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Optimal Shape of a Uniformly Convergent Column Subjected to a Load by the Follower Force Directed Towards the Positive Pole with Regard to the Condition of Permissible Stresses

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The results of theoretical research and numerical computations into the optimization of the shape of a cantilever column subjected to a load by the follower force directed towards the positive pole are presented in this paper. The equation of the bending line and the cross–section of the considered column were derived and an adequate boundary problem was formulated on the basis of the static criterion of stability. The additional condition of permissible stresses of an optimized column was assumed with regard to one of the three classes of the columns. The value of the critical load and optimum shape of the system, which fulfils the condition of compressive strength of the column rod for the chosen case of a load, were determined.

Keywords: Buckling, column, critical force, optimization.

1. Introduction

Theoretical research and numerical computations into the optimization of the shape of a cantilever column subjected to the specific load, formulated by L. Tomski (comp. [5]), are conducted in the paper. Taking into account past results of research into the stability of slender systems subjected to the specific load, the results of theoretical research into the optimization of the columns subjected to the follower load by a force directed towards the positive pole [2, 3, 4] and to generalised load by a force directed towards the positive pole [1, 3] were presented in works [1, 2, 3, 4]. A modified algorithm of simulated annealing was used to determine the maximum value of the critical force [2, 3]. An increase in the critical parameter of the load of 40.64% (the load by the follower force directed towards the positive pole) and of 51.74% (the generalised load by a force directed towards the positive pole) was obtained. In work [4], the equation of motion and cross–section of the column for the load by the follower force directed towards the positive pole was derived and the adequate boundary conditions applying the variational method were formulated. In considerations one of the three classes of the columns – versatile uniformly convergence column – was taken into account. Determination of the shape of the system for the chosen parameters of the head subjected to the load was possible due to solution of the boundary problem. Next, considering the obtained shape, the value of the critical load of the optimized system was determined on the basis of courses of changes in natural frequencies of the considered column in relation to external load. In publication [1], the column subjected to a generalised load by a force directed towards the positive pole and built of two segments was examined. An increase in the critical load of 10.3% in comparison to identical prismatic column was obtained at the condition of constant volume and length of the system.

2. The physical model of the column

The physical model of the column loaded by the follower force directed towards the positive pole (comp. [3, 5]) in the constructional version of loading and receiving heads built of circular elements (constant curvature) is presented in Fig. 1.

The column was loaded by the force P passing through the constant point O – the centre of loading "1" and receiving "2" heads. Pole O is placed in the distance R from the free end of the column. It was assumed that elements of receiving heads are infinitely rigid. Rod of the column "3" was rigidly mounted from one side ($x = 0$) and connected to the receiving head at the free end ($x = l$)

With regard to the value of a radius of curvature R of the loading head, exemplary denotations of the considered system are introduced:

– COi(0.1) – the optimized column with the continuously changeable bending rigidity along the system length at the parameter of loading and receiving heads R^* = 0.1.

– COr(0.4) – the optimized column with the discrete changeable bending rigidity along the system length at the parameter of loading and receiving heads $R^* = 0.4$.

– CP(0.2) – prismatic column (comparative) with the constant bending rigidity along the system length at the parameter of loading and receiving heads $R^* = 0.2$,

where:

$$R^* = \frac{R}{l} \tag{1}$$

Additionally, the following constants: total length l, volume V_{obj}, value of Young's modulus E and material density ρ of the optimized column, and corresponding comparative column were assumed. The column was described by moment of inertia of cross–section $J(x)$, cross–sectional area $A(x)$ and transversal displacement $W(x, t)$.

