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Free Vibrations of Timoshenko Beam with End Mass in the Field of Centrifugal Forces

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An analysis of natural frequencies and modes for a cantilever radial rotating beam with end mass is carried out within framework of Timoshenko beam model, on the base of convenient dimensionless equations of motion depended only on two dimensionless parameters. It is shown that the shear deformations at high angular speeds lead to significant changes in the natural modes, and as a consequence – to relevant qualitative effects for the natural frequencies.

Keywords: Radial rotating beams, vibrations, Timoshenko beam.

1. Introduction

Vibrations of radial rotating beams were studied in connection with various technical applications, in particular, with oscillations of turbine blades. The first studies were performed at beginning of the twentieth century (see, e.g., [1, 2]); later the methods of calculating the natural frequencies and modes of beams in the field of centrifugal forces were refined ([3–9] and others), with focus on numerical methods of determining the dynamical characteristics.

At small relative length of beams (such as blades), one can expect the substantial increase of the role of shear deformation that are neglected in the classical Euler–Bernoulli (E–B) beam model. Therefore in last decades researches of free and forced oscillations of radial rotating beams were also conducted on the basis of Timoshenko beam model, taking into account the shear deformability ([10–17] and others).

The role of centrifugal forces can be especially important in cantilever beams with end masses (e.g., blades with bandages). Free vibration analyses for such beams were conducted in [18–21] and others papers, but, as a rule, these researches were carried out within framework of classical E–B model.

In this work there are analyzed natural frequencies and natural modes of Timoshenko cantilever beam with a mass at the end, rotating about the axis perpendicular to the beam axis, and oscillating in the plane passing through the axis of rotation. The analysis is based on dimensionless equations of motion which depend on two dimensionless parameters – a shear deformability parameter and normalized parameter of tensile centrifugal force (or angular velocity of rotation). The influence of dimensionless parameters of the beam, of the end mass and of motion on the two first natural frequencies and modes is studied. It is shown that the shear deformations at high angular speeds lead to significant changes in the natural modes, and as a consequence – to relevant qualitative effects for the natural frequencies.

2. Statement of the problem. Governing equations

Let a cantilever beam of the length l having at the end a body with mass m and moment of inertia I_d, rotates with angular velocity Ω about the axis passing through the clamped edge of the beam (Fig. 1). Natural frequencies and natural modes are studied for free vibrations of the beam in the plane passing through the axis of rotation. S. P Timoshenko beam model (TB) is used, and centrifugal force generated by the mass at the edge is taken into account as well as moment of inertia forces appearing at turning of the mass.

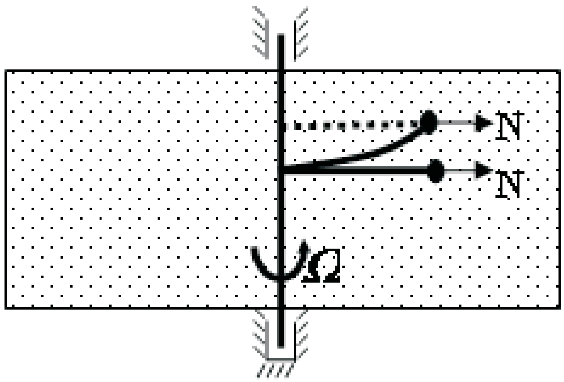

Figure 1 A rotating cantilever beam with end mass

Differential equations of force balance at oscillation of TB with account of longitudinal tensile force N are as follows

$$\frac{\partial Q}{\partial x} - \rho A \frac{\partial^2 y}{\partial t^2} + N \frac{\partial^2 y}{\partial x^2} = 0 \tag{1}$$

$$-\rho J \frac{\partial^2 \psi}{\partial t^2} + Q - \frac{\partial M}{\partial x} = 0 \tag{2}$$

where $y(x,t)$ is the full transverse deflection of the beam, ψ is the angle of rotation of the cross section, connected with y by relation $\partial y/\partial x = \psi + \gamma$, γ is the angle of shear, J is the moment of inertia of the beam cross section, $A$is the cross–sectional area, ρ is the density. The bending moment M and the transverse shear force Q are expressed via y and ψ:

$$M = -EJ\frac{\partial \psi}{\partial x}, \quad Q = k'AG\left(\frac{\partial y}{\partial x} - \psi\right) \tag{3}$$

(k' is the shear factor which depends on shape of the cross–section).

Substituting expressions (3) in (1), (2) we obtain the set of equations for y and ψ:

$$k'AG\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x} - \psi\right) - \rho A\frac{\partial^2 y}{\partial t^2} + N\frac{\partial^2 y}{\partial x^2} = 0 \tag{4}$$

$$-\rho J\frac{\partial^2 \psi}{\partial t^2} + k'AG\left(\frac{\partial y}{\partial x} - \psi\right) + EJ\frac{\partial^2 \psi}{\partial x^2} = 0 \tag{5}$$

