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Optimization of the Control System Parameters with Use of the New Simple Method of the Largest Lyapunov Exponent Estimation

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This text covers application of Largest Lapunov Exponent (LLE) as a criterion for control performance assessment (CPA) in a simulated control system. The main task is to find a simple and effective method to search for the best configuration of a controller in a control system. In this context, CPA criterion based on calculation of LLE by means of a new method [3] is compared to classical CPA criteria used in control engineering [1].

Introduction contains references to previous publications on Lyapunov stability. Later on, description of classical criteria for CPA along with formulae is presented. Significance of LLE in control systems is explained. Moreover, new efficient formula for calculation of LLE [3] is shown. In the second part simulation of the control system used for experiment is described. The next part contains results of the simulation in which typical criteria for CPA are compared with criterion based on value of LLE. In the last part results of the experiment are summed up and conclusions are drawn.

Keywords: Largest Lapunov Exponent, LLE, control system, parameters, optimization, controller, PID, inverted pendulum, IAE, ISE, ITAE

1. Introduction

Typical criteria of control performance assessment (CPA) used in control engineering are widely known and described in many publications. In this text definitions from [1] will be used. The main scope of this article is to investigate application of LLE as CPA criterion using simple method for LLE calculation[3].

Depending on the dynamical system type and kind of the information that is useful in its investigations, there are applied different types of invariants characterizing the system dynamics. One can use for instance Kolmogorov entropy [4] or correlation dimension [5,6], to determine chaotic level or complexity of the system dynamics.[7]. But when there is a need to predict behavior of the real system with possibility of different disturbances existence, Lyapunov exponents are one of the most often applied tools. That is because these exponents determine the exponential convergence or divergence of trajectories that start close to each other. The existence of such numbers has been proved by Oseledec theorem [8],but the first numerical study of the system behavior using Lyapunov exponents has been done by Henon and Heiles [9], before the Oseledec theorem publication. The most important algorithms for calculating Lyapunov exponents for a continuous systems have been developed by Benettin et al. [10] and Shimada and Nagashima [11], later improved by Benettin et al. [12,13] and Wolf [14]. For the system with discontinuities or time delay, one possible approach is the estimation of Lyapunov exponents from the scalar time series basing on Takens procedure [15]. Numerical algorithms for such estimation have been developed by Wolf et al. [16], Sano and Sawada [17], and later improved by Eckmann et al. [18], Rosenstein et al. [19] and Parlitz [20].

The set of Lyapunov exponents contains much physical information characterizing the considered dynamical system, but calculation of the full spectrum demands much time and labor. Hence, only the largest Lyapunov exponent (LLE), which determines the predictability of the dynamical system, is frequently referred. That is because the presence of at least one positive Lyapunov exponent, by definition, is the most important evidence for chaos [21]. The algorithm for calculating the largest Lyapunov exponent was independently presented by Rosentein et al. [19] and Kantz [22] These methods make use of the statistical properties of the local divergence rates of nearby trajectories. An improved algorithm based on Rosentein and Kantz was recently presented by Kim and Choe [23]. The next method of the LLE calculation was introduced by Stefanski [24–27]. This method based on the synchronization phenomena allows the LLE estimation for both, continuous and not continuous systems, and thus can be applied for system with flow and maps dynamics representation.

Nowadays, LLE is employed in many different areas of the scientific research [28–39]. In this paper, LLE is applied to check control performance of a control system. The new method of the LLE estimation [3] is used in this paper.

2. Classical CPA criteria and CPA criterion based on LLE

The basic task for any control system is to minimise error of regulation. Error of regulation is a function of time equal to the difference between value of reference signal and output signal of the system:

$$e(t) = y_0(t) - y(t) \tag{1}$$

where:

$e(t)$ – error of regulation,

$y_0(t)$ – reference signal,

$y(t)$ – output signal.

It is expected that regulation error attains small values and tends to zero quickly.