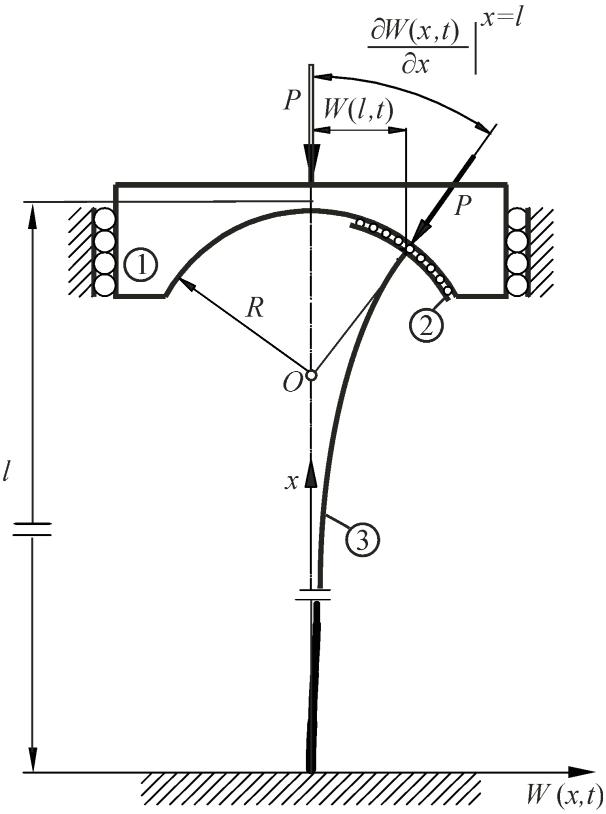

Figure 1 The physical model of the column loaded by the follower force directed towards the positive pole (comp. [2, 3, 4])

3. A rod with continuously variable bending rigidity – COi(R^*) system

The boundary problem of the column COi(R^*) was formulated and solved in work [4]. Fundamental equations for kinetic energy (2), potential energy (3) with an additional condition of a constant volume of the rod (4) and Hamilton's principle (5) are in the following form:

$$T = \frac{\rho}{2}\int_0^l A(x)\left[\dot{W}(x,t)\right]^2 dx \tag{2}$$

$$V = \frac{E}{2}\int_0^l J(x)\left[W''(x,t)\right]^2 dx - \frac{P}{2}\int_0^l \left[W'(x,t)\right]^2 dx + \frac{PR}{2}\left[W'(x,t)|_{x=l}\right]^2 \tag{3}$$

$$H = V + \lambda_1(t)\left(V_{obj} - \int_0^l A(x)\,dx\right) \tag{4}$$

$$\delta \int_{t_1}^{t_2} (T - H)\, dt = 0 \tag{5}$$

Combine expressions (2) and (4) in (5), after separation of variables of function $W(x,t)$ in relation to x and t the system of equations was received.

$$E\left[J(x)\; y''(x)\right]'' + Py''(x) - \omega^2 \rho A(x)\, y(x) = 0 \tag{6}$$

$$\rho \left[J(x)\right]^{-0.5} \omega^2 \left[y(x)\right]^2 + E\left[y''(x)\right]^2 + \lambda_3 \left[J(x)\right]^{-0.5} = 0 \tag{7}$$

where: $\lambda_i(t)$ at $i \in \mathbf{N}$ are optional constants, while $\delta(.)$ is operator of variation.

For the considered case of the load, the column was rated as divergence or divergence pseudo–flutter type of the two types of the systems (comp.[3, 5]). Therefore the value of the critical load is obtained for the condition ω =0. Distribution of moment of inertia along the column length in relation to the maximum of critical force for the assumed criterion of the constant volume of the system was received on the basis of relationships (6, 7). Adequate relationships were written in the parametric form:

$$J(\phi) = J_o \sin^4(\phi) \tag{8}$$

$$x(\phi) = l\left(B_3 + B_2\left(\phi - \frac{1}{2}\sin(2\phi)\right)\right) \tag{9}$$

where: J_o – moment of inertia in relation to the neutral bending axis at the reference point $x_{od} = x(\pi/2)$; B_2, B_3 – integration constants.

The values of constants B_2, B_3 and the range of changes in parameter φ were determined on the basis of the boundary conditions.

Exemplary shapes of the optimized models are presented in Fig. 2.

