Mechanics and Mechanical Engineering
Vol. 17, No. 3 (2013) 251–259

Effect of Galactic Rotation on Radial Velocities and Proper Motion Part I

Osman M. Kamel
Astronomy and Space Science Dept.
Faculty of Sciences
Cairo University, Giza, Egypt

Adel S. Soliman
Theoretical Physics Dept.
National Research Center Dokki, Giza, Egypt
adelssoliman7@yahoo.com

Medhat K. Ammar
Mathematics Dept.
Faculty of Sciences
Helwan University, Helwan, Egypt

Received (10 March 2013)
Revised (16 May 2013)
Accepted (20 July 2013)

We expand $\Delta\rho$ the radial velocity of a group of stars moving around the center of galaxy, firstly in circular orbits. The expansion of $\Delta\rho$ is performed up to the third order of $O(r/R_0)^3$. A new result is encountered. The Oort constant is splitted into 3 parts A_1, A_2, A_3 instead of one constant A. Moreover we verify the problem when the motion of the stars is elliptic. For proper motion components, there is no split of the second Oort's constant B. In all involved expansions orders of magnitude higher than the third in ΔR or r/R_0 are neglected.

Keywords: Orbital mechanics of the stars, stellar kinematics, galactic dynamics

1. Introduction

We devote the introduction to the denotation of the assumptions concerned with Fig. 1 & 2 later, in Part II, because it is the basis of the whole of our treatment.

1. The radial velocity of the stars is freed from the effect of solar motion.
2. We assume a group of stars at position X, moving in a circular motion, firstly, with velocity V around the galactic center C, that means perpendicular to CX; C is the center of the galaxy.

3. V_0 is the mean circular velocity perpendicular to CO of a group of stars close to our Sun which belongs to our galaxy.

4. X; O situated in the galactic plane.

5. OC = R_0 ; CX = R; G = NOX = G_0 + L , where G is the galactic longitude, G_0 the longitude of C; G the galactic longitude of X in the (G, g) system.

(vi) V inclined to OX by an angle (90 – L – θ); the radial component of V is V sin(L+θ), that of V_0 is V_0 sinL; $\Delta\rho$ the radial velocity of X relative to O is given by

$$\Delta\rho = V\sin(L+\theta) - V_0 \sin L \tag{1}$$

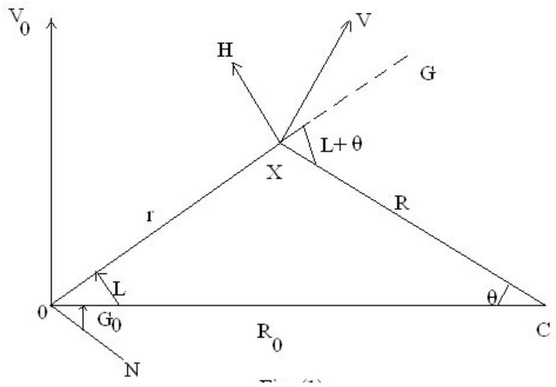

Figure 1

2. Oort's constants for radial Velocities

By Taylor's theorem for the case of one variable:

$$f\left(x+\Delta x\right) = f\left(x\right) + \Delta x\left(\frac{\partial f}{\partial x}\right) + \frac{1}{2}\left(\Delta x\right)^2\left(\frac{\partial^2 f}{\partial x^2}\right) + \frac{1}{6}\left(\Delta x\right)^3\left(\frac{\partial^3 f}{\partial x^3}\right) + ... \tag{2}$$

$\frac{d}{dx} = \frac{\partial}{\partial x}$ since we deal with one variable only.

Accordingly we may write

$$V = f\left(R\right) = f\left(R_0 + \Delta R\right) = f\left(R_0\right) + \Delta R\left(\frac{df}{dR}\right)_0 + \left(\Delta R\right)^2\left(\frac{d^2 f}{dR^2}\right) + ... \tag{3}$$

The note means that the differential coefficients are evaluated at R=R_0; z=0.

$$\begin{aligned} V = V_0 + \Delta R\left(\tfrac{dV}{dR}\right)_0 + \tfrac{1}{2}\left(\Delta R\right)^2\left(\tfrac{d^2V}{dR^2}\right) + \tfrac{1}{6}\left(\Delta R\right)^3\left(\tfrac{d^3V}{dR^3}\right)_0 + \\ + \tfrac{1}{24}\left(\Delta R\right)^4\left(\tfrac{d^4V}{dR^4}\right)_0 + ... + Remainder \end{aligned} \tag{4}$$

