

## Influence of Inherent Material Damping on the Dynamic Buckling of Composite Columns with Open Cross-Sections

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Received (10 March 2013)

Revised (17 April 2013)

Accepted (23 May 2013)

In this paper the analysis of the damping behaviour of thin-walled composite columns with open stiffened cross-sections subjected to in-plane pulse loading is described. The pulse loading of a rectangular shape is concerned. The discussed problem of the dynamic interactive buckling is solved by the analytical-numerical method (ANM) using the Koiter's perturbations method. A critical value of the dynamic load factors is determined according to the Budiansky-Hutchinson's criterion for different value of the viscous damping ratio. The detailed calculations confirm that small damping does not affect the dynamic buckling of the thin-walled composite columns under the impact in-plane loading.

*Keywords:* Damping, buckling, composite, thin-walled column, pulse loading, in-plane impact.

### 1. Introduction

Damping of the fiber-reinforced lamina is a very important parameter in the design of composite structures. Energy dissipation mechanisms in this structure can be divided into two classes: those associated with the material damping, and those associated with additional sources of dissipation such as friction at joints [1]. In metal structures usually dominates the latter, but with the fibre-reinforced laminate structures the situations is different. The inherent material damping contributes

significantly to the overall damping. Composites have a material damping capacity ratio 10–100 times higher than metals. But it is often too low for many applications, for example in dynamic buckling. In recent years, viscoelastic damping materials in composites have been used to increase the damping of composite structures with little reduction in stiffness and strength [2–4].

There are several mechanisms of dissipative behaviour in this type of composite material: viscoelastic behaviour of matrix and/or fibres, thermoelastic damping due to cyclic heat flow, Coulomb friction due to slip in the fibre–matrix interface, dissipation caused by damage in the composite and so on [2]. Several analytical approaches are available in the form of micromechanical [5–6], macromechanical [7] and structural models/theories as a result of investigations carried out for both static and dynamic performance of composites [8–10]. There are conducted experimental [8, 11–14] research and FEM calculations additionally [4, 15–16]. A review of the available publications on composite material damping one can find in paper [2] with regard to different aspects such as mechanisms of damping, methods of predicting the damping and damping models/theories.

## 2. Damping modeling in linear system

The mechanical structures explicitly are described by the mass (denote as  $m$ ), the stiffness (denote as  $k$ ) and the energy dissipation phenomena. Many different effects contribute to the damping simultaneously and even complex models describe only the aspects of the observed structural response. Therefore, this study follows the most common procedure for the use of a viscous damping with damping forces proportional to the velocity. In the case of the viscous single degree of freedom linear system, one gets [17]:

$$m \ddot{z} + c \dot{z} + k z = 0 \quad (1)$$

or

$$\ddot{z} + 2\omega_o h \dot{z} + \omega_o^2 z = 0 \quad (2)$$

where:  $\omega_o = \sqrt{k/m}$  is the eigenfrequency,  $c$  or  $h$  the viscous damping ratio,  $z$  the displacement and  $\ddot{z} = d^2z/dt^2$ ,  $\dot{z} = dz/dt$  denote the second- and the first-order time derivatives. Between parameters describing damping the relation is as follows:

$$2\omega_o h = c/m \quad (3)$$

Many scientists [2] apply the hysteretic damping model to the description of damping, where the dissipative force is proportional to the displacement  $z$ :

$$\ddot{z} + \omega_o^2(1 + j\eta) z = 0 \quad (4)$$

where:  $\eta$  is the hysteretic loss factor, which is defined through the complex Young's modulus  $E^* = E' + jE''$ :

$$\eta = \frac{E''}{E'} \quad (5)$$

where:  $j = \sqrt{-1}$ . The viscous damping ratio  $h$  is related to the hysteretic loss factor  $\eta$ :

$$\eta = 2h \quad (6)$$

Solving Eq. (4) can lead to many problems, while the differential equation (2) is easily to calculate. Thus, hysteretic damping should be applied only for steady state harmonic excitation. What is important, while using Eq. (2) the hysteretic damping model can be substituted by an equivalent viscous damping mechanism.

Loss factor  $\eta$  can be determined experimentally using the half-power bandwidth method [13, 17]. One has to measure frequency bandwidth, between points on the response curve, where the amplitude of response of these points is  $1/\sqrt{2}$  times the maximum amplitude. The bandwidth for small damping correspond to the frequencies:  $\omega_1 = \omega_n(1 - \eta)$  and  $\omega_2 = \omega_n(1 + \eta)$ .