If reference signal $y_0(t)$ or disturbance signal acting on the object of regulation are in the form of step function, error of regulation can be presented as a sum:

$$e(t) = e_u + e_p(t) \tag{2}$$

where:

$e_u = \lim_{t\to\infty} e(t)$ – fixed component of regulation error,

$e_p(t)$ – transient component of regulation error [1].

There are different criteria used for control performance assessment (CPA). Classical criteria calculated on the basis of regulation error signal include [1]:

1. Steady state error
$$e_u = \lim_{t\to\infty} e(t) \tag{3}$$
2. Maximum value of transient error
$$e_1 = \max_t |e_p(t)|\, dt \tag{4}$$
3. Overshoot
$$\chi = \left|\frac{e_2}{e_1}\right| \tag{5}$$
where:
e_1 – defined in formula (4),
e_2 – maximum value of transient error with sign opposite to e_1;
4. Regulation time with allowed value of regulation error equal Δe:
$$t_r = \max_i \{t_i\} \tag{6}$$
where: $|e_p(t_i)| = \Delta e$
5. Integral of absolute error (IAE):
$$IAE = \int_0^\infty |e_p(t)|\, dt \tag{7}$$
6. Integral of squared error (ISE):
$$ISE = \int_0^\infty e_p^2(t) dt \tag{8}$$
7. Integral of time multiplied by absolute error ($ITAE$):
$$ITAE = \int_0^\infty t\,|e_p(t)|\, dt \tag{9}$$

On the other hand, any control system can be analysed as a dynamic system (in general it can be nonlinear). Step disturbance acting on the system in time $t = 0$ can be treated as a change of initial conditions of the dynamic system.

Lapunov exponent is a measure of system's sensibility to variations of initial conditions. Lapunov exponents are generalisation of system's eigenvalues in critical point. Positive values of Lapunov exponent imply exponential divergence of phase trajectories starting in slightly different initial points. Negative Lapunov exponents imply convergence of phase trajectories.

In general, calculations of Lapunov exponents can be complicated. However, using a simple and efficient method proposed in [3], it is possible to calculate approximate value of largest Lapunov exponent (LLE) on the basis of state vector of the system easily. The following formula is used:

$$\lambda^* = \frac{\mathbf{z} \cdot \frac{d\mathbf{z}}{dt}}{|\mathbf{z}|^2} \tag{10}$$

where $\mathbf{z}$(t) – perturbation vector. In case of automatic regulation system, perturbation vector can be defined as difference between vector of desirable (reference) values of state variables and state vector of the system (the system should be observable). Average value $\hat{\lambda}^*$ of parameter λ^* is an approximate value of LLE.

Value of LLE is particularly interesting from the point of view of control performance assessment. Negative values of Lapunov exponent should imply correct control in the system, because phase trajectories starting in slightly different initial points are convergent. From the point of view of control performance, negative value of LLE should result in ability of a system to react properly on a disturbance or variation of reference value. The smaller the value of LLE, the better performance of control can be expected.

1. Simulation of regulation system

In order to test applicationof LLE as CPA index, a simulation has been written using C# programming language. Program simulates behaviour of an inverted pendulum with PID controller. Scheme of the control system is presented in Fig. 1, whereas scheme of the inverted pendulum (control object) is presented in Fig. 2.

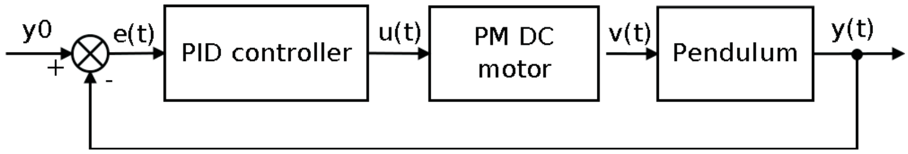

Figure 1 Scheme of the control system

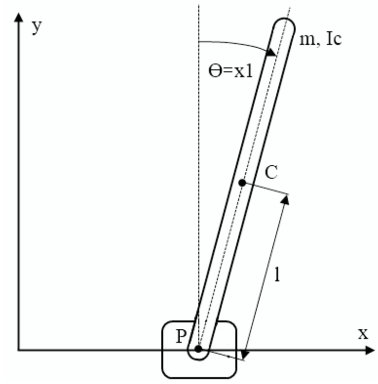

Figure 2 Scheme of the inverted pendulum – control object

The bar of mass m, reduced length and mass moment of inertia I_c (with respect to its centre of the mass C)is fixed to a bearing in point P on a mobile cart and is enabled to rotate freely around point P.