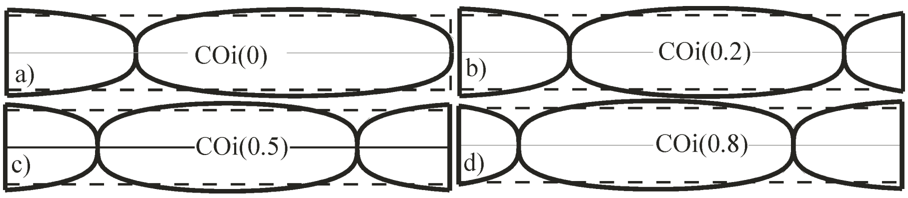

Figure 2 The shape of the optimized column COi(R^*) for the chosen values of parameter R^* of the loading head (comp. [3])

4. A rod with the stepwise variable bending rigidity – COr(R*) system

As a result of carried out theoretical research and numerical computations into the optimization of the system COi(R^*), the shapes of the column were obtained, which were characterised by a presence of the zero cross–section along the length of the column (comp. Fig. 2). To exclude this phenomenon, prof. L. Tomski proposed to introduce the additional condition (criterion) in the model of the optimization problem, i.e. resistance stresses of the optimised column σ_o in relation to the critical stress of the prismatic column (equation 10).

$$\sigma_o = \chi \, \sigma_p \tag{10}$$

$$A_{min} = \frac{(P_c)_o}{\sigma_o} \tag{11}$$

where χ – coefficient which value is dependent on the geometrical and physical parameters of the optimized column and adequate prismatic column.

The assumed condition (10) leads into determination of the value of critical load $(P_c)_o$ of the optimised column and its shape with regard to minimal cross–section of the rod A_{min} described by a relationship (11).

The results of numerical computations of the column COi(R^*) were applied to determine the parameters of the system COr(R^*) – comp. section 3. The obtained distribution of moment of inertia described by relationships (8, 9) is a base for further research into the optimization of the considered system.

4.1. *The bending line and the boundary conditions of the column COr(R^*)*

In the case of COr(R^*) system, a model of the column (Fig. 3) as a set of prismatic segments with the stepwise variable bending rigidity was assumed. Every segment with circular section was described by the moment of inertia J_i, length l_i and transversal displacement $y_i(x)$. The following constants were accepted: total length l, total volume V_{obj}, the value of Young's modulus E of individual segments of the optimised column COr(R^*) and adequate comparative column CP(R^*):

$$l_i = x_i - x_{i-1} \tag{12}$$

$$V_{obj} = \sum_{i=1}^{n} (V_{obj})_i \tag{13}$$

The potential energy of the presented model of the column is a sum of elastic strain energy of the bending of individual segments forming the system and the potential energy of external load (comp. [3]):

$$V = \sum_{i=1}^{n} \left\{ \frac{EJ_i}{2} \int_{x_{i-1}}^{x_i} \left[y_i''(x) \right]^2 dx - \frac{P}{2} \int_{x_{i-1}}^{x_i} \left[y_i'(x) \right]^2 dx \right\} + \frac{PR}{2} \left[y_n'(l) \right]^2 \tag{14}$$

Considering the determined range of changes in moment of inertia $J(x)$ of the system COi(R^*) (comp. equation (8, 9)), the moment of inertia of the i – th segment of the rod of the column COr(R^*) was defined as follows:

$$J_i = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} J(x) \, dx \tag{15}$$

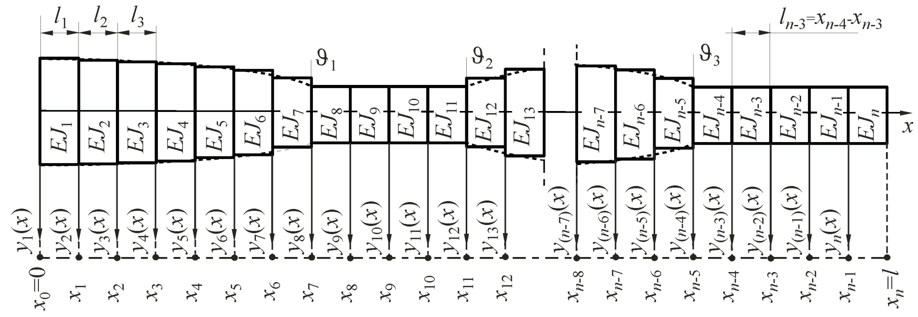

Figure 3 Division of the optimised column into prismatic segments (system COr(R^*))