Loss factor  $\eta$  of this method is defined as:

$$\eta = \frac{\omega_2 - \omega_1}{\omega_n} \quad (7)$$

The specific damping capacity  $\Psi$  (SDC) of a material is defined as the ratio:

$$\Psi = \frac{D}{U} \quad (8)$$

of the dissipated energy  $D$  per cycle of vibration and the maximum stored energy  $U$  per cycle [8–10, 17]. It is possible to derive the relation between parameters describing models of damping above:

$$\Psi = 4\pi h = 2\pi\eta \quad (9)$$

In experimental studies the concept of a logarithmic decrement is used to describe damping properties of mechanical system. There are two definition of the decrement [17]:

$$\delta = \ln \frac{a_n}{a_{n+1}} = \ln \frac{a_t}{a_{t+0.5T}} \quad \text{or} \quad \delta_T = \ln \frac{a_n}{a_{n+2}} = \ln \frac{a_t}{a_{t+T}} \quad (10)$$

where:  $a_n = a_t$  and  $a_{n+1} = a_{t+0.5T}$  are absolute values of two successive extreme deflections,  $a_{n+2} = a_{t+T}$  – deflections after the vibration period  $T$ . The relationship is as follows:

$$\delta_T = 2\delta \quad (11)$$

Finally, the relation between all parameters describing damping can be defined as:

$$\delta_T = 2\delta = 0.5\Psi = 2\pi h = \pi\eta \quad (12)$$

For the multiple degree of freedom system the Rayleigh damping model is used [4, 11, 15–16]. In this case equations of the motions have the form:

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{D}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = 0 \quad (13)$$

where:  $\mathbf{M}$ ,  $\mathbf{D}$ ,  $\mathbf{K}$  are the mass, viscous damping and stiffness matrices respectively,  $\mathbf{z}$  is the displacement vector. In order to avoid an explicit expression of the damping matrix  $\mathbf{D}$  a linear composition of  $\mathbf{M}$  and  $\mathbf{K}$  is introduced instead:

$$\mathbf{D} = \alpha \mathbf{M} + \beta \mathbf{K} \quad (14)$$

where: two constant parameters  $\alpha$  and  $\beta$  control the damping. This approach has no real physical meaning and is chosen for mathematical convenience. Despite this disadvantage, it is a frequently applied method of introducing the dissipative properties into an analysis on the structural level. For example, in ANSYS [18] its general form is

$$\mathbf{D} = \alpha \mathbf{M} + \beta \mathbf{K} + \sum_{k=1}^{n_{mat}} \beta_k \mathbf{K}_k + \beta_c \mathbf{K} + \mathbf{D}_\omega + \sum_{l=1}^{n_{ele}} \mathbf{D}_l \quad (15)$$

where:  $\beta_k$  is constant stiffness material multiplier,  $\beta_c$  – variable stiffness matrix multiplier, expressed as  $\beta_c = 2h/\omega_o = \eta/\omega_o$ ,  $\mathbf{D}_\omega$  – frequency-dependent damping matrix,  $\mathbf{D}_l$  – element damping matrix.

After transformation from original coordinates of  $\mathbf{z}$  to generalized coordinates of  $\xi$  using transformation operator obtained by the eigenvectors  $\phi_i$  of the generalized eigenproblem:

$$\mathbf{K} \phi_i = \omega_i^2 \mathbf{M} \phi_i. \quad (16)$$

The coupled set of differential equations (12) yields a single equation for every degree of freedom  $i$ :

$$\ddot{\xi}_i + 2 \omega_i h_i \dot{\xi}_i + \omega_i^2 \xi_i = 0. \quad (17)$$

The damping formulation (13) leads to:

$$2 \omega_i h_i = \alpha + \beta \omega_i^2 \quad (18)$$

for the  $i$ -th eigenmode. Using this approach, damping of two eigenmodes can be specified exactly by the free parameters  $\alpha$  and  $\beta$ . Subsequently, the damping ratios  $h_i$  of all other modes are given by this relation (17).

