.42	0.47	<b>0.68</b>	1.11

**Table 2** Values of the coefficient  $\chi$  in case of bars with internal walls (for  $n = 1$  and  $\nu = 0.33$ )

		$l/a$					
		4	8	10	12	16	20
$a/\delta$	20	1.36	20.37	49.47	102.34	322.70	787.08
	50	0.27	<b>3.32</b>	7.99	16.47	51.83	126.33
	100	0.11	<b>0.87</b>	2.04	4.16	13.00	31.64
	150	0.08	0.42	<b>0.94</b>	1.88	5.81	14.09
	200	0.07	0.26	<b>0.55</b>	<b>1.08</b>	3.29	7.95

$$K_g \gamma^{IV} + \left\{ \frac{Pa^2}{12} \left[ 4 + \left( \frac{\delta}{a} \right)^2 \right] - K_s \right\} \gamma'' + K_\gamma \gamma = 0 \quad (12)$$

The final form of the governing differential equation (12) is:

$$\gamma^{IV} + 2\alpha \gamma'' + \beta^2 \gamma = 0 \quad (13)$$

where

$$2\alpha = \frac{1}{K_g} \left\{ \frac{Pa^2}{12} \left[ 4 + \left( \frac{\delta}{a} \right)^2 \right] - K_s \right\} = \frac{P(4a^2 + \delta^2) - 16a\delta^3 G}{a^5 \delta E}$$

and

$$\beta^2 = \frac{K_\gamma}{K_g} = \frac{4E\delta^3}{a} \frac{12}{Ea^5\delta} = 48 \frac{\delta^2}{a^6}$$

The general solution of the equation (13) is:

$$\gamma(z) = C_1 \cosh(t_1 z) + C_2 \cosh(t_1 z) + C_3 \sinh(t_2 z) + C_4 \sinh(t_2 z) \quad (14)$$

where  $t_1$  and  $t_2$  are:

$$t_1 = \sqrt{-\alpha - \sqrt{\alpha^2 - \beta^2}}, \quad t_2 = \sqrt{-\alpha + \sqrt{\alpha^2 - \beta^2}}$$

**Table 3** The critical load [MN] (for  $n = 1$ ,  $E = 70$  GPa,  $\nu = 0.33$ ,  $\delta = 0.01$  m,  $a = 0.4$  m)

	Without internal walls	FEM	Eq. (16)	Eq. (21)
$l = 3$ m	$\chi = 0.75$	23.092	24.510	23.466
<b>Difference</b>			<b>5.8 %</b>	<b>1.6 %</b>
$l = 2.3$ m	$\chi = 0.5$	25.125	28.190	26.132
<b>Difference</b>			<b>10.9 %</b>	<b>4 %</b>

**Table 4** The critical load [MN] (for  $n = 1$ ,  $E = 70$  GPa,  $\nu = 0.33$ ,  $\delta = 0.01$  m,  $a = 0.4$  m)

With internal walls		FEM	Eq. (17)
$l = 2$ m	$\chi = 0.85$	73.642	70.549
<b>Difference</b>			<b>4.2 %</b>
$l = 1.73$ m	$\chi = 0.5$	72.718	56.231
<b>Difference</b>			<b>22.7 %</b>

### 3. Buckling of simply supported bar

Let us consider a simply supported bar with or without internal walls, compressed by axial loads  $P$  (Fig. 1c). Substituting the boundary conditions of the form:

$$\begin{aligned} z = 0; \quad \gamma = 0; \quad \gamma'' = 0 \\ z = l; \quad \gamma = 0; \quad \gamma'' = 0 \end{aligned}$$

to the equation (14), solving the resulting system of equations and equating the resolving to zero, we receive:

$$2\alpha = \frac{\pi^2}{l^2} + \beta^2 \frac{l^2}{\pi^2} \quad (15)$$

From the equation (15) we obtain the formula for the distortional critical load in case of any number of deformed state  $n$ :

$$P_{kr}^n = \frac{\delta(n^4 a^6 E \pi^4 + 48 l^4 E \delta^2 + 16 a^2 G l^2 n^2 \pi^2 \delta^2)}{a l^2 n^2 \pi^2 (4a^2 + \delta^2)} \quad (16)$$