Equations of the bending lines (17), the boundary conditions in relations to the mounting point (18) ($x_0 = 0$), free end of the system (19, 20) ($x_n = l$) and the continuity conditions between neighbouring segments of the column (21–24) at the points with coordinates x_j (where $j = 1, 2 \ldots\ n - -1$) were obtained on the basis of the static criterion of stability expressed by relationship:

$$\delta V = 0 \tag{16}$$

hence:

$$\bar{y}_i^{IV}(\xi) + \beta_i \bar{y}_i''(\xi) = 0 \qquad i = 1, 2 \ldots n \tag{17}$$

$$\bar{y}_1(0) = \bar{y}_1'(0) = 0 \tag{18}$$

$$\bar{y}_n(1) = R^* \bar{y}_n'(1) \tag{19}$$

$$R^* \bar{y'''}_n(1) - \bar{y''}_n(1) = 0 \tag{20}$$

$$\bar{y}_j(\xi_j) = \bar{y}_{(j+1)}(\xi_j) \tag{21}$$

$$\bar{y'}_j(\xi_j) = \bar{y'}_{(j+1)}(\xi_j) \tag{22}$$

$$EJ_j \bar{y''}_j(\xi) = EJ_{j+1} \bar{y''}_{(j+1)}(\xi_j) \tag{23}$$

$$EJ_j \bar{y'''}_j(\xi) = EJ_{j+1} \bar{y'''}_{(j+1)}(\xi_j) \tag{24}$$

Dimensionless quantities were introduced in the equations (17–24):

$$\xi = \frac{x}{l} \tag{25}$$

$$\bar{y}_i(\xi) = \frac{y_i(x)}{l} \tag{26}$$

$$\beta_i = \frac{Pl^2}{EJ_i} \tag{27}$$

4.2. *The strength condition of the optimized column*

Optimal shape of the column COi(R^*) in relation to an independent variable φ connected to the reduced parameter ξ is presented in the section 3.

The shapes of the optimized column COi(R^*) displaying the change in the moment of inertia of section $J(\varphi)$, presented in Fig. 2, do not fulfil the condition of permissible stresses in all the tested range φ:

$$\forall\left(\phi \in \langle \phi_0, \phi_1 \rangle\right) \quad \frac{(P_c)_o}{A\left(\phi\right)} \leq \sigma_o \tag{28}$$

Considering minimal value of the cross–section A_{min} (comp. equation (10)), the condition (28, 29) can be presented as:

$$\forall\left(\phi \in \langle \phi_0, \phi_1 \rangle\right) \quad A\left(\phi\right) \geq A_{min} \tag{29}$$

So, the function of moment of inertia of column section resulting from the condition (29) will be composed of:

– "old" $J(\varphi)$ when the condition (28) is fulfilled in the given range of independent variable φ,

– J_{min} – in the case when the function $A(\varphi)$ does not fulfil the condition (28) in the given range, and for the considered column – versatile uniformly convergent – one can write:

$$J_{min} = \frac{A_{min}^2}{4\pi} \tag{30}$$

where $J_{min} = J_{min}(P)$, $P = P(J(\varphi))$.

Solution to the boundary problem (equations (17–24)) is determined numerically by the iterative method with the assumed accuracy ε. The function of moment of inertia of the system section is given by a formula (31):

$$J\left(\phi\right) = \begin{cases} J_{ow}\sin^4\left(\phi\right) & if \quad \phi \in \langle \phi_0, \vartheta_1 \rangle \cup \langle \vartheta_2, \vartheta_3 \rangle \cup \langle \vartheta_4, \phi_1 \rangle \\ J_{min} & if \quad \phi \in \langle \vartheta_1, \vartheta_2 \rangle \cup \langle \vartheta_3, \vartheta_4 \rangle \end{cases} \tag{31}$$

where: J_{ow} – moment of inertia in relation to neutral axis in the reference point x_0 of the column, with regard to the condition of resistance stresses σ_o.

φ_0, φ_1 – the values of variable φ corresponding to the mounting and loading points of the column according to the relationship (9).

After obtaining the consecutive approximation of the function $J(\varphi)$ in the given iterative step, the value of the parameter of critical force $(\beta_c)_{ow}$ was determined, whereas:

$$(\beta_c)_{ow} = \frac{(P_c)_o\, l^2}{EJ_p} \tag{32}$$

J_p – moment of inertia in relation to the neutral bending axis of the prismatic column.