Eliminating ψ from (4)

$$\frac{\partial \psi}{\partial x} = \frac{1}{k'AG}\left(k'AG\frac{\partial^2 y}{\partial x^2} - \rho A\frac{\partial^2 y}{\partial t^2} + N\frac{\partial^2 y}{\partial x^2}\right) \tag{6}$$

we come to the equation for deflection $y(x,t)$ (in the case of constant force N):

$$\begin{aligned} &EJ\frac{\partial^4 y}{\partial x^4} - \rho J\left(1 + \frac{E}{k'G}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2 J}{k'G}\frac{\partial^4 y}{\partial t^4} + \rho A\frac{\partial^2 y}{\partial t^2} \\ &+N\left(\frac{EJ}{k'AG}\frac{\partial^4 y}{\partial x^4} - \frac{\rho J}{k'AG}\frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^2 y}{\partial x^2}\right) = 0 \end{aligned} \tag{7}$$

We introduce the dimensionless variables and parameters [23, 24]:

$$\xi = \frac{x}{r_0} \quad \tau = \frac{c}{r_0}t \quad Y = \frac{y}{r_0} \quad c^2 = \frac{E}{\rho} \quad r_0^2 = \frac{J}{A} \quad \chi = \frac{E}{k'G} \quad \bar{N} = \frac{N}{EA} \tag{8}$$

where c is the sound speed in the material of the beam, r_0 is the radius of inertia of the cross–section, χ is the shear deformability parameter (for the Euler–Bernoulli and Rayleigh beam models one has $\chi = 0$, which corresponds to an infinitely large shear stiffness). In these parameters equation (7) takes the form:

$$\frac{\partial^4 Y}{\partial \xi^4} - (1+\chi)\frac{\partial^4 Y}{\partial \xi^2 \partial \tau^2} + \chi\frac{\partial^4 Y}{\partial \tau^4} + \frac{\partial^2 Y}{\partial \tau^2} + \bar{N}\left(\chi\frac{\partial^4 Y}{\partial \xi^4} - \chi\frac{\partial^4 Y}{\partial \xi^2 \partial \tau^2} - \frac{\partial^2 Y}{\partial \xi^2}\right) = 0 \tag{9}$$

The equation for the cross-sectional angle ψ in dimensionless parameters (8) is as follows:

$$\frac{\partial \psi}{\partial \xi} = \left(\frac{\partial^2 Y}{\partial \xi^2} - \chi\frac{\partial^2 Y}{\partial \tau^2} + \chi\bar{N}\frac{\partial^2 Y}{\partial \xi^2}\right) \tag{10}$$

For the formulation of boundary conditions is also advisable to have an uncoupled equation for angle ψ. Excluding y from equations (4), (5) and passing to the dimensionless parameters, we obtain the equation

$$\frac{\partial^4 \psi}{\partial \xi^4} - (1+\chi)\frac{\partial^4 \psi}{\partial \tau^2 \partial \xi^2} + \chi\frac{\partial^4 \psi}{\partial \tau^4} + \frac{\partial^2 \psi}{\partial \tau^2} - \bar{N}\left(\frac{\partial^2 \psi}{\partial \xi^2} + \chi\frac{\partial^4 \psi}{\partial \tau^2 \partial \xi^2} - \chi\frac{\partial^4 \psi}{\partial \xi^4}\right) = 0 \tag{11}$$

Equation (11) is similar to equation (17) for the total deflection (but the boundary conditions for them are different). Equation (6) in dimensionless variables is as follows:

$$\frac{\partial\psi}{\partial\xi} = \left(\frac{\partial^2 Y}{\partial\xi^2} - \chi\frac{\partial^2 Y}{\partial\tau^2} + \chi\bar{N}\frac{\partial^2 Y}{\partial\xi^2}\right) \tag{12}$$

In the case of rotating beam with end mass the longitudinal tensile force is $N = m\Omega^2 l$ (we assume that the end mass is large in comparison with mass of the beam and neglect the centrifugal forces of the beam itself). Dimensionless axial force $\bar{N}$ is then equal to

$$\bar{N} = \frac{m\,\Omega^2 l}{E\,A} \tag{13}$$

2.1. *Boundary conditions*

Formulation of boundary conditions in TB differs from E–B model. At the clamped end $\xi = 0$ boundary conditions are

$$Y(\xi = 0) = 0, \quad \psi(x = 0) = 0 \tag{14}$$

At the free end $x = l$ ($\xi = L/R$) the bending moment (3) is equal to the moment of inertia forces for the end mass $M^i = -I_d\varepsilon$ and it is directed oppositely (due to usually assumed sign rule for the moment, ε is the angular acceleration of the mass). We obtain the following boundary condition for the bending moment:

$$-EJ\frac{\partial\psi}{\partial x}\bigg|_{x=l} = I_d\,\frac{\partial^2\psi}{\partial t^2}\bigg|_{x=l}$$

which in dimensionless parameters (8) takes the form

$$\frac{\partial\psi}{\partial\xi}\bigg|_{\xi=l/r_0} = -I_d^*\,\frac{\partial^2\psi}{\partial\tau^2}\bigg|_{\xi=l/r_0} \qquad \left(I_d^* = \frac{I_d}{r_0^\rho J}\right) \tag{15}$$

The second boundary condition at $x = l$ is stated for the transverse force. In case of oscillations of the beam in the plane passing through the rotation axis (perpendicular to the axis of the undeformed beam), the direction of the longitudinal force N remains unchanged, i. e it is parallel to the axis of the beam (Fig. 2).