The cart can move in the direction of "x" axis. The cart is driven by a permanent magnet DC motor. It is assumed that inertia forces acting on the cart due to pendulum bar's rotation are negligible. The task of the system is to control velocity of the cart in such a way that the pendulum's bar remain in vertical position (with point C above point P) due to inertia forces.

Output signal of the control object is pendulum's bar deflection angle from its vertical position $y(t) = \theta(t)$. Reference value in this system corresponds to vertical position of the pendulum's bar (with point above point P): $y_0 = \theta_0 = 0$. Therefore regulation error, according to (1), is equal to $e(t) = -\theta(t)$. The value of regulation error is the input value of PID controller. PID controller generates input voltage signal for the motor[1], which drives the cart. Variation of cart velocity values cause inertia forces to act on the pendulum's bar.

2.1. *Dynamics of the pendulum's bar*

Equation of motion of the pendulum's bar is presented below:

$$\frac{d^2\theta}{dt^2} = \frac{mgl\sin\theta - mal\cos\theta - b\frac{d\theta}{dt}}{I_C + ml^2} \tag{11}$$

where

g – gravitational acceleration,

$a = \frac{d^2x}{dt^2}$ – acceleration of the cart,

b – damping coefficient.

[1]For compatibility of signs it has been assumed that negative voltage applied to the motor causes move of the cart in the direction of "x" axis

After transformation to state space, equation can be presented as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{mgl\sin\theta - mal\cos\theta - b\frac{d\theta}{dt}}{I_C + ml^2} \end{bmatrix} \tag{12}$$

where: $x_1 = \theta,\ x_2 = \dot{\theta}$.

2.2. *Motor equations*

The motor which drives the pendulum's cart is an astatic object. Such object can be described using following substitute model [1]:

$$G_s(s) = \frac{k_s\, e^{-sT_0}}{s} \tag{13}$$

This transfer function is a ratio:

$$G_s(s) = \frac{X(s)}{U(s)} \tag{14}$$

where $X(s)$ is Laplace transform of motor's output signal (corresponding to horizontal coordinate $x(t)$ of P point of the cart) and $U(s)$ is Laplace transform of input signal applied to the motor (voltage). Therefore, for initial conditions equal to 0 position of the cart is equal tothe convolution of inverse Laplace transform of transfer function of the motor and voltage signal applied to the motor:

$$x(t) = g_s(t)\, u(t) = \int_0^t g_s(\tau)\, u(t-\tau)\, d\tau \tag{15}$$

where: $g_s(t)$ – inverse Laplace transform of transfer function of the motor, $u(t)$ – voltage signal applied to the motor.

However, in the case of this control object, according to (12) it is not required to know exact value of $x(t)$. It is enough to find derivatives of this signal. By means of shift in time domain theorem, using Laplace transfom tables [1], function $g_s(t)$ can be found:

$$g_s(t) = L^{-1}\{G_s(s)\} = L^{-1}\left\{\frac{k_s}{s}e^{-sT_0}\right\} = k_s\, 1(t - T_0) \tag{16}$$

Cart's velocity can be calculated by differentiating formula (15). Using (16), the following result is obtained:

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\int_0^t g_s(\tau)u(t-\tau)d\tau = \frac{d}{dt}\int_0^t k_s\, 1(\tau - T_0)u(t-\tau)d\tau$$

From the definition of unit step function $1(\tau - T_0) = 0$ for $\tau < T_0$. Therefore, limits of integration can be changed:

$$v(t) = \frac{d}{dt}\int_0^t k_s\, 1(\tau - T_0)u(t-\tau)d\tau = \frac{d}{dt}\int_{T_0}^t k_s\, u(t-\tau)d\tau$$