In the range of changes in independent variable φ (comp. equation (31)) "contact" points ϑ_k ($k \in$ N) were considered. "Contact" points (comp. Fig. 3) are the boundary places, where the the moment of inertia function of the cross–section fulfils the condition:

$$J\left(\vartheta_k\right) = J_{min}. \tag{33}$$

4.3. *The results of numerical computations*

Numerical computations were carried out to determine the value of critical load and column shape in the frame of research into the considered system COr(R^*). The value of critical parameter of load $(\beta_c)_{ow}$ (comp. equation (32)) and the range of changes in moment of inertia in relation to the neutral bending axis (function $J(\varphi)$) were determined numerically using the iterative method with regard to the volume occupied by segments with minimal cross–section A_{min} in the assumed model of column. The computations were carried out for eleven chosen parameters R^* of the loading head in the range $R^* \in \langle 0,\, 1\rangle$ and the value of resistance stresses σ_o was assumed (comp. equation (10)) for $\chi = 3$.

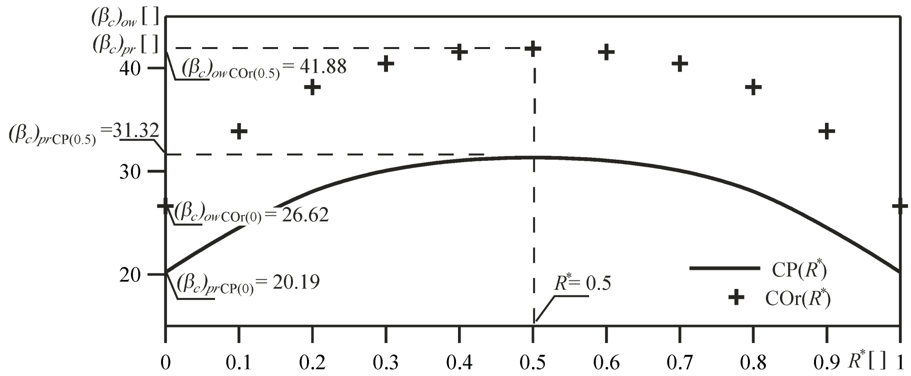

Figure 4 Critical load parameters $(\beta_c)_{ow}$ and $(\beta_c)_{pr}$ change depending on the head parameter R^* of the systems COr(R^*) for $\chi = 3$ and CP(R^*)

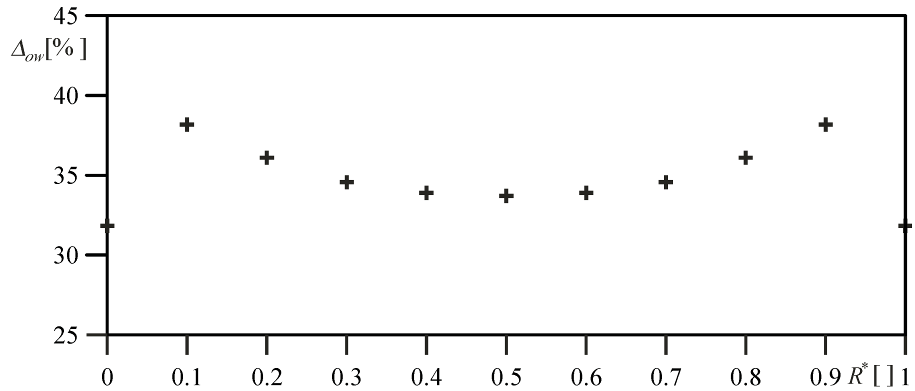

Figure 5 Percentage increase of the critical load Δ_{ow} with regard to the head parameter R^* values of the system COr(R^*) for $\chi = 3$