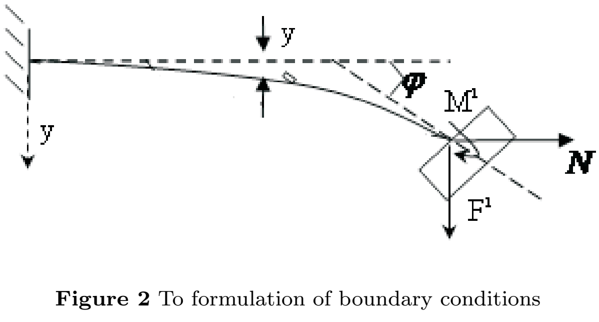

Figure 2 To formulation of boundary conditions

The transverse force at the edge is equal to the projection of the longitudinal forces N on the plane perpendicular to the bent axis: $Q = -N \sin \varphi$. This force is added to the projection on the same plane of relative inertia force $-m\ddot{y}(l) \cos \varphi$. With account of (3), we obtain nonlinear boundary condition

$$k'AG \left(\frac{\partial y}{\partial x} - \psi \right) = -m\ddot{y}(l) \cos \varphi - N \sin \varphi$$

which after linearization takes the form:

$$k'AG \left(\frac{\partial y}{\partial x} - \psi \right) = -m \frac{\partial^2 y}{\partial t^2} - N \frac{\partial y}{\partial x}$$

where $x = l$

This condition in dimensionless parameters (8) takes the form:

$$\frac{\partial Y}{\partial \xi} (1 + \chi \bar{N}) + \frac{\chi}{\mu} \frac{l}{r_0} \frac{\partial^2 Y}{\partial \tau^2} = \psi \text{ at } \xi = l/r_0 \tag{16}$$

where the parameter of weight ratio is introduced:

$$\mu = \frac{\rho A \, l}{m} \tag{17}$$

3. Solution for the free oscillation problem

Assuming the total dynamic deflection and angle ψ in the form

$$Y(\xi, \tau) = F(\xi) e^{i\omega_0 \tau} \qquad \psi(\xi, \tau) = e^{i\omega_0 \tau} \Psi(\xi) \tag{18}$$

and substituting (18) into (9) we obtain the ordinary differential equation

$$\left(1 + \bar{N}\chi\right) F^{IV} + \left((1 + \chi)\, \omega_0^2 + \bar{N}\chi\omega_0^2 - \bar{N}\right) F^{II} + \left(\omega_0^4 \chi - \omega_0^2\right) F = 0 \tag{19}$$

and similar equation for $\Psi(\xi)$. The characteristic equation

$$\left(1 + \bar{N}\chi\right) k^4 + \left((1 + \chi)\, \omega_0^2 + \bar{N}\chi\omega_0^2 - \bar{N}\right) k^2 + \left(\omega_0^4 \chi - \omega_0^2\right) = 0 \tag{20}$$

has roots

$$k_{1,2}^2 = \frac{-\left((1 + \chi)\, \omega_0^2 + \bar{N}\chi\omega_0^2 - \bar{N}\right) \pm \sqrt{D}}{2\left(1 + \bar{N}\chi\right)}, \tag{21}$$

where

$$D = \left((1 + \chi)\, \omega_0^2 + \bar{N}\chi\omega_0^2 - \bar{N}\right)^2 + 4\omega_0^2 \left(1 + \bar{N}\chi\right) \left(1 - \omega_0^2 \chi\right)$$

The discriminant can also be written in the form $D = \left((1 - \chi)\, \omega_0^2 + \bar{N}\chi\omega_0^2 - \bar{N}\right)^2 + 4\omega_0^2$, whence follows that always $D > 0$ and that $\sqrt{D} > \left|(1 - \chi)\, \omega_0^2 + \bar{N}\chi\omega_0^2 - \bar{N}\right|$. To define the sign of the numerator in (20), consider the expression:

$$D - \left((1 + \chi)\, \omega_0^2 + \bar{N}\chi\omega_0^2 - \bar{N}\right)^2 = 4\omega_0^2 \left(1 + \bar{N}\chi\right) \left(1 - \chi\omega_0^2\right) \tag{22}$$

This expression is positive for $\chi\omega_0^2 < 1$ (first, or "low frequency" case) and negative for $\chi\omega_0^2 > 1$ (second, "high frequency" case).

In the first case ($\chi\omega_0^2 < 1$) equation (20) has two real and two imaginary roots, $\pm k_1,\ \pm i\, k_{*2}$, where

$$k_1^2 = \frac{\sqrt{D} - \left[(1+\chi)\,\omega_0^2 - \bar{N}(1-\chi\omega_0^2)\right]}{2\left(1+\bar{N}\chi\right)},$$
$$k_{*2}^2 \equiv -k_2^2 = \frac{\left[(1+\chi)\,\omega_0^2 - \bar{N}(1-\chi\omega_0^2)\right] + \sqrt{D}}{2\left(1+\bar{N}\chi\right)} \tag{23}$$