But we have

$$\Delta\rho = \left(V \frac{R_0}{R} - V_0 \right) \sin L \tag{5}$$

and

$$\frac{R_0}{R} = 1 - \frac{\Delta R}{R_0} \tag{6}$$

whence

$$\Delta\rho = \left[\begin{array}{l} \Delta R \left\{ \left(\frac{dV}{dR}\right)_0 - \frac{V_0}{R_0} \right\} + (\Delta R)^2 \left\{ \frac{1}{2} \left(\frac{d^2V}{dR^2}\right)_0 - \frac{1}{R_0} \left(\frac{dV}{dR}\right)_0 \right\} + \\ + (\Delta R)^3 \left\{ \frac{1}{6} \left(\frac{d^3V}{dR^3}\right)_0 - \frac{1}{2} \frac{1}{R_0} \left(\frac{d^2V}{dR^2}\right)_0 \right\} + ... \end{array} \right] \sin L \tag{7}$$

neglecting powers > 3 in ΔR.

From the following equality:

$$(R_0 + \Delta R)^2 = R_0^2 - 2rR_0 \cos L + r^2 \tag{8}$$

we may write

$$\left(\frac{\Delta R}{R_0}\right)^2 + 2\left(\frac{\Delta R}{R_0}\right) + 2\left(\frac{r}{R_0}\right)\cos L - \left(\frac{r}{R_0}\right)^2 = 0 \tag{9}$$

i.e.

$$(\Delta R)^2 + 2\Delta R R_0 + \left(2rR_0 \cos L - r^2\right) = 0 \tag{10}$$

By the solution of this second degree equation (9), in ΔR, we get

$$\Delta R = -R_0 \pm R_0 \left[1 - 2\left(\frac{r}{R_0}\right)\cos L + \left(\frac{r}{R_0}\right)^2 \right]^{1/2} \tag{11}$$

i.e.

$$R = \pm R_0 \left[1 - 2\left(\frac{r}{R_0}\right)\cos L + \left(\frac{r}{R_0}\right)^2 \right]^{1/2} \tag{12}$$

By the binomial theorem expansion, and after some calculations, and reductions, we find after the neglection of powers higher than the third in $\left(\frac{r}{R_0}\right)$,

$$\begin{aligned} \Delta R = R_0 \Bigg[& -\left(\frac{r}{R_0}\right)\cos L + \frac{1}{2}\left(\frac{r}{R_0}\right)^2 \left(1 - \cos^2 L\right) \\ & + \frac{1}{2}\left(\frac{r}{R_0}\right)^3 \left(\cos L - \cos^3 L\right) \Bigg] \end{aligned} \tag{13}$$

Evidently up to $\mathrm{O}\left(\frac{r}{R_0}\right)$

$$\Delta R = -r \cos L \tag{14}$$

Alternatively, Eq.(14) may be written as

$$\Delta R = R_0 \left[-\left(\frac{r}{R_0}\right)\cos L + \left(\frac{r}{R_0}\right)^2 \left\{\frac{1}{4}(1-\cos 2L)\right\} + \left(\frac{r}{R_0}\right)^3 \left\{\frac{1}{8}(\cos L - \cos 3L)\right\}\right] \tag{15}$$

where $L = G - G_0$, that we may write

$$\Delta\rho = A_1.\,(\Delta R) + A_2.\,(\Delta R)^2 + A_3.\,(\Delta R)^3 \tag{16}$$

Where

$$\begin{aligned} A_1 &= \left(\tfrac{dV}{dR}\right)_0 - \tfrac{V_0}{R_0} \\ A_2 &= \tfrac{1}{2}\left(\tfrac{d^2V}{dR^2}\right)_0 - \tfrac{1}{R_0}\left(\tfrac{dV}{dR}\right)_0 \\ A_3 &= \tfrac{1}{6}\left(\tfrac{d^3V}{dR^3}\right)_0 - \tfrac{1}{2}\tfrac{1}{R_0}\left(\tfrac{d^2V}{dR^2}\right)_0 \end{aligned} \tag{17}$$

A_i (i= 1, 2, 3) are the modified new Oort's constants.