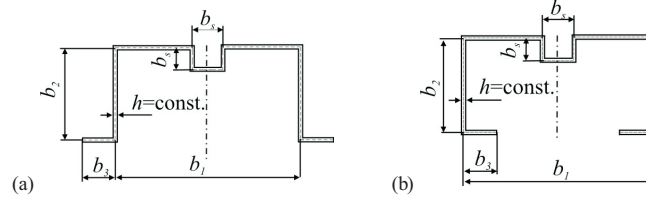
### 3. Results and discussion

The prismatic thin-walled columns with open cross-sections (Fig. 1), subjected to axial compression, are considered. The detailed analysis of the calculations was conducted for the composite columns with the following dimensions:  $b_1 = 100$  mm,  $b_2 = 5$  mm,  $b_3 = 15$  mm,  $b_s = 15$  mm,  $h = 1.5$  mm and four various lengths:  $l = 2500, 2000, 1500$  and  $1000$  mm. Each column is made of a twelve-layer composite with the symmetric ply alignment  $[45/-45/0_4]_S$  [19–21]. Each layer of the thickness  $h_{lay}=0.125$  mm is characterized by the following mechanical properties:  $E_1 = 140$  GPa,  $E_2 = 10.3$  GPa,  $G_{12} = 5.15$  GPa,  $\nu_{12} = 0.29$ ,  $\rho = 1600$  kg/m<sup>3</sup> [22].

For the thin-walled structures with initial deflections and the viscous damping, the non-linear Lagrange's equations of motion for this case of an interaction of  $N$  eigenmodes can be defined as [19–21, 23–34]:

$$\ddot{\xi}_r + 2\omega_r h_r \dot{\xi}_r + \omega_r^2 \left[ \left( 1 - \frac{\sigma}{\sigma_r} \right) \xi_r + a_{rpq} \xi_p \xi_q - \frac{\sigma}{\sigma_r} \xi_r^* \right] = 0 \quad \text{for } r = 1, \dots, N \quad (19)$$

where:  $\xi_r$  – dimensionless amplitude of the  $r$ -th buckling mode (maximum deflection referred to the thickness of the first plate),  $\sigma_r$ ,  $\omega_r$ ,  $\xi_r^*$  – critical stress, circular frequency of free vibrations and dimensionless amplitude of the initial deflection corresponding to the  $r$ -th buckling mode and  $h_r$  – the damping ratios corresponding to the  $r$ -th frequency.



**Figure 1** Open cross-sections of columns with a central intermediate stiffener [19-21], a) C-shape with outside stiffeners, b) C-shape with inside stiffeners

The last part in the equation (19) results from including the non-linear model of the thin-walled structures and the amplitude of imperfections. In the case of one mode buckling, the linear equation of motion (19) has the form:

$$\ddot{\xi}_r + 2\omega_r h_r \dot{\xi}_r + \omega_r^2 \left(1 - \frac{\sigma}{\sigma_r}\right) \xi_r = 0 \quad (20)$$

The third part of the equation (20) compared with Eq. (2) was modified by taking into account the influence of the compressing load on the frequency of free vibration of the real construction.

The expressions for the postbuckling coefficients  $a_{rpq}$  are to be found in papers [27-31]. In the equations of motion (19), inertia forces of the pre-buckling state and second order approximation have been neglected [23]. The initial conditions have been assumed in the following form [23-34]:

$$\xi_r(t=0) = 0 \quad \dot{\xi}_r(t=0) = 0 \quad (21)$$

The static problem of interactive buckling of thin-walled multilayer columns (i.e., for  $\dot{\xi}_r = 0$  in Eqs. (19)) has been solved with the method presented in paper [28-30].

The frequencies of free vibrations have been determined analogously as in paper [35]. The problem of interactive dynamic buckling (Eqs. (19)) has been solved by means of the Runge-Kutta numerical method modified by Hairer and Wanner [36].

Values of the critical stresses and the circular frequencies of free vibrations corresponding to the buckling modes under analysis for different column lengths  $l$  are presented in Tab. 1 and Tab. 2, respectively.

In these tables, the following index symbols were introduced: 1 - flexural-distortional mode for  $m = 1$ ; 2 - flexural-torsional-distortional mode for  $m = 1$ ; 3 - flexural-distortional mode for  $m = 3$ ; 4 - flexural-torsional-distortional mode for  $m = 3$ , where  $m$  is the  $m$ -th harmonic mode.

Further on, the analysis of dynamic interactive buckling of the columns under consideration was conducted. Analysis was limited to the interaction of four buckling modes (i.e.  $N = 4$ ). Solving the set of equations (19) one receives the amplitudes of the buckling mode (i.e.  $\xi_1, \xi_2, \xi_3, \xi_4$ ) as a function of time and amplitude of the load pulse.