Then the general solution of (19) and of similar equation for $\Psi(\xi)$ can be written as:

$$F(\xi) = C_1 shk_1\xi + C_2 chk_1\xi + C_3 \sin k_{*2}\xi + C_4 \cos k_{*2}\xi$$
$$\Psi(\xi) = D_1 shk_1\xi + D_2 chk_1\xi + D_3 \sin k_{*2}\xi + D_4 \cos k_{*2}\xi \tag{24}$$

and the general solutions of equations (9) and (11) have the form

$$Y(\xi,\,\tau) = e^{i\omega_0\tau}\left[C_1 shk_1\xi + C_2 chk_1\xi + C_3 \sin k_{*2}\xi + C_4 \cos k_{*2}\xi\right]$$
$$\psi(\xi,\,\tau) = e^{i\omega_0\tau}\left[D_1 shk_1\xi + D_2 chk_1\xi + D_3 \sin k_{*2}\xi + D_4 \cos k_{*2}\xi\right] \tag{25}$$

In the second case ($\chi\omega_0^2 > 1$), equation (20) has four imaginary roots $\pm\, ik_{*1},\ \pm i\, k_{*2}$, where

$$k_{*1}^2 \equiv -k_1^2 = \frac{\left[(1+\chi)\,\omega_0^2 + \bar{N}(\chi\omega_0^2 - 1)\right] - \sqrt{D}}{2\left(1+\bar{N}\chi\right)} \tag{26}$$

Then the general solutions of equation (19) and of similar equation for $\Psi(\xi)$ are:

$$F(\xi) = C_1 \sin k_{*1}\xi + C_2 \cos k_{*1}\xi + C_3 \sin k_{*2}\xi + C_4 \cos k_{*2}\xi$$
$$\Psi(\xi) = D_1 \sin k_{*1}\xi + D_2 \cos k_{*1}\xi + D_3 \sin k_{*2}\xi + D_4 \cos k_{*2}\xi \tag{27}$$

and solutions of equations (9) and (11) are:

$$Y(\xi,\,\tau) = e^{i\omega_0\tau}\left[C_1 \sin k_{*1}\xi + C_2 \cos k_{*1}\xi + C_3 \sin k_{*2}\xi + C_4 \cos k_{*2}\xi\right]$$
$$\psi(\xi,\,\tau) = e^{i\omega_0\tau}\left[D_1 \sin k_{*1}\xi + D_2 \cos k_{*1}\xi + D_3 \sin k_{*2}\xi + D_4 \cos k_{*2}\xi\right] \tag{28}$$

Eight constants C_k and D_k are linked by four conditions that are derived from the equation (12). Substituting into (12) the obtained solutions for$Y(\xi,\,\tau)$ and $\Psi(\xi,\,\tau)$, we obtain the following relations.

Case $\chi\omega_0^2 < 1$:

$$D_1 = \nu_1 C_2, \quad D_2 = \nu_1 C_1, \quad D_3 = \nu_2 C_4, \quad D_4 = -\nu_2 C_3 \tag{29}$$

Here

$$\nu_1 = \frac{\left(1+\chi\bar{N}\right)k_1^2 + \chi\omega_0^2}{k_1}, \quad \nu_2 = \frac{-\left(1+\chi\bar{N}\right)k_{*2}^2 + \chi\omega_0^2}{k_{*2}} \tag{30}$$

Case $\omega_0^2\chi > 1$

$$D_1 = \nu_3 C_2, \quad D_2 = -\nu_3 C_1, \quad D_3 = \nu_2 C_4, \quad D_4 = -\nu_2 C_3 \tag{31}$$

where

$$\nu_3 = \frac{-\left(1+\chi\bar{N}\right)k_{*1}^2 + \chi\omega_0^2}{k_{*1}} \tag{32}$$

4. Frequency equation

Let us substitute the general solutions for Y and ψ (25) or (28) into boundary conditions (14), (15), (16), with account for relations between C_i and D_i (29) or (31). For the case $\chi\omega_0^2 < 1$ conditions (14) give

$$C_4 = -C_2 \quad D_4 = -D_2 \tag{33}$$

Then in view of (29) one has

$$\frac{D_3}{\nu_2} = -\frac{D_1}{\nu_1}, \quad \nu_2 C_3 = \nu_1 C_1$$

These relations allow to exclude two arbitrary constants from the general solution and express it via the constants C_1, C_2:

$$\begin{aligned} Y &= e^{i\omega_0\tau}\left[C_1 g_1(\xi) + C_2 g_2(\xi)\right] \\ \psi &= e^{i\omega_0\tau}\nu_1\left[C_1 g_2(\xi) + C_2 g_3(\xi)\right] \end{aligned} \tag{34}$$

where the following functions are introduced:

$$\begin{aligned} g_1(\xi) &= shk_1\xi + \frac{\nu_1}{\nu_2}\sin k_{*2}\xi \\ g_2(\xi) &= chk_1\xi - \cos k_{*2}\xi \\ g_3(\xi) &= shk_1\xi - \frac{\nu_2}{\nu_1}\sin k_{*2}\xi \end{aligned} \tag{35}$$

The boundary condition for the bending moment (15) leads to equation

$$C_1\left[g_5\left(l/r_0\right) - I_d^*\omega_0^2 g_2\left(l/r_0\right)\right] + C_2\left[g_4\left(l/r_0\right) - I_d^*\omega_0^2 g_3\left(l/r_0\right)\right] = 0 \tag{36}$$

where

$$\begin{aligned} g_4(\xi) &\equiv \frac{\partial g_3}{\partial\xi} = k_1 chk_1\xi - \frac{\nu_2}{\nu_1}k_{*2}\cos k_{*2}\xi \\ g_5(\xi) &\equiv \frac{\partial g_2}{\partial\xi} = k_1 shk_1\xi + k_{*2}\sin k_{*2}\xi \end{aligned} \tag{37}$$

