Mechanics and Mechanical Engineering
Vol. 18, No. 1 (2014) 5–10

Investigation on the Film Flow of a Third Grade Fluid on an Inclined Plane Using HPM

Mohammadreza Azimi
Department of Mechanical Engineering,
Tarbiat Modares University,
Tehran, IRI

Alireza Azimi
Department of Chemical Engineering,
Collaege of Chemical Engineering,
Mahshahr Branch, Islamic Azad University,
Mahshahr, IRI
meysam.azimi@gmail.com

Received (10 March 2014)
Revised (16 October 2014)
Accepted (23 October 2014)

This work concerns the study of the thin film flow problem arising in non–Newtonian fluid mechanics using analytical approach. The governing equations are reduced to ordinary nonlinear boundary value problem by applying the transformation method. Homotopy Perturbation Method (HPM) has been applied to obtain solution of reduced nonlinear boundary value problem. The analytical solutions of the flow velocity distributions for different cases have been presented. The effect of material constant has also discussed. Finally, analytical results have been compared with numerical one obtained by forth order Runge Kutta method. High accuracy and validity are the advantages of present study.

Keywords: Homotopy Perturbation Method, thin film flow, third grade fluid, boundary value problem.

1. Introduction

The study of non–Newtonian fluids offers many interesting and exciting challenges due to their technical relevance in modeling fluids with complex rheological properties (such as polymer melts, synovial fluids, paints, etc). These fluids cannot be described by the classical Navier-Stokes theory. In general, the equations governing the flow of non–Newtonian fluids are more non–linear and are of a higher order than the Navier-Stokes equations. Therefore, one needs additional boundary conditions for the unique solution of such equations. One of the subclasses of non—

Newtonian fluids is a third grade. These different–type fluid was the main topic of many pervious researches [1-3].

Most scientific problems such as present study, are inherently of nonlinearity that do not admit analytical solution, due to the difficulties of numerical solutions [4] so these equations should be solved using special techniques. Homotopy Perturbation Method (HPM) proposed first by Ji–Huan He [5, 6], which does not require a small parameter in an equation, has a significant advantage in that provides an analytical approximate solution to a wide range of linear and nonlinear problems in applied sciences. This method is a powerful mathematical technique for solving differential and integral equations, and has already been applied to several nonlinear problems [7–11]. In this work we will investigate the thin film flow of third grade fluid problem by means of homotopy perturbation method.

2. Problem describtion

The thin film flow of a third grade fluid down an inclined plane of inclination $\alpha \neq 0$ is governed by the following nonlinear boundary value problem:

$$\frac{d^2u}{dy^2} + \frac{6(\beta_2+\beta_3)}{\mu}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} + \frac{\rho g \sin\alpha}{\mu} = 0 \tag{1}$$

With following boundary conditions:

$$u(0) = 0, \qquad \left.\frac{du}{dy}\right|_{y=\delta} = 0 \tag{2}$$

Introducing the parameters

$$\begin{aligned} y &= \delta y^* \\ u &= \frac{\delta^2 \rho g \sin\alpha}{\mu} u^* \\ \beta^* &= \frac{3\delta^2\rho^2 g^2 \sin^2\alpha}{\mu^3}(\beta_2+\beta_3) \end{aligned} \tag{3}$$

The problem, after omitting asterisks, takes the following form

$$\frac{d^2u}{dy^2} + 6\beta\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} + 1 = 0 \tag{4}$$

And boundary conditions can be obtained as follows:

$$u(0) = 0, \qquad \left.\frac{du}{dy}\right|_{y=1} = 0 \tag{5}$$

Here, is the dynamic viscosity, g is the gravity, is the fluid density and $\beta > 0$ is the material constant of a third grade fluid.