By substituting $w = t - \tau$, $dw = -d\tau$ the following formula is obtained:

$$v(t) = \frac{d}{dt} \int_{t-T_0}^{0} k_s\, u(w)(-dw) = k_s \frac{d}{dt} \int_{0}^{t-T_0} u(w)dw$$

Finally, using the definition of derivative function (when dt tends to zero):

$$\begin{aligned} v(t) &= k_s \frac{d}{dt} \int_0^{t-T_0} u(w)dw = k_s \frac{\int_0^{t+dt-T_0} u(w)dw - \int_0^{t-T_0} u(w)dw}{dt} \\ &= k_s \frac{\int_{t-T_0}^{t+dt-T_0} u(w)dw}{dt} = k_s \frac{u(t-T_0)\, dt}{dt} = k_s\, u(t-T_0) \end{aligned} \tag{17}$$

It means that in any time $t > 0$, velocity of motor described by transfer function (13) is equal to the product of its amplification factor $k_s$and value of voltage signal on its input in time $t - T_0$.

In the implementation of the simulation it has been assumed that velocity and acceleration of the cart are limited, whereas horizontal position of the cart $x(t)$ is not limited.

2.3. *Controller*

PID controller is described by the following equation [1]:

$$u(t) = k_p \left[e(t) + \frac{1}{T_l} \int_0^t e(\tau)d\tau + T_D \frac{de(t)}{dt} \right] + u(0) \tag{18}$$

where k_p, T_l, T_D are constant coefficients. Performance of the control system depends strongly on proper adjustment of these values.

2.4. *Control system simulation program*

As it was mentioned before, whole control system is simulated by an application written in C#. Fundamental action performed by the program is numerical integration of the set of equations (12) by means of Runge–Kutta method of the fourth order (RK4). After each step of integration, on the basis of calculated values of θ angle, output signal of the controller is computed according to the formula (18). On the basis of controller output signal, using formula (17), velocity and acceleration of the cart are calculated. New acceleration of the cart is provided to the next step of procedure which integrates the set of equations (12). On the basis of the output signal of the object of regulation (pendulum's bar angle of deflection $\theta(t)$) values of IAE (7), ISE (8) and $ITAE$ (9) are calculated. In each step of integration value of parameter λ^* (10) is also computed. Its average value $\hat{\lambda}^*$ is calculated after each period of pendulum's bar swing. When $\hat{\lambda}^*$ stabilises, it is assumed to be equal to LLE.

3. Experiment

During the experiment action of the control system was tested for different coefficients of PID regulator (k_p, T_l, T_D). For each combination of PID coefficients values of IAE (7), ISE (8), $ITAE$ (9) and LLE were calculated. Later on, it was checked whether these criteria are compatible, i. e. whether for smaller values of LLE, values of IAE, ISE and $ITAE$ are also smaller. Moreover, graphs presenting $\theta(t)$ were compared for a few sets of PID coefficients in order to compare typical CPA criteria with criterion based on LLE.

3.1. *Parameters of the system, parameters of integration procedure, initial conditions*

Parameters of the system are chosen on the basis of an existing inverted pendulum, which is currently being prepared.

For calculations of dynamics of the pendulum's bar (12), following values are assumed: $m = 0,1$ kg; $g = 9,81$ m/s; $l = 0.115$ m; $b = 0,0001$ Nms; $I_C = \frac{1}{12}(0,1)(0,25)^2$ kgm$^2 \approx 5,2\,10^{-4}$ kgm^2

For simulation of the motor (17) values are chosen as follows:

$$k_s = 0,3\,\frac{\text{m/s}}{\text{V}} \qquad T_0 = 0,04\ \text{s}$$

It is assumed that motor voltage (equal to output voltage of the controller), velocity and acceleration of the cart are limited:

$$|u(t)| \le 12\,\text{V} \qquad \left|\frac{dx}{dt}\right| \le 4\,\frac{\text{m}}{\text{s}} \qquad \left|\frac{d^2x}{dt^2}\right| \le 20\,\frac{\text{m}}{\text{s}^2}$$