Proceeding similarly but in case of the bar with internal walls (now we have 12 walls instead 4 as in the previous case (Fig. 1)) we get the formula for critical load for any  $n$ :

$$P_{kr}^n = \frac{3 \delta(n^4 a^6 E \pi^4 + 3072 l^4 E \delta^2 + 512 a^2 G l^2 n^2 \pi^2 \delta^2)}{64 a l^2 n^2 \pi^2 (a^2 + \delta^2)} \quad (17)$$

In both cases, the critical load  $P_{kr}$  may be expressed in terms of critical load of flexural buckling  $P_E$  by using the coefficient  $\chi$ :

$$P_{kr} = \chi P_E = \chi \frac{\pi^2 E J_c}{l^2} \quad (18)$$

In the case of bars without internal walls we have:

$$\chi = \frac{6 l^2 \left( \frac{a^6 \pi^2 n^2}{l^2} + \frac{48 l^2 \delta^2}{n^2 \pi^2} + \frac{8 a^2 \delta^2}{1 + \nu} \right)}{a^2 \pi^2 (4a^2 + \delta^2)^2} \quad (19)$$

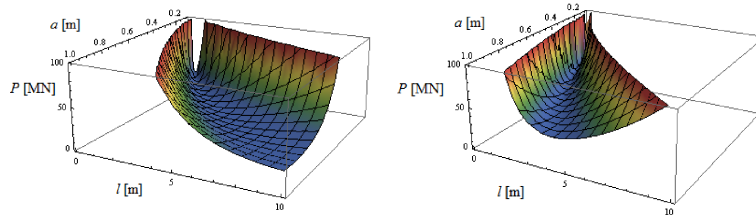
and in the case of bars with internal walls:

$$\chi = \frac{3 l^2 \left( \frac{a^6 \pi^2 n^2}{l^2} + \frac{3072 l^2 \delta^2}{n^2 \pi^2} + \frac{256 a^2 \delta^2}{1 + \nu} \right)}{16 a^2 \pi^2 (a^2 + \delta^2) (3a^2 + \delta^2)} \quad (20)$$

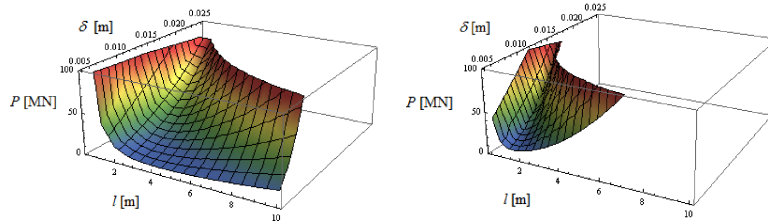
If  $\chi > 1$  the Euler critical load  $P_E$  is smaller than the load corresponding the distortional buckling. The derived formulas for critical loads are valid only if  $\chi < 1$ . Example values of the coefficient  $\chi$  obtained from equations (19) and (20) are shown in Tab. 1. and Tab. 2. These values depend on the relationships  $\frac{a}{\delta}$  and  $\frac{l}{a}$  in case of constant value of the Poisson ratio  $\nu$ . In Figs. 2, 3, 4 are shown the relation of change of values of critical force  $P$  in the dependence on dimensions  $l$ ,  $a$  and  $\delta$  in case of bars with/without internal walls.