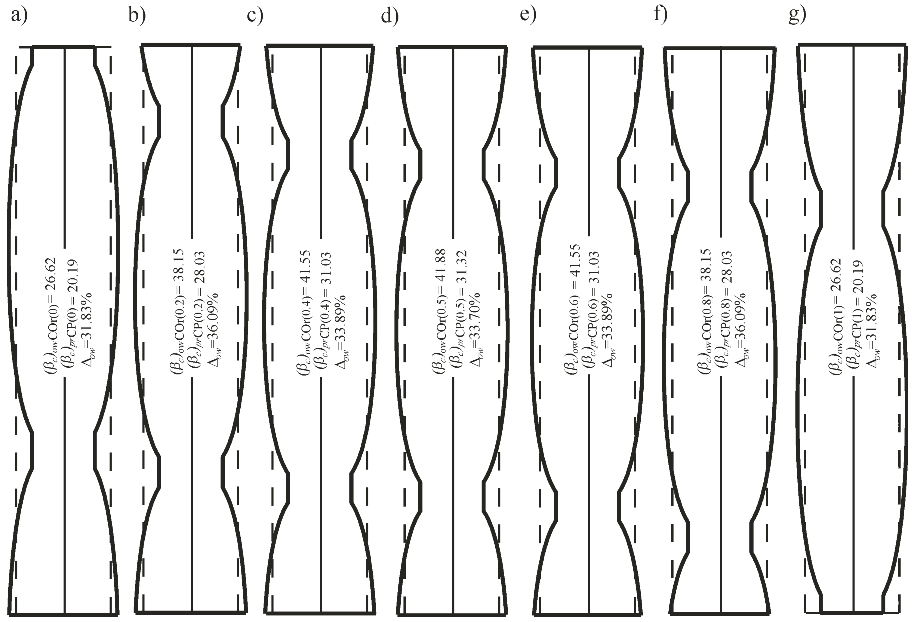

Figure 6 Optimal shapes of columns which fulfils strength condition (14) with regard to value of parameter R^* of the system COr(R^*) for $\chi = 3$

The range of changes in critical parameter of column load in relation to parameter R^* (comp. equation (1)) of loading head are presented in Fig. 4. The results of numerical computations were presented in the case of the system COr(R^*) with the optimized shape (points) and prismatic column (comparable) CP(R^*) with the constant bending rigidity along the length of the system (solid line). For the considered values of radius R of the loading head, the change in critical load was characterised by a presence of the maximal value of the critical parameter of a load: $(\beta_c)_{ow}$, $(\beta_c)_{pr}$ (comp. [2, 3]).

Percentage increase in the critical load of the column with variable crosssectional area COr(R^*) in relation to the system CP(R^*) is presented in Fig. 5, where:

$$\Delta_{ow} = \frac{(\beta_c)_{ow} - (\beta_c)_{pr}}{(\beta_c)_{pr}} 100\ \% \tag{34}$$

Shapes of the optimized columns COr(R^*) for given values of radius R of loading head were determined (Fig. 6) on the basis of the solution of the boundary problem and relationship (31) describing the range of changes in moment of inertia in relation to the neutral bending axis $J(\varphi)$. Contour of the prismatic column (comparable) CP(R^*) was marked by dashed lines considering the assumed criterion of the constant volume of the system. For the chosen values of parameter R^*, the value of critical parameter of load of the optimised $(\beta_c)_{ow}$ and prismatic $(\beta_c)_{pr}$ columns and a percentage increase in the critical load Δ_{ow} were additionally given.

5. Conclusion

The problems connected with the optimization of the shape of a column subjected to a load by the follower force directed towards the positive pole were analysed and examined in this paper. Functional of potential energy with the added side condition of the constant volume of the system was described with regard to the static criterion of stability. The system of equations for the bending line and function of the cross–section was determined and the adequate boundary conditions were obtained. The circular section of the considered column was taken into consideration (versatile uniformly convergence system). The shape of the column with a zero cross–section along the length of the column was obtained as a result of the conducted theoretical research and numerical computations. The location of the discussed points depends on the value of the geometrical parameter of the head carrying the considered case of the load. To exclude the discussed phenomenon, the introduction of the additional condition into the model of the optimization problem, i.e. resistance stresses of the optimized column, was proposed. The values of the critical load and shapes of the system with regard to the minimal cross–section of rod A_{min} were obtained considering the assumed criterion. An increase in the critical load from 31.83% to 38.17% in comparison to the critical load of a comparable column was obtained in dependence on the geometry of the loading and receiving heads. Additionally, it was stated that there is a value of the geometrical parameter of the head for which the maximal value of critical parameter of the load ($R^* = 0.5$) is received.

Acknowledgements

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