Following initial conditions were chosen:

$$x_1(0) = \theta(0) = 1^o = \frac{\pi}{180}\,\text{rad} \qquad x_2(0) = \dot{\theta}(0) = \frac{1^o}{s} = \frac{\pi}{180}\,\frac{\text{rad}}{\text{s}}$$

Integration step of RK4 procedure is equal to 10^{-5} s. Moreover, it is assumed that value of $\hat{\lambda}^*$ parameter is stable (and is approximately equal to LLE) if during 10 subsequent periods of pendulum bar swing absolute value of difference of $\hat{\lambda}^*$ does not exceed 0,0001.

Parameters of the controller were chosen so as to observe mainly behaviour of stable system or system on the border of stability. Simulation was run for all possible combinations of following k_p, T_l, T_D values:

$$k_p \in \{36;\ 37;\ 38\}$$

$$T_l \in \{0,05;\ 0,06;\ 0,07; ...;\ 0,50\}$$

$$T_D\{0;\ 0,01;\ 0,02; ...;\ 0,20\}$$

3.2. *Results – comparison between typical,integral CPA criteria and criterion based on LLE*

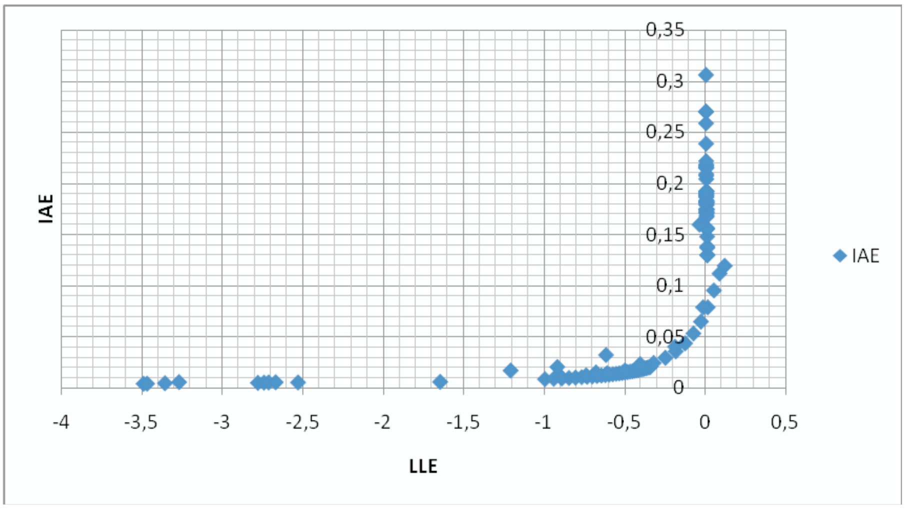

Figure 3 Dependence between IAE and LLE for all combinations of PID coefficients k_p, T_l, T_D