3. Application of HPM

HPM provides an analytical approximate solution for problems at hand as other explained techniques. Brief theoretical steps for the equation of following type can be given:

$$A(u) - g(r) = 0, \quad r \in \Omega \tag{6}$$

With the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \tag{7}$$

Where A, B , $g(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function, and the boundary of domain Ω. Generally speaking the operator A can be divided into a linear part L and a nonlinear part $N(u)$. Eq. 6 can therefore, be rewritten as:

$$L(u) + N(u) - g(r) = 0 \tag{8}$$

We construct a homotopy of Eq. 6 $f(r, p) : \Omega \times [0, 1] \to R$, which satisfies

$$H(f, p) = (1 - p)[L(f) - L(u_0)] + p[A(f) - g(r)] = 0, \quad p \in [0, 1], r \in \Omega \tag{9}$$

Or

$$H(f, p) = L(f) - L(u_0) + pL(u_0) + p[N(f) - g(r)] = 0 \tag{10}$$

So, we can construct a homotopy of problem as follows:

$$H(f, p) = (1 - p)(f'' + 1) + p(6\beta f'' f'^2) \tag{11}$$

Where $p \in [0, 1]$ is an embedding parameter, while f_0 is an initial approximation of Eq. 9 which satisfies the boundary condition.

We consider f as follows:

$$f = f_0 + pf_1 + p^2 f_2 + p^3 f_3 + ... \tag{12}$$

Setting $p = 1$ yields in the approximate solution of equation to:

$$f = \lim_{p \to 1} f = f_0 + f_1 + f_2 + f_3 + ... \tag{13}$$

Substituting f from Equation.13 into Equation.11 and rearranging based on powers of p–terms, we have:

$$p^0 : f_0'' + 1 = 0 \qquad f_0(0) = 0, \quad f_0'(0) = 0 \tag{14}$$

$$p^1 : f_1'' + 6\beta f_0'' f_0'^2 = 0 \qquad f_1(0) = 0, \quad f_1'(1) = 0 \tag{15}$$

$$p^2 : f_2'' + 6\beta f_1'' f_0'^2 + 12\beta f_0' f_1' f_0'' = 0 \qquad f_2(0) = 0, \quad f_2'(1) = 0 \tag{16}$$

Solving the above equations results in the following answers:

$$f_0(t) = -\frac{1}{2}y^2 + y \tag{17}$$

$$f_1(t) = \frac{1}{2}\beta(y-1)^4 - \frac{1}{2}\beta \tag{18}$$

$$f_2(t) = -2\beta^2 y^6 + 12\beta^2 y_3^5 0\beta^2 y^4 + 40\beta^2 y^3 - 30\beta^2 y^2 + 12\beta^2 y \tag{19}$$

We avoid listing the other components. However, according to equation, we can obtain f as follows:

$$f(y) = f_0(y) + f_1(y) + f_2(y) \tag{20}$$

4. Results and discussions

In this section we want to investigate analytical results in some numerical cases. Tab. 1 presents the analytical solution obtained by HPM for different values of material constant. As it can be seen in Tab. 1, the increase in material constant value causes reduction in velocity.

Table 1 Effect of material constant variation on velocity field

	$\beta = 0$	$\beta = 0.04$	$\beta = 0.08$	$\beta = 0.1$
y	$u(y)$	$u(y)$	$u(y)$	$u(y)$
0	0	0	0	0
0.1	0.095	0.089621	0.087242	0.087176
0.2	0.18	0.170553	0.165829	0.165237
0.3	0.255	0.242626	0.235898	0.234652
0.4	0.32	0.305643	0.297387	0.295547
0.5	0.375	0.3594	0.3501	0.347813
0.6	0.42	0.403699	0.393772	0.391198
0.7	0.455	0.43836	0.428115	0.42539
0.8	0.48	0.463232	0.452863	0.450079
0.9	0.495	0.478202	0.467804	0.465005
1	0.5	0.4832	0.4728	0.47

Fig. 1 presents a comparison between analytical results obtained by HPM and numerical solutions achieved from forth order Runge Kutta method when $\beta = 0.03$.

According to Fig. 1, the obtained results have good agreement with numerical solutions.