ΔR is given by Equation (14), so by squaring and cubing Eq.(14), we acquire $(\Delta R)^2$, $(\Delta R)^3$ after some simple trigonometric calculations, and we can assign $\Delta\rho$. We notice that:

$$\Delta\rho = \Delta R\left[\left(\frac{dV}{dR}\right)_0 - \frac{V_0}{R_0}\right]\sin L = rA\sin 2L \tag{18}$$

where A is Oort constant, up to first order in r/R_0, we have

$$A = \frac{1}{2}\left[\frac{V_0}{R_0} - \left(\frac{dV}{dR}\right)_0\right] \tag{19}$$

We notes that Oort's constant splits into three A_1, A_2, A_3 parts if we expand up to power $O(r/R_0)^3$,hich is an important result of the analysis.

3. Case of Elliptic Orbits

We proceed now to consider X; O to be at peri – apse of ellipses with focus at C the center of the galaxy, or in other words to assume that the motion of the group of stars at X, O is elliptic around the center C. We can deduce the following formulas geometrically:

$$\frac{R_0}{R} = \frac{a_2(1-e_2)}{a_1(1-e_1)} \tag{20}$$

where a_1, a_2, e_1, e_2 are the semi – major axes and eccentricities of the two elliptic orbits. We have

$$\Delta\rho = \left(\frac{R_0}{R}V - V_0\right)\sin L \tag{21}$$

$$R = XC = a_1(1-e_1) = b_1 \tag{22}$$

$$R_0 = OC = a_2(1-e_2) = b_2 \tag{23}$$

$$V = \sqrt{\frac{\mu(1+e_1)}{a_1(1-e_1)}} \qquad V_0 = \sqrt{\frac{\mu(1+e_2)}{a_2(1-e_2)}} \tag{24}$$

whence we may write

$$\Delta\rho = \left[\frac{b_2}{b_1}\sqrt{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}} - \sqrt{\frac{\mu\,(1+e_2)}{a_2\,(1-e_2)}}\right]\sin\left(G-G_0\right) \tag{25}$$

where

$$L = G - G_0 \tag{26}$$

and

$$\Delta R = a_1\,(1-e_1) - a_2\,(1-e_2) = b_1 - b_2 \tag{27}$$

since

$$R = R_0 + \Delta R \qquad \Delta R = R - R_0 \tag{28}$$

Alternatively by simple geometry, we have

$$R^2 = R_0^2 + r^2 - 2rR_0\cos L \tag{29}$$

i.e.

$$a_1^2\,(1-e_1)^2 = a_2^2\,(1-e_2)^2 + r^2 - 2ra_2\,(1-e_2)\cos\left(G-G_0\right) \tag{30}$$

i.e.

$$r^2 - 2rb_2\cos L + \left(b_2^2 - b_1^2\right) = 0 \tag{31}$$

Solving the above second degree equation (31) in r , we get

$$\frac{r}{R_0} = \left[b_2\cos L \pm \sqrt{b_1^2 + \frac{b_2^2}{2}\left(\cos 2L - 1\right)}\right]/R_0 \tag{32}$$

But we have from above neglecting terms of order $> O\left(r/R_0\right)^3$

$$\Delta R = R_0\left[-\frac{r}{R_0}\cos L + \left(\frac{r}{R_0}\right)^2\left\{\frac{1-\cos 2L}{4}\right\} + \left(\frac{r}{R_0}\right)^3\left\{\frac{\cos L - \cos 3L}{8}\right\}\right] \tag{33}$$

By substitution for the value of r/R_0 given by Eq.(32), we obtain the expression for ΔRin terms of a_1, a_2, e_1, e_2, R_0, L.

Whence from Eq.(16), we may find, by the substitution of the value of ΔR and its second and third power, the value of $\Delta\rho$ algebraically, when we neglect powers higher than the third in r/R_0 in our expansions, and when the orbits of the two groups of stars are elliptic and not circular. The A$_i$'s ; i = 1, 2, 3, are determined from observations. As a result of our algebraic computations, we find the following set

$$\Delta R = R_0 \left[-\frac{r}{R_0}\cos L + \left(\frac{r}{R_0}\right)^2 \left\{\frac{1-\cos 2L}{4}\right\} + \left(\frac{r}{R_0}\right)^3 \left\{\frac{\cos L - \cos 3L}{8}\right\} \right] \tag{34}$$

$$(\Delta R)^2 = R_0^2 \left[\left(\frac{r}{R_0}\right)^2 \left\{\frac{1+\cos 2L}{2}\right\} - \left(\frac{r}{R_0}\right)^3 \left\{\frac{\cos L - \cos 3L}{4}\right\} \right] \tag{35}$$

$$(\Delta R)^3 = R_0^3 \left[-\left(\frac{r}{R_0}\right)^3 \left\{\frac{3\cos L + \cos 3L}{4}\right\} \right] \tag{36}$$