Results of theoretical and numerical analysis were also compared with the solution using the Timoshenko energy method [1], [6] (for  $n = 1$  and  $m = 1$ ) in the case of cross-section without internal walls:

$$P_{kr}^n = \left( \frac{2a^6n^4 - \frac{(l^2m^2 + a^2n^2)^2\delta^2}{\nu^2 - 1}}{6a^6n^2} \right) \frac{\pi^2 E J_c}{l^2} \quad (21)$$



**Figure 2** The critical load  $P$  vs. height of cross-section  $a$  and bar length  $l$ , for the cross-section without and with internal walls, respectively



**Figure 3** The critical load  $P$  vs. thickness of walls  $\delta$  and length of bar  $l$ , for the cross-section without and with internal walls, respectively

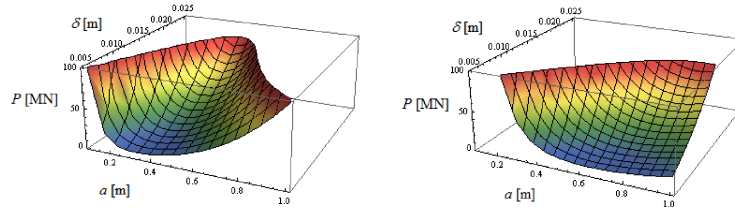
In Tab. 3. and Tab. 4. presented results of calculation of critical loads for different values of the coefficient  $\chi$ , in cases:

- proposed solutions Eqs. (16), (17),
- analytical solution by Chudzikiewicz [6] Eq. (21),
- FEM,

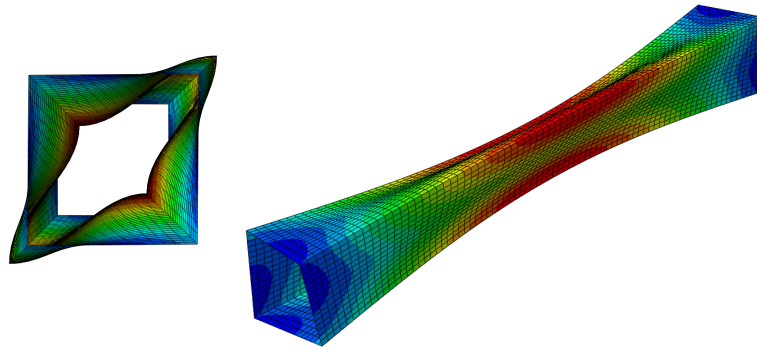
for cross-sections with/without internal walls (Fig. 1).

#### 4. Conclusions

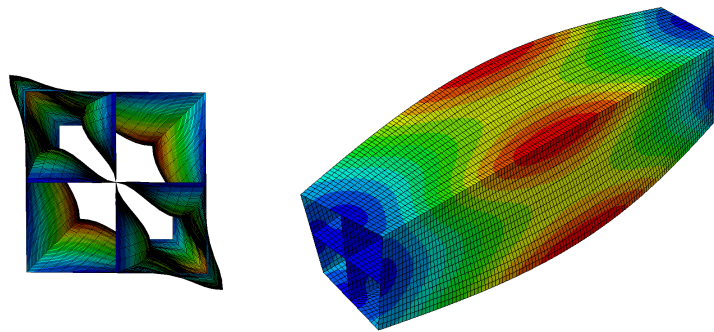
The distortional mode of elastic stability loss of the thin-walled box-type bars is discussed in the paper. We should remember that exceeding the Euler force is more



**Figure 4** The critical load  $P$  vs. thickness of walls  $\delta$  and height of cross-section  $a$ , for the cross-section without and with internal walls, respectively



**Figure 5** First mode of distortional buckling - Abaqus (Tab. 3.) -  $P_{kr} = 23.092$  MN)



**Figure 6** First mode of distortional buckling - Abaqus (Tab. 4.) -  $P_{kr} = 73.642$  MN)

significant because is generally closer to the failure load than the considered critical load (16), (17). However, in all studied cases the solution it is dominated by plate buckling.

The results summarized in Tab. 3. and Tab. 4. show that the distortional buckling in elastic range is not possible for bars made of standard metals used in construction (steel, aluminum). On the other hands, how Chudzikiewicz notices in his article [6]: for bars made of materials with low modulus of elasticity and high elastic limit the buckling under consideration may be of some importance also in practical cases. Some aluminum alloys can meet these conditions. It is worthwhile noticing that

using the derived formulas (16), (17) we can calculate the critical loads with the sufficient accuracy. In the case of bars for which the value of  $\chi$  coefficient is close to 1 the error should not exceed 5%.

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