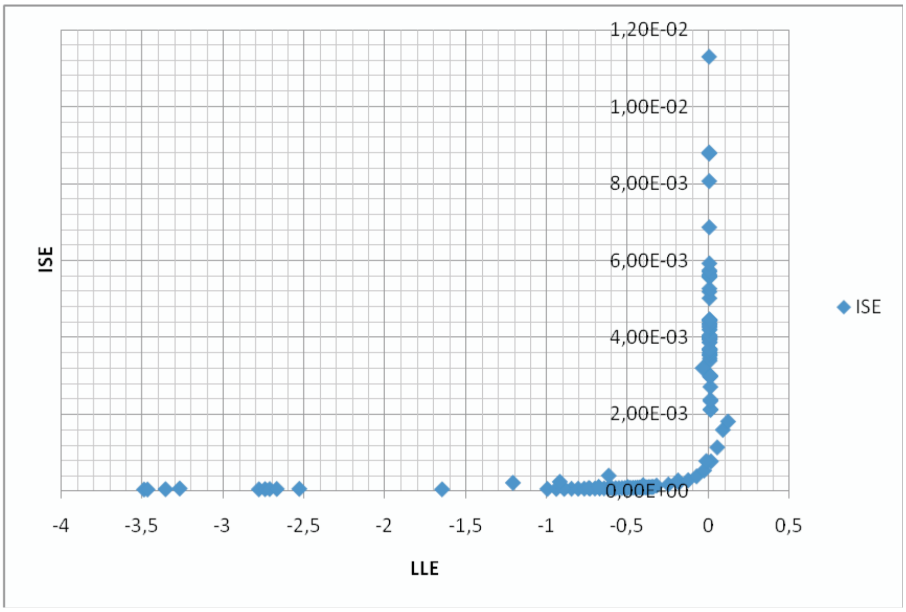

Figure 4 Dependence between ISE and LLE for all combinations of PID coefficients k_p, T_l, T_D

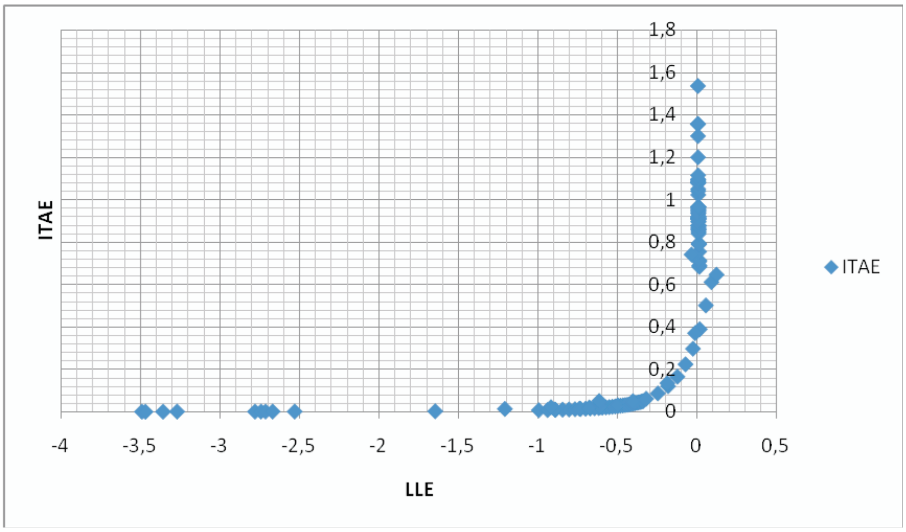

Figure 5 Dependence between ITAE and LLE for all combinations of PID coefficients k_p, T_l, T_D

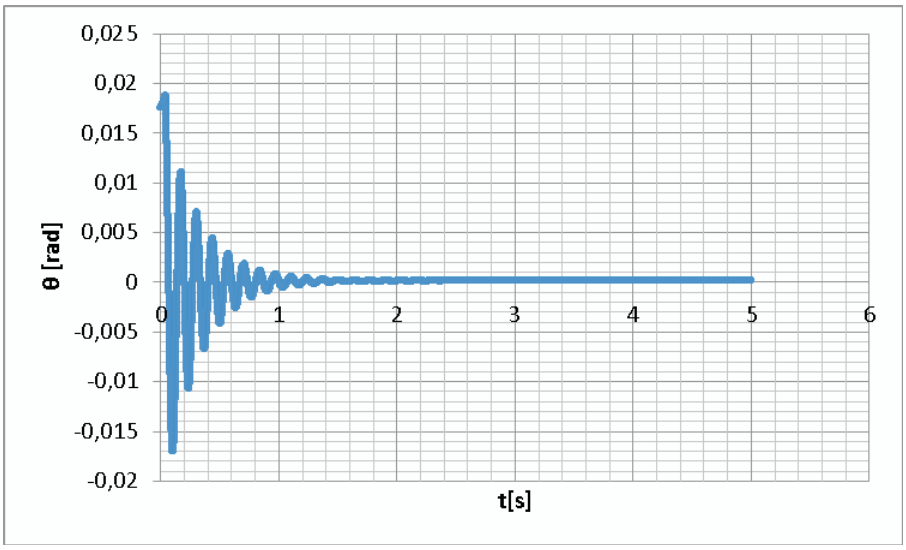

Figure 6 Graph of output signal of regulation object for PID coefficients for which LLE value is smallest (-3,491) and integral CPA criteria values are also smallest ($IAE = 0,004$; $ISE = 4,13E-05$; $ITAE = 0,001$)