The boundary condition (16) leads to equation

$$\begin{aligned} &C_1\left[g_6\left(\frac{l}{r_0}\right)(1+\chi\bar{N}) - \nu_1 g_2\left(\frac{l}{r_0}\right) - \frac{\chi}{\mu}\frac{l}{r_0}\omega_0^2 g_1\left(\frac{l}{r_0}\right)\right]h_{11} \\ &+C_2\left[g_5\left(\frac{l}{r_0}\right)(1+\chi\bar{N}) - \nu_1 g_3\left(\frac{l}{r_0}\right) - \frac{\chi}{\mu}\frac{l}{r_0}\omega_0^2 g_2\left(\frac{l}{r_0}\right)\right] = 0 \end{aligned} \tag{38}$$

where

$$g_6(\xi) = \frac{\partial g_1}{\partial \xi} = k_1 chk_1 \xi + \frac{\nu_1}{\nu_2} k_{*2} \cos k_{*2} \xi \tag{39}$$

Two homogeneous linear equations (36) and (38) give the frequency equation

$$\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = 0 \tag{40}$$

where

$$\begin{aligned}
h_{11} &= g_5 \left(l/r_0\right) - I_d^* \omega_0^2 g_2 \left(l/r_0\right) \\
h_{21} &= g_6 \left(\frac{l}{r_0}\right) (1 + \chi \bar{N}) - \nu_1 g_2 \left(\frac{l}{r_0}\right) - \frac{\chi}{\mu} \frac{l}{r_0} \omega_0^2 g_1 \left(\frac{l}{r_0}\right) \\
h_{12} &= g_4 \left(l/r_0\right) - I_d^* \omega_0^2 g_3 \left(l/r_0\right) \\
h_{22} &= g_5 \left(\frac{l}{r_0}\right) (1 + \chi \bar{N}) - \nu_1 g_3 \left(\frac{l}{r_0}\right) - \frac{\chi}{\mu} \frac{l}{r_0} \omega_0^2 g_2 \left(\frac{l}{r_0}\right)
\end{aligned} \tag{41}$$

5. Results of numerical analysis

Equation (40) was solved numerically in the Maple–10 package. All quantities in the frequency equation (40) can be expressed in the following dimensionless parameters: parameter of relative length of the beam l/r_0, the mass ratio parameter μ (17), the shear deformability parameter χ (8), the dimensionless moment of inertia of the mass I_d^* (16), the relative angular velocity $\bar{\Omega} = \frac{\Omega}{\omega_{0*}}$ (here $\omega_{0*}^{=}\sqrt{\frac{3EJ}{ml^3}}$). The parameter of the longitudinal force $\bar{N}$ is expressed through the parameters l/r_0 and $\bar{\Omega}$: $\bar{N} = \bar{\Omega}^2 \frac{3r_0^2}{l^2}$. These parameters determine the natural frequency of the normalized beam in time τ (8) $\bar{\omega} = \frac{\omega_0}{\omega_{0*}^{(\tau)}}$, where $\omega_{0*}^{(\tau)} = \sqrt{3\mu}\left(\frac{r_0}{l}\right)^2$.

Testing the program was carried out by comparing results of the calculations for the case $\chi = 0$ with the earlier calculations according to the classical Euler–Bernoulli model [25]. As was noted above, for $\chi = 0$ the model Timoshenko reduces to Rayleigh model, which usually gives results that do not differ practically from the Euler–Bernoulli model (except for very short beams). In all our calculations for $\chi = 0$ the results almost identical to the data [25].

Then the dependencies of natural frequencies and natural modes on dimensionless parameters in a wide ranges were studied using the package Maple–13. The main attention was paid to analysis of the influence of the shear deformability on different levels of angular velocity parameter $\bar{\Omega}$. For the shear deformability parameter χ we assumed the following values: $\chi = 0$; $\chi = 3$ (which, as can be seen from (8), approximately corresponds to isotropic material for many typical cross–sections of beam), and $\chi = 10$, $\chi = 20$ (increased shear deformability, for example, a composite material, thin–walled beams, etc.). Parameter $\bar{\Omega}$ was assumed in the range from 0 to 2 (the value $\bar{\Omega} = 1$ corresponds to the angular velocity equaled to the natural cyclic frequency of beam, with account only the end mass). For the relative length l/r_0 there were taken values l/r_0= 20, 10 and 5; for the mass ratio parameter μ - values 0.1, 0.5 and 1.0.

To simplify the analysis we first neglected with the moment of inertia forces of the end mass, and then the effect of the parameter I_d^* was studied separately.

Results for beams with neglecting the moment of inertia of the end mass ($I_d^* = 0$). There were constructed plots of the normalized first eigenfrequency $\bar{\omega}_0$ via the normalized angular velocity $\bar{\Omega}$ at $\mu = 1$ for three χ values: $\chi = 0$, 3 and 10 at $l/r_0 = 20$, 10, and 5 (Fig. 3, $a - c$, respectively).