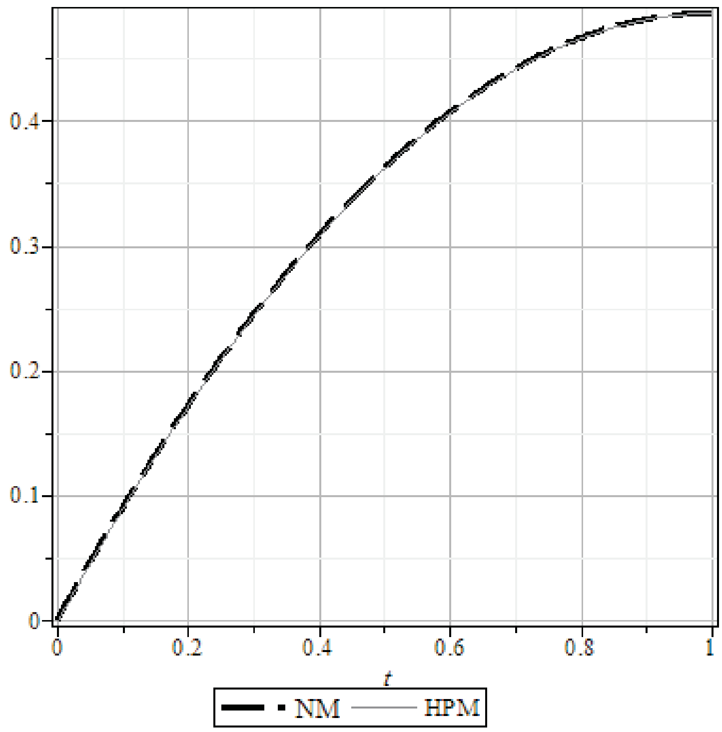

Figure 1 Comparison between HPM and Runge Kutta method when $\beta = 0.03$

5. Conclusions

In this work, we have used an approximate analytical method called Homotopy Perturbation Method (HPM) in order to investigate the thin film flow of third grade fluid on an inclined wall. The partial differential equations have been reduced to ordinary boundary value problem. Approximate analytical results have been compared with numerical ones in some cases. Variations in velocity y with for different values of has been presented. The analytical results are found to be in good agreement with numerical solutions which reveals the effectiveness and convenience of the Homotopy Perturbation Method.

References

[1] **Sajid, M. and Hayat, T.**: The application of homotopy analysis method to thin film flows of a third order fluid, *Choas Solitons and Fractals*, in press.

[2] **Talay Akyildiz, F., Bellout, H. and Vajravelu, K.**: Exact solutions of nonlinear differential equations arising in third grade fluid flows, *Int. J. Non–Linear Mech.*, 39, 1571–1578, **2004**.

[3] **Vajravelu, K., Cannon, J. R., Rollins, D. and Leto, J.**: On solutions of some non–linear differential equations arising in third grade fluid flows, *Int. J. Engrg. Sci.*, 40, 1571–1578, **2004**.

[4] **Kaushik, V. V. R., Ghosh, S., Das, G. and Das, P. K.**: CFD modeling of water flow through sudden contraction and expansion in a horizontal pipe, *Chemical Engineering Journal*, 45, 30–36, **2011**.

[5] **He J. H.**: An approximation solution technique depending upon an artificial parameter, *Commun. Nonlinear Sci. Numer. Simulat.*, 3(2), 92, **1998**.

[6] **He J. H.**: Homotopy perturbation techniques comput. methods., *Appl. Mech. Engrg*, 178, 257, **1999**.

[7] **Shakeri, F., Ganji, D. D. and Azimi, M.**: Application of HPM–Pade' to Jeffery Hamel Flow Problem, *International Review of Mehanical Engineering*, Vol. 6, No. 3, pp. 537–540, **2012**.

[8] **Ganji, D. D. and Azimi, M.**: Application of DTM on MHD Jeffery Hamel Problem with Nanoparticles, U.P.B., *Scientific Bulletin Series A*, vol. 75, no. 1, pp. 223–230, **2013**.

[9] **Azimi, M., Azimi, A. and Mirzaei, M.**: Investigation of the unsteady graphene oxide nanofluid flow between two moving plates, *J. Comput. Theor. Nanos.*, 11(10), 1–5, **2014**.

[10] **Ayaz, F.**: Solutions of the systems of differential equations by differential transform method, *Applied Mathematics and Computation*, 147(2), 547-567, **2004**.

[11] **Ganji, D. D. and Rajabi, A.**: Assessment of homotopy-perturbation and perturbation methods in heat radiation equations, *International Communications in Heat and Mass Transfer*, Vol. 33, No. 3, pp. 391-400, **2006**.