Graphs visible in Fig. 3, 4 and 5 show that CPA criteria based on integration (IAE, ISE, $ITAE$) are generally compatible with criterion based on value of LLE (for higher values of LLE, values of IAE, ISE and ITAE are also higher for most of PID coefficients combinations). What is more, optimal values of PID coefficients chosen on the basis of IAE, ISE and $ITAE$ are the same as PID coefficients chosen on the basis of LLE (Fig. 6).

However, graphs on Figs 3, 4 and 5 exhibit a characteristic common feature: for wide range of LLE (approximately from -3,5 to -0,5) values of integral CPA criteria change very slightly, whereas for values of LLE close to 0 many different values of integral CPA criteria are observed. In order to explain the reason for this phenomenon, a few graphs should be analysed.

Fig. 7 and 8 depict graphs of output signal when the control system is close to the border of stability. For graph in Fig. 7 values of integral CPA criteria are few times higher than for graph in Fig. 8 just due to the fact, that amplitude of vibrations in Fig. 7 is higher than in Fig.8. However, value of LLE for both Fig. 7 and 8 is close to zero.

Comparing Fig. 6 and 9 it can be observed that for two signals whose regulation performance differ a lot (in the first case time of regulation is smaller than 2,5 s, in the second case it is more than 5 s) differences of integral CPA criteria are small. On the other hand, difference in LLE is significant (-3,491 for Fig. 6 and -0,516 for Fig. 9).

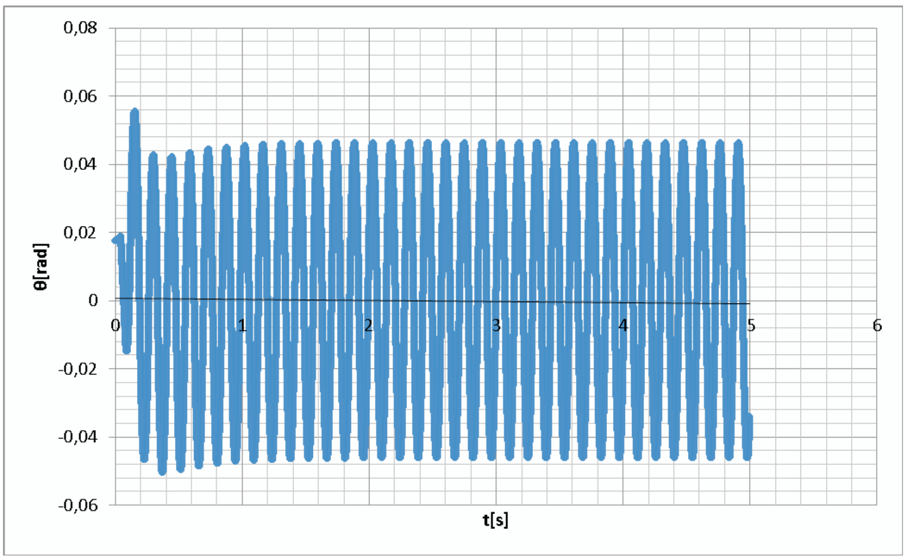

Figure 7 Graph of output signal of regulation object for PID coefficients for whichintegral CPA criteria values are largest ($IAE = 0,306$; $ISE = 0,011$; $ITAE = 1,537$); LLE=0,003