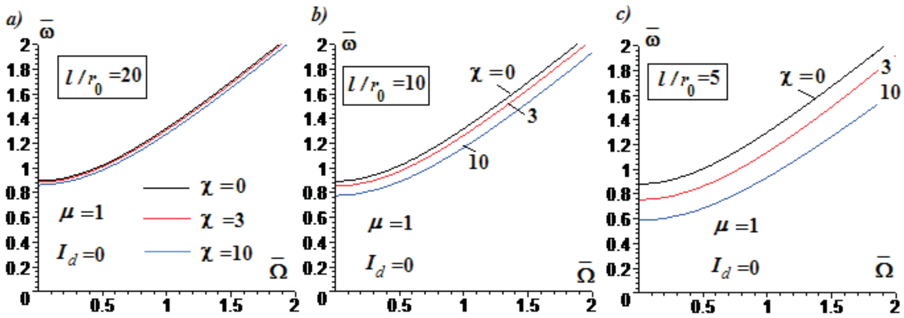

Figure 3 Normalized first natural frequency via the normalized angular velocity for three values of the shear parameter χ= 0 , 3, 10: a) $l/r_0 = 20$; b) $l/r_0 = 10$; c) $l/r_0 = 5$ ($\mu = 1$, I_d^*=0)

The shear deformability leads to a decrease of the natural frequency of oscillation $\bar{\omega}$, and the effect of rotation on $\bar{\omega}$ in these parameters is approximately the same for different values of the parameter χ (curves $\bar{\omega} - \bar{\Omega}$ for different χ are almost equidistant) . As might be expected, for a relatively long beam ($l/r_0 = 20$) influence of χ on $\bar{\omega}$ is weak, for short beams this effect becomes significant (at $l/r_0 = 5$ reduction of $\bar{\omega}$ for $\chi = 10$ in comparison to $\chi = 0$ is about 30%).

It is interesting to compare the first natural modes and their flexural components for different values of the parameters χ and $\bar{\Omega}$, represented for $l/r_0 = 10$ in Fig. 4. For non–rotating beams the total deflection is almost pure bending (the shear component is small); with increasing angular velocity the shear deflection increases, and the portion of the bending deflection at the total deflection decreases. Even more pronounced increase of the shear component of deflection is caused by the increase of the parameter χ from 3 to 10 (Fig. 4, a, b).

With rising angular velocity and shear deformability the eigenmode profile rectified, and the turn angle at the initial section becomes markedly different from zero.

There were also performed calculations of the second natural frequencies and the corresponding natural modes. Dependencies of the second natural frequency on the angular velocity of rotation are shown in Fig. 5 for l/r_0=10; $\mu = 1$, J_d^*=0 at three χ-values, along with corresponding curves for the first natural frequency (in this scale the curves for different χ almost merge).

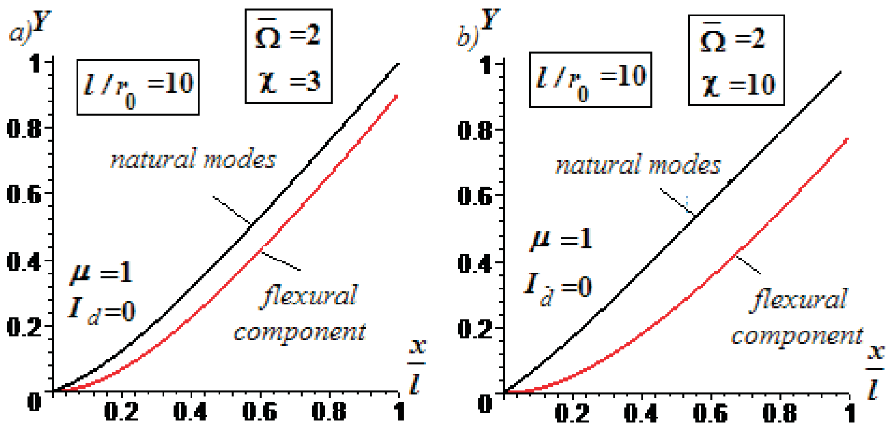

Figure 4 The first natural modes and their flexural components for the beam l/r_0=10, $\mu = 1$, , I_d^*=0 $\bar{\Omega} = 2$ at twoχ- values: χ=3 (*a*) and χ=10 (*b*)

The shear deformability affects the higher natural frequencies substantially stronger than the first frequency. Neglecting this effect for l/r_0=10 leads to overestimation of the natural frequency approximately by 20%.

The shear deformability relatively weakly change the second natural modes (as well as the first one), but has a significant effect on the flexural – shear deflections ratio. These modes for non-rotating beams are depicted in Fig. 6, *a*, *b* (l/r_0=10; $\mu = 1$, J_d^*=0, $\bar{\Omega} = 0$).

Whereas the first natural mode at χ= 3 almost wholly is a flexural one (Fig. 4, *a*), in the second mode at a given l/r_0 the flexural deflection is about half of the total deflection.

The rotation has a relatively weaker effect on higher natural modes and the different deflection components ratio (Fig. 7, *a*, *b*, for the same beam (l/r_0=10; $\mu = 1$, J_d^*=0), but at χ=10; (*a*) $\bar{\Omega} = 0$; (*b*) $\bar{\Omega} = 2$). In this case the increased shear deformability results in prevailing shear component of the total deflection.

