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Parametric Study on Thick Plate Vibration Using FSDT

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The prime objective of the research is to investigate the influence of various structural parameters like aspect ratio, boundary condition, size of cut-out etc. on the free vibration frequencies of a thick rectangular plate. Plates being one the most common structural elements has always enticed the interest of many researchers towards this problem. In here a general first order shear deformation theory (FSDT) is used to analyse the free vibration behaviour of rectangular isotropic plates. A finite element program has been developed using 9 node isoparametric element. A number of numerical examples are presented here. Two different sets of mass lumping scheme are considered to carry the analysis using and without using rotary inertia. The definite advantage of this work over other similar works done by using FEM pakages is its exceptional accuracy. At most the error calculated for convergence study with published literature is 1%.

Keywords: Finite element method, FSDT, rectangular plate, rotary inertia, natural frequency.

1. Introduction

Vibration is the mechanical phenomenon of oscillations of any system about an equilibrium point. The frequency at which a system tends to oscillate when not under any external force is its natural frequency. Vibration is generally undesirable since it is waste of energy and creates noise. If the forced frequency (frequency of the system at some applied force) is equal to the natural frequency of the system, phenomenon of resonance occurs which increases the amplitude of the vibration.

There are several instances like "Galloping Gertie" which bear testimony to the fact. Hence it often becomes a necessity to know the natural frequency of the structure while designing.

Plate structures are one of the most common structural element. Its wide applicability has captured the attention of many a researchers. In fact the analysis of plates first started way back in the 1800s. Euler [1] was responsible for solving free vibrations of a flat plate using a mathematical approach for the first time. Then it was the German physicist Chladni [2] who discovered the various modes of free vibrations. Then later on the theory of elasticity was formulated. Navier [3] can be considered as the originator of the modern theory of elasticity. Navier's numerous scientific activities included the solution of various plate problems. He was also responsible for deriving the exact differential equation for rectangular plates with flexural resistance. For the solution to certain boundary value problems Navier introduced exact methods which transformed differential equations to algebraic equations. Poisson in 1829 [4] extended the use of governing plate equation to lateral vibration of circular plates.

Many theories with different assumptions have been developed over the years to accurately predict the response of plates. The earliest plate theory suggested for the plates was the Kirchhoff plate theory or Classical Plate Theory (CPT). In this theory the normal of the plane is assumed to be straight and normal in the deformed configuration. The CPT can be applied to a fine degree of accuracy to analyse the plates whose thickness is small by two orders of magnitude as compared to the planar dimensions. Such an assumption neglects the transverse shear effects, which have significant impact on the behaviour of thick plates. This limits the usage of the theory for only thick plates. Mindlin refined the CPT by including the transverse shear effects in his model and the theory when applied to thick plates is called the First–order Shear Deformation Theory (FSDT). In this model, the normal of the plate is assumed to be straight but no longer normal in the deformed configuration. This assumption makes the transverse shear strains and stresses to be constant in the thickness direction of the thick plate and thus requires the shear correction factor. The shear correction factor is a dimensionless quantity, which accounts for the difference between the constant state of shear strains and stresses in the First-order Shear Deformation Theory and the actual distribution of shear strains and stresses according to the elasticity theory.

In the late 1900s, the theory of finite elements was evolved which is the basis for all the analysis on complex structures. However the analyses using finite elements are now being carried out using comprehensive software which requires high CPU resources to compute the results. A number of excellent comprehensive review and bibliographical information by several researchers provide a valuable insight in this area [5–13].

Yu [14] used the Gorman method to calculate the dynamic repose of cantilever plates with attached point mass. Very recently bending and free vibration behaviour of laminated soft core skew sandwich plate with stiff laminate face sheets was investigated using a recently developed C_0 finite element (FE) model based on higher order zigzag theory [15]. A new implementation of the ancient Chinese method called the Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM) was presented by Bayat et al. [16] to obtain natural frequency and corresponding displacement of tapered beams. Amabili and Carra [17] experimentally studied the large-amplitude forced vibrations of a stainless-steel thin rectangular plate carrying different concentrated masses. Lorenzo [18] used the trigonometric Ritz method for general vibration analysis of rectangular Kirchhoff plates. Mantari et al. [19] performed bending and free vibration analysis of multilayered plates and shells using higher order shear deformation theory.