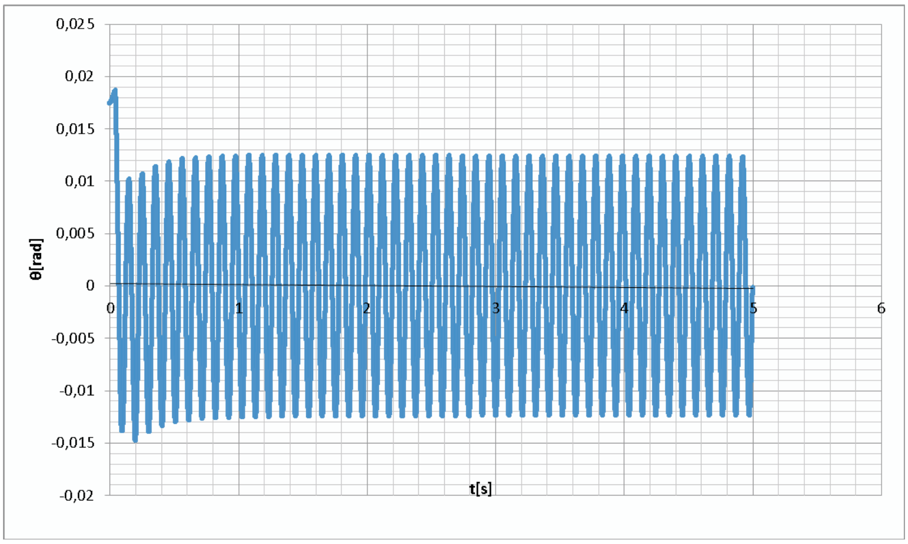

Figure 8 Graph of output signal of regulation object for PID coefficients , for which integral CPA criteria values are ($AE = 0,079$, $ISE = 0,001$, $ITAE = 0,390$, whereas LLE = 0,015

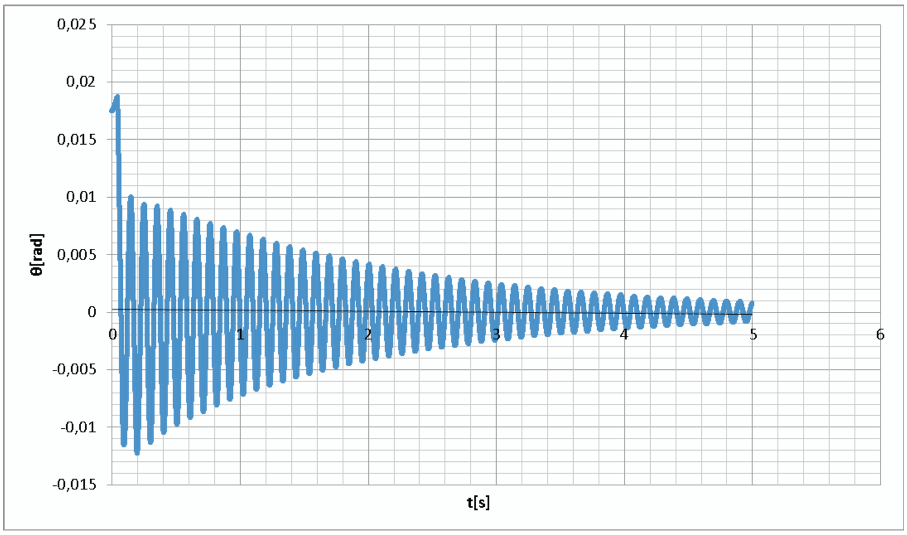

Figure 9 Graph of output signal of regulation object for PID coefficients , for which integral CPA criteria values are $IAE = 0,014$; $ISE = 8,17E-05$; $ITAE = 0,026$, whereas LLE = -0,516

4. Conclusions

According to Figs 3, 4 and 5 assessment of control performance by means of LLE is in most cases compatible with integral CPA criteria. However, Figs 6, 7, 8 and 9 show that integral CPA criteria have significant disadvantages, which do not appear when LLE is used as CPA criterion. Integral CPA criteria, which measure area under graphs of regulation error, change mainly due to variations of regulation error amplitude, whereas value of LLE decreases mainly when transient error disappears faster. Due to the fact that fast reduction of regulation error is the most important feature of any control system, it can be deduced that LLE is better criterion for assessment of control performance than integral CPA criteria.

Moreover, it is important that calculation of LLE by means of method [3] does not require much more computational power than calculation of integral CPA criteria. However, it is expected that control system is observable.

Acknowledgments

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