6. Effect of the moment of inertia of the end mass.

Results of calculations based on the mass moment of inertia at the end of ($I_d^* \neq 0$) shown in Fig. 8–10. In all calculations was assumed l/r_0=10; $\mu = 1$; the first and second natural frequencies and modes were considered. Fig. 8 illustrates the first and second natural frequencies of the angular speed for three values χ=0; 3 and 10 at I_d^*=50 and I_d^*=200 (Figs. 8, *a* and *b*, respectively).

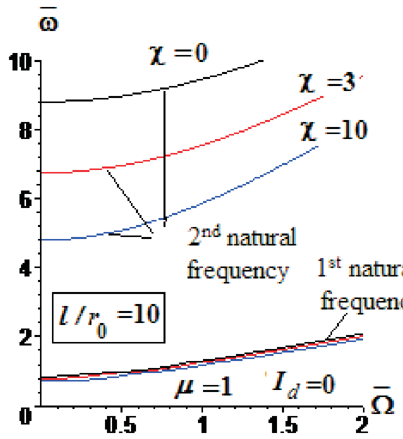

Figure 5 The first and second normalized eigenfrequency $\bar{\omega}_0$ via the angular velocity of rotation at l/r_0=10; $\mu = 1$, J_d^*=0, for different χ-values (χ=0; 3 and 10)

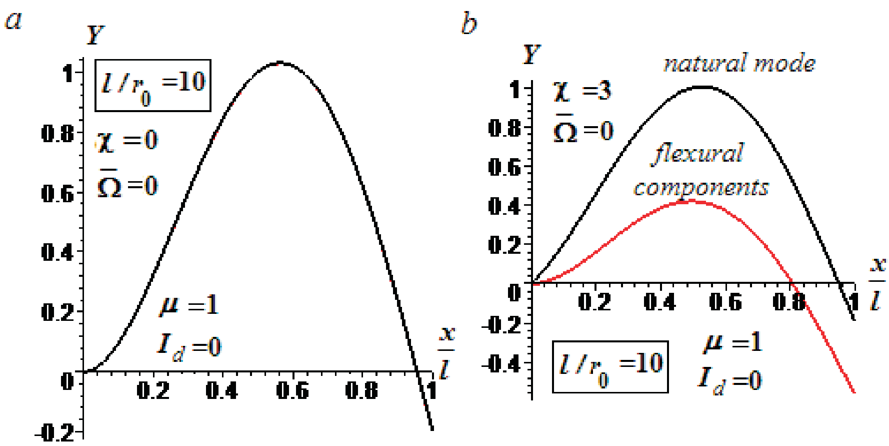

Figure 6 The second natural mode of non–rotating beam l/r_0=10; $\mu = 1$, J_d^*=0: a) $\chi = 0$, b) $\chi = 3$

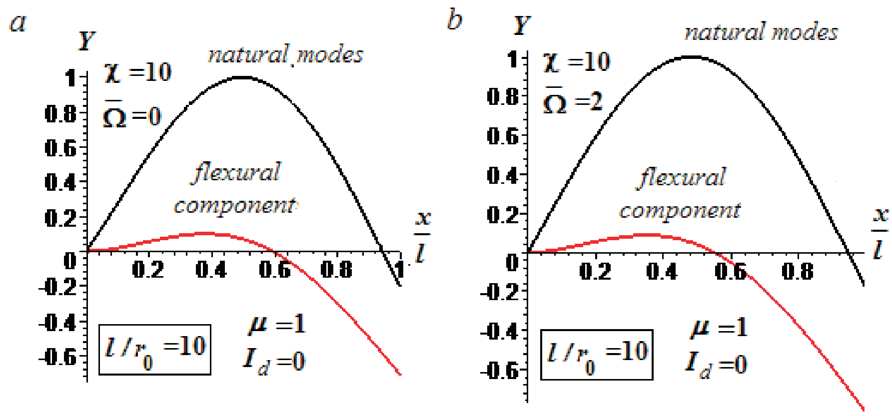

Figure 7 The second natural modes of the beam with high shear deformability (χ=10 (a) $\bar{\Omega} = 0$; (b) $\bar{\Omega} = 2$ (l/r_0=10; $\mu = 1$, I_d^*=0)

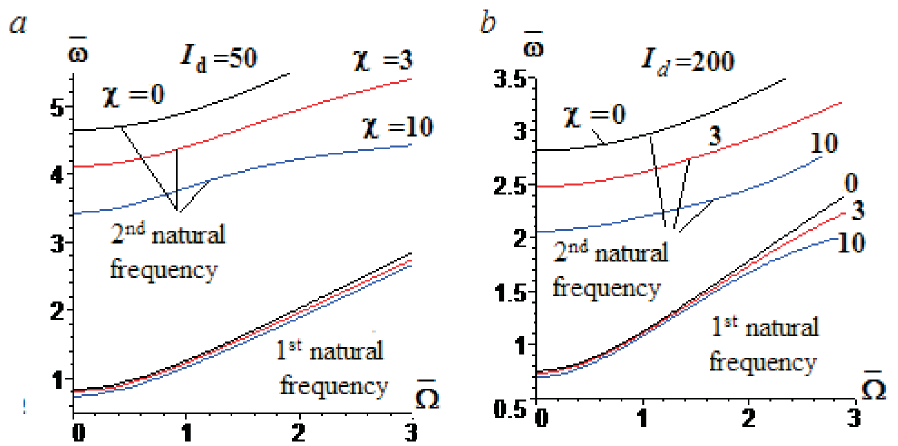

Figure 8 Normalized first and second natural frequency $\bar{\omega}$ via the normalized angular velocity for three χvalues at I_d^*=50 (a) and I_d^*=200(b); (l/r_0=10; $\mu = 1$)