The primary objective of the current paper is evaluate the effect of various parameters like thickness ratio, aspect ratio, boundary condition etc. on the frequency of isotropic thick plates. Also many a times it is seen that central cutouts are present in this plates to fulfill some design or aesthetic requirement. An attempt has been to study the effect of increase in the central cutout on the natural frequencies. The special feature of the formulation is that Rotary inertia is included in the consistent mass matrix for the analysis.

2. Finite Element Formulation

A general first order shear deformation theory (FSDT) is developed to analyse the free vibration behaviour of rectangular isotropic plates. For the analysis a finite element program is developed using 9 node isoparametric element. The element is capable of handling plate of any shape. This is possible by using a simple mapping technique defined as

$$x = \sum_{r=1}^{9} N_r x_r \quad \text{and} \quad y = \sum_{r=1}^{9} N_r y_r \tag{1}$$

where (x, y) are the coordinates of any point within the element, (x_r, y_r) are the coordinates of the rthnodal point and N_r is the corresponding interpolation function of the element. In this element, Lagrangian interpolation function has been used for N_r. Taking the bending rotations as independent field variables, the effect of shear deformation may be expressed as

$$\left\{ \begin{array}{c} \phi_x \\ \phi_y \end{array} \right\} = \left\{ \begin{array}{c} \theta_x - \frac{\partial w}{\partial x} \\ \theta_y - \frac{\partial w}{\partial y} \end{array} \right\}$$

where ϕ_x and ϕ_y are the average shear rotation over the entire shell thickness and θ_x and $\theta_y$are the total rotations in bending. The other independent field variable is w, where w is the transverse displacement.

The interpolation functions used for the representation of element geometry, in eqns (1) has been used to express the displacement field at a point within the element in terms of nodal variables as

$$w = \sum_{r=1}^{9} N_r w_r \quad \theta_x = \sum_{r=1}^{9} N_r \theta_{xr} \quad \text{and} \quad \theta_y = \sum_{r=1}^{9} N_r \theta_{yr} \tag{2}$$

The generalized stress–strain relationship with respect to its reference plane may be expressed as

$$\{\sigma\} = [D]\{\varepsilon\} \tag{3}$$

where:

$$\{\sigma\} = [M_x \;\; M_y \;\; M_{xy} \;\; Q_x \;\; Q_y] \tag{4}$$

$$\{\varepsilon\} = \left\{ \begin{array}{l} -\frac{\partial\theta_x}{\partial x} \\ -\frac{\partial\theta_y}{\partial y} \\ -\frac{\partial\theta_x}{\partial x} - \frac{\partial\theta_y}{\partial y} \\ \frac{\partial w}{\partial x} - \theta_x \\ \frac{\partial w}{\partial y} - \theta_y \end{array} \right\} \tag{5}$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & 0 & 0 & 0 \\ D_{21} & D_{22} & 0 & 0 & 0 \\ 0 & 0 & D_{33} & 0 & 0 \\ 0 & 0 & 0 & D_{44} & D_{45} \\ 0 & 0 & 0 & D_{54} & D_{55} \end{bmatrix} \tag{6}$$

where:

$$\begin{aligned} D_{11} &= D_{22} = \frac{E}{1-\nu^2} \\ D_{12} &= D_{21} = \nu D_{11} \\ D_{33} &= \frac{E}{2(1+\nu)} \\ D_{44} &= D_{55} = \frac{Eh^3}{12(1-\nu^2)} \\ D_{45} &= D_{54} = \nu D_{44} \end{aligned}$$

With the help of eqns (2) and (5) the strain vector may be expressed in terms of the nodal displacement vector $\{\delta\}$ as

$$\varepsilon = \sum_{r=1}^{9} [B]_r \left[\{\delta_r\}\right]_e \tag{7}$$

where $[B]$ is the strain displacement matrix containing interpolation functions and their derivatives.

Once the matrices $[B]$ and $[D]$ are obtained, the stiffness matrix of an element $[K]_e$ can be easily derived with the help of virtual work method which may be expressed as

$$[K] = t \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] |J| d\xi d\eta \tag{8}$$

where $|J|$ is the determinant of the Jacobian matrix.