Also from above, we can cite

$$r = b_2 \cos L \pm \sqrt{b_1^2 + \frac{b_2^2(\cos 2L - 1)}{2}} \tag{37}$$

By substitution for the value of r we may write

$$\Delta R = R_0 \left[-\left\{\frac{b_2\cos L \pm \sqrt{b_1^2 + \frac{1}{2}b_2^2(\cos 2L - 1)}}{R_0}\right\}\cos L + \left\{\frac{b_2\cos L \pm \sqrt{b_1^2 + \frac{1}{2}b_2^2(\cos 2L - 1)}}{R_0}\right\}^2 \left\{\frac{1-\cos 2L}{4}\right\} + \left\{\frac{b_2\cos L \pm \sqrt{b_1^2} + \frac{1}{2}b_2^2(\cos 2L - 1)}{R_0}\right\}^3 \left\{\frac{\cos L - \cos 3L}{8}\right\} \right] \tag{38}$$

and similarly for $(\Delta R)^2$, $(\Delta R)^3$. In addition, we have for r, r^2

$$r^2 = b_1^2 + b_2^2\cos 2L \pm 2b_2\cos L \left\{b_1^2 + \frac{b_2^2(\cos 2L - 1)}{2}\right\}^{1/2} \tag{39}$$

$$r^3 = 3b_1^2 b_2 \cos L + b_2^3 \cos 3L \pm \left\{b_1^2 + \frac{b_2^2(\cos 2L - 1)}{2}\right\}^{1/2} \left\{b_1^2 + b_2^2 + 2b_2^2\cos 2L\right\} \tag{40}$$

4. Oort's Second constant for proper motion component

We may write

$$T = V.\,R^{-1}\left(R_0 \cos L - r\right) - V_0 \cos L$$
$$= \left[V_0 + \Delta R\left(\frac{dV}{dR}\right)_0 + \frac{1}{2}\left(\Delta R\right)^2\left(\frac{d^2V}{dR^2}\right)_0 + \frac{1}{6}\left(\Delta R\right)^3\left(\frac{d^3V}{dR^3}\right)_0\right] \tag{41}$$
$$\times\left(R_0 + \Delta R\right)^{-1}\left(R_0 \cos L - r\right) - V_0 \cos L$$

(See end of article)

But

$$\left(R_0 + \Delta R\right)^{-1} = \frac{1}{R_0} - \frac{\Delta R}{R_0^2} + \frac{\left(\Delta R\right)^2}{R_0^3} - \frac{\left(\Delta R\right)^3}{R_0^4} + ... \tag{42}$$

Whence

$$\begin{aligned} T = &\left[\frac{1}{R_0}V_0 + \frac{\Delta R}{R_0}\left(\frac{dV}{dR}\right)_0 + \frac{1}{2}\frac{\left(\Delta R\right)^2}{R_0}\left(\frac{d^2V}{dR^2}\right)_0\right.\\ &+\frac{1}{6}\frac{\left(\Delta R\right)^3}{R_0}\left(\frac{d^3V}{dR^3}\right)_0 - -\frac{\Delta R}{R_0^2}V_0 - \frac{\left(\Delta R\right)^2}{R_0^2}\left(\frac{dV}{dR}\right)_0\\ &-\frac{1}{2}\frac{\left(\Delta R\right)^3}{R_0^2}\left(\frac{d^2V}{dR^2}\right)_0 + V_0\frac{\left(\Delta R\right)^2}{R_0^3} + \frac{\left(\Delta R\right)^3}{R_0^3}\left(\frac{dV}{dR}\right)_0\\ &\left.-V_0\frac{\left(\Delta R\right)^3}{R_0^4}\right]\left(R_0 \cos L - r\right) - V_0 \cos L \end{aligned} \tag{43}$$

Performing the expansions for the expression of T up to the third power in ΔR, we find