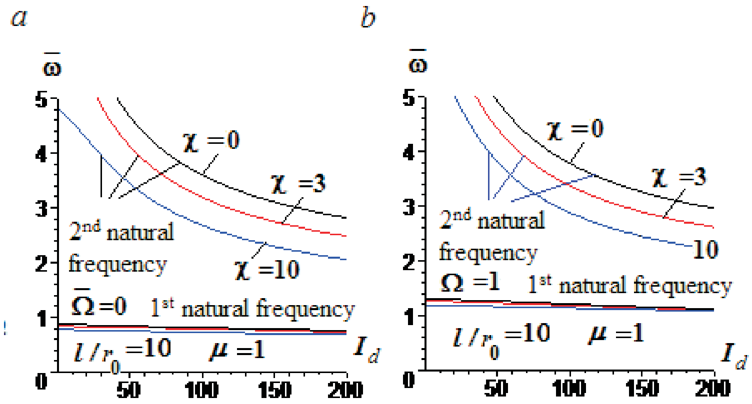

Figure 9 The normalized frequency $\bar{\omega}$ via the normalized moment of inertia of the end mass at l/r_0=10; $\mu = 1$, with different value; (a) $\bar{\Omega} = 0$, (b) the angular velocity $\bar{\Omega} = 1$

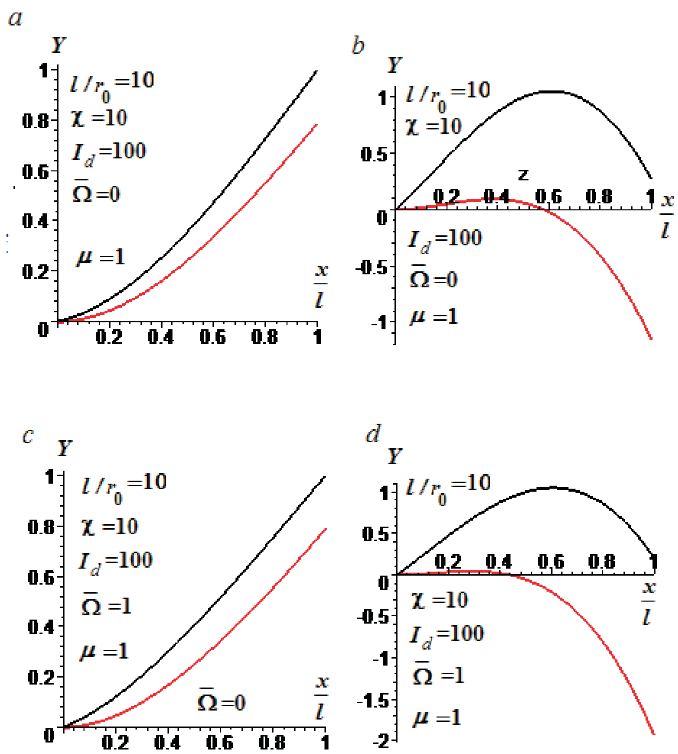

Figure 10 The first and second natural shapes and the flexural components for a non–rotating (a, b) and rotating ($\bar{\Omega} = 1$) beam (c, d) with I_d^*=100 (l/r_0=10, $\mu = 1$, χ=10)

Fig. 9 shows the dependence of the normalized frequency $\bar{\omega}$ on normalized moment of inertia of the end mass at different values χ for a nonrotating beams (a) and at angular velocity $\bar{\Omega} = 1$ (b).

The first and second eigenmodes and their flexural components at χ=10 are shown in Fig. 10 for non–rotating (a, b) and rotating ($\bar{\Omega} = 1$) beam (c, d) with a normalized moment of inertia end mass I_d^*=100.

Comparing the results presented in Fig. 8–10 with those given above for the case $I_d^* = 0$, we can draw the conclusion that the moment of inertia of the end mass lowers the natural frequencies, especially the higher ones, and markedly changes the dependencies of natural frequencies on the angular velocity.

7. Conclusions

There has been conducted analysis of natural frequencies and natural modes of cantilever beam with the end mass, rotating about an axis perpendicular to the beam axis, and oscillating in the plane passing through the axis of rotation, within framework of Timoshenko beam model. The peculiarity of the analysis is the use of dimensionless equations of motion which depend only on two dimensionless parameters – a shear deformability parameter and normalized angular velocity of rotation. In the numerical experiment the influence of the angular velocity of rotation on the first and second natural frequencies and natural modes has been studied, as well as effects of relative length, shear deformability, relative end mass and the dimensionless moment of inertia of the end mass. For the first time, the ratio of bending and shear components in the natural modes has been studied for different combinations of dimensionless parameters.

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