In the similar manner, the consistent mass matrix of an element can be derived and it may be expressed as

$$[M] = \rho t \int_{-1}^{1} \int_{-1}^{1} \left([N_w]^T [N_w] + \frac{h^2}{12} [N_{\theta x}]^T [N_{\theta x}] + \frac{h^2}{12} [N_{\theta y}]^T [N_{\theta y}] \right) |J| d\xi d\eta \tag{9}$$

The stiffness matrix and mass matrix are evaluated for all the elements and they are assembled together to form the overall stiffness matrix [K_0] and mass matrix [M_0]. Once [K_0] and [M_0] are obtained the equation of motion may be expressed as

$$[K_0] = \omega^2[M_0] \tag{10}$$

The boundary conditions used are:

Simply supported condition (denoted by S):

$w = \theta_x = 0$ – at boundary line parallel to x–axis.

$w = \theta_y = 0$ – at boundary line parallel to y–axis.

Clamped condition (denoted by C):

$$w = \theta_x = \theta_y = 0$$

Free boundary condition (denoted by F):

$$w \neq 0 \quad \theta_x \neq 0 \qquad \theta_y \neq 0$$

3. Results and discussions

3.1. *Convergence study*

The convergence and validation of the proposed finite element model are presented, taking various examples from literature. The problem under investigation is explained and the numerical results of the considered problem are discussed in this section. A thick square plate simply supported at all the edges is studied here for dynamic response. The 1st eight modes are studied here. Several mesh sizes are considered and convergence in calculated frequencies is observed. It is seen that the solution converges and is within less than 1% error of published literature at mesh size of 18 × 18. Hence mesh size of 18 × 18 is used throughout the paper.

Table 1 Convergence Study [SSSS Square plate, $h/a = 0.1$]

	Modes							
Source	1	2	3	4	5	6	7	8
RI (8*8)	19.0646	45.4871	45.4871	69.7696	85.1083	85.1086	106.6288	106.6290
RI (10*10)	19.0648	45.4845	45.4845	69.7842	85.0670	85.0670	106.6605	106.6609
RI (12*12)	19.0649	45.4836	45.4835	69.7894	85.0522	85.0524	106.6722	106.6735
RI (14*14)	19.0649	45.4832	45.4831	69.7916	85.0457	85.0462	106.6777	106.6792
RI (16*16)	**19.0650**	45.4830	45.4835	69.7929	85.0434	85.0426	106.6805	106.6845
RI (18*18)	19.0650	**45.4829**	**45.4829**	**69.7936**	**85.0408**	**85.0414**	**106.6837**	**106.6850**
RI (20*20)	19.0650	45.4829	45.4829	69.7936	85.0401	85.0427	106.6839	106.6851
Mindlin [20]	19.065	45.482	45.482	69.794	85.038	85.038	-	-

3.2. *Effect of rotary inertia*

A literature survey revealed that in most cases rotary inertia is not considered by researchers. Herein an attempt is made to observe the effect of rotary inertia on the vibration characteristics of a thick square plate simply supported at all the edges. Tab. 2 presents the comparison between values calculated by considering and without considering rotary inertia. The published results are in excellent sync with the present solution calculated by considering rotary inertia. The percent variation in frequencies calculated using and without using rotary inertia are presented here. It is seen that for higher mode numbers the variation in RI (rotary inertia) and WRI (without rotary inertia) is higher. Hence for thick plates the inclusion of rotary inertia in finite element formulation is very essential.

Table 2 Effect of Rotary inertia [SSSS Square plate, $h/a = 0.1$]

Source	Modes					
	1	2	3	4	5	6
RI (18*18)	19.065	45.483	45.483	69.794	85.041	85.041
WRI (18*18)	19.205	46.199	46.199	71.320	87.172	87.172
Mindlin [20]	19.065	45.482	45.482	69.794	85.038	85.038
% Variation (RI & WRI)	**0.730**	**1.550**	**1.549**	**2.140**	**2.444**	**2.444**

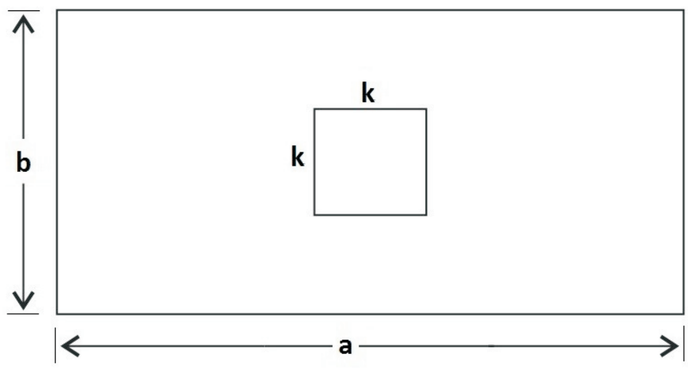


Figure 1 Rectangular plate with central cut–out

3.3. *Effect of aspect ratio*

Tab. 3 presents a study on the effect of aspect ratio on the free vibration frequency of simply supported thick rectangular plates. It is seen that as aspect ratio of the plates increases the natural frequency decreases. This is because with increase in area the mass of the plate increases which decreases the frequency.