**Table 1** Critical stresses for the column shown in Fig. 1 [19-21]

$l$	<b>Example 1 (Fig. 1a)</b>				<b>Example 2 (Fig. 1b)</b>			
	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$
[mm]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]
2500	52.25	29.36	131.97	119.04	50.93	42.24	106.37	189.15
2000	79.54	42.11	127.08	112.83	74.58	62.50	112.25	175.77
1500	129.08	66.90	138.74	136.74	104.99	103.51	135.71	174.22
1000	143.95	109.24	208.16	210.21	107.73	180.40	219.32	242.19

**Table 2** Circular frequency of free vibration of the columns shown in Fig. 1 [19-21]

$l$	<b>Example 1 (Fig. 1a)</b>				<b>Example 2 (Fig. 1b)</b>			
	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$
[mm]	[rad/s]	[rad/s]	[rad/s]	[rad/s]	[rad/s]	[rad/s]	[rad/s]	[rad/s]
2500	227	170	1082	1027	224	204	971	1295
2000	350	254	1327	1305	339	310	1247	1561
1500	594	428	1849	1835	536	532	1828	2072
1000	942	820	3396	3412	814	1054	3485	3662

A detailed analysis was conducted for a rectangular pulse load:

$$\sigma = \begin{cases} \sigma_D & \text{for } 0 \leq t \leq t_0 \\ 0 & \text{for } t > t_0 \end{cases} \quad (22)$$

where:  $t_0$  is a duration of the pulse load, and  $\sigma_D$  is an amplitude of the dynamic load. The duration  $t_0$  equal to the period of fundamental flexural free vibrations  $T_1 = 2\pi/\omega_1$ . The computation time is equal to  $1.5t_0$ . The level of imperfections was assumed as:

$$\xi_1^* = \xi_2^* = |l/(3000h)| \quad \xi_3^* = \xi_4^* = |l/(6000h)| \quad (23)$$

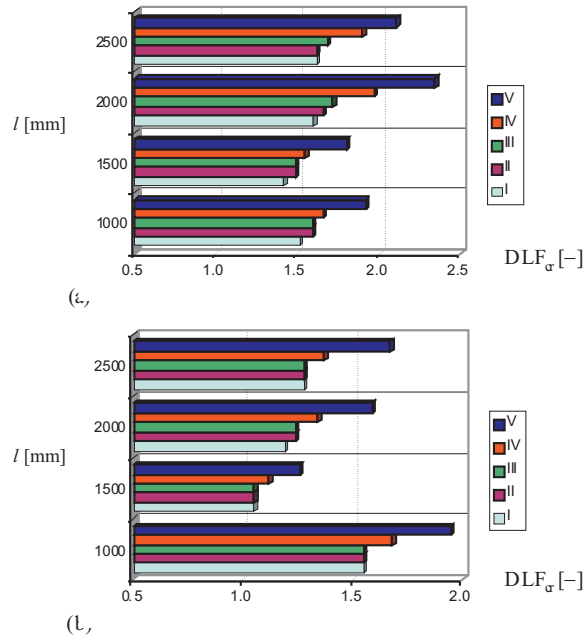
The dynamic buckling is possible only when the geometric imperfections are not equal to zero. When the displacement growth is assessed with time for different amplitude of load, buckling occurs when the dynamic load reaches a critical value associated with a maximum acceptable deformation or strain, stress, the magnitudes of which are defined arbitrarily. So it appears to be no perfect criterion as yet for dynamic buckling. Therefore, this study follows the most widely used the Budiansky–Hutchinson’s criterion [20–21, 25–26, 31–34]. In order to find a critical value of the dynamic load factors:  $DLF_{cr} = (\sigma_D/\sigma_{min})_{cr}$ , one has to find out which of the displacements’ growth is the highest for certain force amplitude:  $DLF = (\sigma_D/\sigma_{min})$  (where:  $\sigma_{min} = \min(\sigma_1; \sigma_2; \sigma_3; \sigma_4)$ ). The value of  $DLF_{cr}$  depends on the step of calculations and restrictions imposed on deflections. In order to find the critical value of dynamic load factor one should draw the graph of deflection amplitude as a function of dynamic load. Values of the critical dynamic load factors  $DLF_{cr}$  for different column lengths  $l$  are presented in Tab. 3 and Fig. 2.

**Table 3** The critical values of the dynamic load factors  $DLF_{cr}$  determined from the Budiansky–Hutchinson’s criterion for the columns shown in Fig. 1

$l$ [mm]	Example 1 (Fig. 1a)					Example 2 (Fig. 1b)				
	I	II	III	IV	V	I	II	III	IV	V
2500	1.62	1.62	1.69	1.90	2.11	1.28	1.28	1.28	1.37	1.67
2000	1.60	1.66	1.72	1.97	2.34	1.19	1.24	1.24	1.34	1.59
1500	1.42	1.49	1.49	1.55	1.80	1.05	1.05	1.05	1.12	1.26
1000	1.52	1.59	1.59	1.66	1.92	1.19	1.24	1.24	1.34	1.59