$$\begin{aligned} T = &-\frac{V_0}{R_0}r + \Delta R\left[\left\{\left(\frac{dV}{dR}\right)_0 - \frac{V_0}{R_0}\right\}\cos L - \frac{r}{R_0}\left(\frac{dV}{dR}\right)_0 + \frac{1}{R_0}\frac{r}{R_0}V_0\right]\\ &+\left(\Delta R\right)^2\left[\left\{\frac{1}{2}\left(\frac{d^2V}{dR^2}\right)_0 - \frac{1}{R_0}\left(\frac{dV}{dR}\right)_0 + \frac{1}{R_0^2}V_0\right\}\cos L - \frac{1}{2}\frac{r}{R_0}\left(\frac{d^2V}{dR^2}\right)_0\right.\\ &\left.+\frac{r}{R_0^2}\left(\frac{dV}{dR}\right)_0 - \frac{r}{R_0^3}V_0\right] + \left(\Delta R\right)^3\left[\left\{\frac{1}{6}\left(\frac{d^3V}{dR^3}\right)_0 - \frac{1}{2}\frac{1}{R_0}\left(\frac{d^2V}{dR^2}\right)_0 + \frac{1}{R_0^2}\left(\frac{dV}{dR}\right)_0\right.\right.\\ &\left.\left.-\frac{V_0}{R_0^3}\right\}\cos L - \frac{r}{R_0}\left\{\frac{1}{6}\left(\frac{d^3V}{dR^3}\right)_0 - \frac{1}{2}\frac{1}{R_0}\left(\frac{d^2V}{dR^2}\right)_0 + \frac{1}{R_0^2}\left(\frac{dV}{dR}\right)_0 - \frac{V_0}{R_0^3}\right\}\right] \end{aligned} \tag{44}$$

i.e.

$$\begin{aligned} T = & -\frac{V_0}{R_0} r + \Delta R \left[A_1 \cos L - \frac{r}{R_0} \left\{ \left(\frac{dV}{dR} \right)_0 - \frac{V_0}{R_0} \right\} \right] \\ & + (\Delta R)^2 \left[\left(A_2 + \frac{V_0}{R_0^2} \right) \cos L - \frac{r}{R_0} \left\{ A_2 + \frac{V_0}{R_0^2} \right\} \right] \\ & + (\Delta R)^3 \left[\left\{ A_3 + \frac{1}{R_0^2} A_1 \right\} \left(\cos L - \frac{r}{R_0} \right) \right] \end{aligned} \tag{45}$$

i.e.

$$\begin{aligned} T = & -\omega_0 r + \Delta R \left[A_1 \left(\cos L - \frac{r}{R_0} \right) \right] \\ & + (\Delta R)^2 \left[\left(A_2 + \frac{\omega_0}{R_0} \right) \left(\cos L - \frac{r}{R_0} \right) \right] \\ & + (\Delta R)^3 \left[\left(A_3 + \frac{1}{R_0^2} A_1 \right) \left(\cos L - \frac{r}{R_0} \right) \right] \end{aligned} \tag{46}$$

where the angular velocity $\omega_0 = V_0/R_0$.

We neglect in our article powers higher than the third in $O(r/R_0)$.

We notice that A_1, A_2, A_3 appear in the above expression of T.

In addition, we have

$$\frac{T}{r} = 2A\cos^2 L - \frac{V_0}{R_0} = A\cos 2L + B \tag{47}$$

where

$$B = A - \frac{V_0}{R_0} = A - \omega_0 \tag{48}$$

B is Oort's second constant for proper motion.

T denotes the transverse linear velocity of X.

The circular motion at X yield V $\cos(L+\theta)$ along XH perpendicular to OX.

Similarly $V_0 \cos L$ is parallel to XH.

5. Discussion

We used Taylor's theorem and the Binomial theorem to execute the expansions involved in this Part I paper, neglecting orders higher than the third in ΔR and r/R_0. This treatment yields three constants of Oort: A_1, A_2, A_3. Through the development we should evaluate (ΔR), $(\Delta R)^2$, $(\Delta R)^3$; r, r^2, r^3. For the proper motion component we deduced the expansion for T, neglecting orders higher than $O(\Delta R)^3$ and in terms of A_i(i=1, 2, 3) represented by Eq. (46). $\omega_0 = V_0/R_0$ also appear in this equation. We preferred stress on the dynamical aspects of the problem discarding the purely astronomical ones. In part II we shall consider the influence of the third order terms arising from the perpendicular distance z above the galactic plane. This will be considered as a generalization for the case investigated in this present Part I. We assume that the two groups of stars at X; O to be situated in the galactic plane.

References

[1] **Smart, W. M.**: Stellar Kinematics, Ch. 8, *Longmans*, **1968**.

[2] **Roy, A. E.**: Orbital Motion, Fourth Edition, *IOP Publishing Ltd.*, Bristol and Philadelphia, **2005**.

[3] **Smart, W. M.**: Spherical Astronomy, *Cambridge University Press*, fifth Edition, **1965**.

[4] **Binney, J and Termaine, S.**: Galactic Dynamics, *Princeton University Press*, **1987**.

[5] **Stewart, C. A.**: Advanced Calculus, Methuen, **1951**.