Five cases of the parameter of damping were considered (Table 3 and Fig. 2):

- I – the viscous damping ratio  $h_r$  for  $r = 1, \dots, 4$  in Eqs. (19) is equal to 0, so there are not damping;
- II – the viscous damping ratio  $h_r$  is equal to 2% for all modes;
- III – the viscous damping ratio  $h_r$  is equal to 0.5% for the global modes ( $\sigma_1$  and  $\sigma_2$ ) and 3% for the next ones ( $\sigma_3$  and  $\sigma_4$ );
- IV – the viscous damping ratio  $h_r$  is equal to 10% for the global modes ( $\sigma_1$  and  $\sigma_2$ ) and 2.5% for the next ones ( $\sigma_3$  and  $\sigma_4$ );
- V – the viscous damping ratio  $h_r$  is bigger from 20% for all modes.



**Figure 2** The critical values of the dynamic load factors  $DLF_{cr}$  determined from the Budiansky–Hutchinson’s criterion for columns shown in Fig. 1: (a) Example 1 (Fig. 1a), (b) Example 2 (Fig. 1b)

In the presented case of the material damping, it is assumed that the viscous parameter of damping  $h_r$  for the carbon–epoxy, composite plate element amounted to 0.5% for frequencies lower than 2000 Hz and 1.5% for frequencies from 2500 Hz to 4000 Hz and 3% for frequencies higher than 5000 Hz [14]. The behaviour of the viscous model and the hysteretic one is different. The hysteretic model led to a frequency independent damping. In this case, for the carbon–epoxy composite plate the loss factor  $\eta$  is less than 1% [14] so the viscous parameter of damping  $h_r$  is less than 0.5% (Eq. 6). It can be shown [37–39] that for low damping the frequency increase of the material damping is very weak. The joints damping is weak for composite structures as well. For the glass–epoxy composite beam the viscous parameter of damping  $h_r$  is less than 1% for all fibre orientation and frequency less than 1200 Hz [8, 11–12]. In the case of Kevlar fibre–epoxy composite beam the viscous parameter of damping  $h_r$  is less than 2% for all fibre orientation and frequency less than 1200 Hz [7, 9–10]. Dissertations above allow to accept two cases for further analysis: Case II where damping is frequency independent and Case III where damping is the function of frequency. If the damping is high, the relationship between the loss factor  $\eta$  and the natural frequency  $\omega_o$  is hyperbolic [40]:

$$\eta = \frac{a_o}{\omega_o^\alpha} \quad (24)$$

where:  $a_o$  and  $\alpha$  are constant. In this model, the loss factor  $\eta$  for glass–epoxy composite beam is lower than 20% ( $h = 10\%$ ) for global mode and lower than 5% ( $h = 2.5\%$ ) for the local ones [40], what was described in Case IV.

The last case (Case V) regards the composite structures with viscoelastic layer [3–4]. For the 3M ISD–112 damping material the viscous parameter of damping  $h_r$  is greater from 25% for the frequency less than 2000 Hz [4].

The small damping (Case II and Case III) doesn't affect the value of critical dynamic load factor for all columns. In this case, the value of  $DLF_{cr}$  grew less than 5%, compared to the column without damping (Case I). It is possible to reduce differences in received results by narrowing the step of calculations. So it is possible to say that one can get the same results for all cases with or without small damping.

If damping is greater (Case IV), the value of  $DLF_{cr}$  grew more than 10% but less than 25%, compared to the column without damping. The growth is so low that damping has secondary meaning for the phenomenon of the dynamic buckling. Only if damping is very strong (Case V), the value of  $DLF_{cr}$  grew more than the 25% but less than 45% and it should be taken into account in calculating the critical value of the dynamic load factors:  $DLF_{cr}$ .

#### 4. Conclusion

In this paper the analysis of the damping behaviour of thin–walled composite columns with open stiffened cross–sections subjected to in–plane pulse loading was described. The detailed calculations confirmed that small damping didn't affect the dynamic buckling of the columns. The influence on the value of the dynamic load factors was observed only in case of composite with one or more layers made of the viscoelastic damping materials.



## 5. Acknowledgements

This publication was partially supported by the project, which has been funded by National Center for Science – a decision number UMO-2011/01/B/ST8/07441. The co-author Zbigniew Kołakowski is thankful for the support.

The present paper was partially supported by Structural Funds in the Operational Programme – Innovative Economy (IE OP) financed from the European Regional Development Fund – Project ”*Modern material technologies in aerospace industry*”, Nr POIG.01.01.02–00–015/08–00. The co-author Andrzej Teter is thankful for the support.

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