Table 3 Effect of aspect ratio (b/a) [SSSS rectangular plate, $h/a = 0.1$]

Aspect Ratio b/a	Modes					
	1	2	3	4	5	6
0.2	188.011	203.598	204.615	217.380	230.915	265.281
0.5	45.483	69.794	106.683	133.623	152.610	152.612
0.8	24.203	50.187	63.890	86.894	89.205	120.828
1	19.065	45.483	45.483	69.794	85.041	85.041
1.5	13.898	26.145	40.764	45.484	51.968	69.792
2	12.067	19.065	30.361	39.095	45.482	45.486
2.5	11.216	15.728	23.091	33.097	38.320	42.434
3	10.752	13.898	19.064	26.145	35.007	37.898
5	10.077	11.216	13.104	15.725	19.061	23.089
10	9.791	10.076	10.551	11.212	12.060	13.090

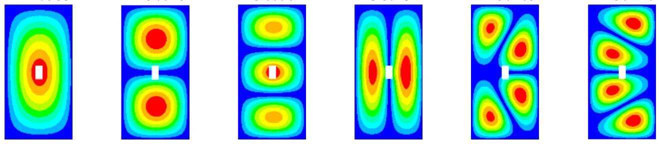
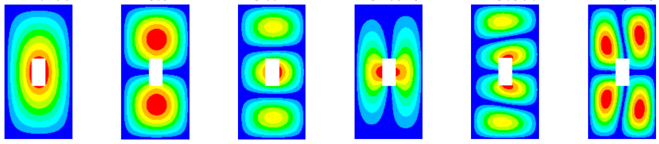
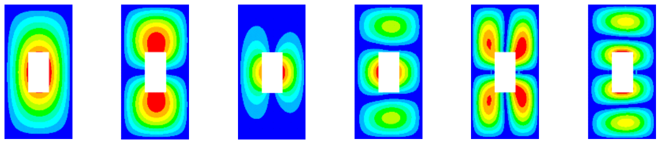
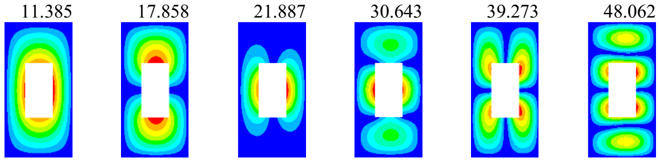
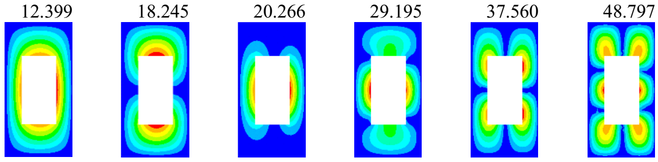
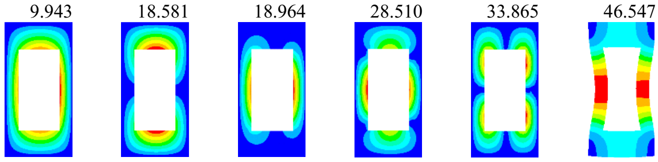
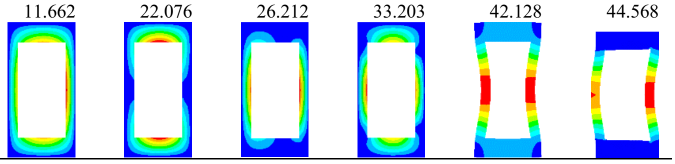
3.4. *Effect of boundary condition*

Tab. 4 presents the influence of boundary conditions on natural frequency a rectangular plate with aspect ratio (b/a) 2 and thickness ratio (h/a) 0.1. Here "S" means simply supported, "C" means clamped and "F" means free. The obtained results clearly show that frequency parameters increases if more constraints are included. For example SSSS have lower frequency than SSCS due to clamping in two sides in the later. This implies that higher constraints on the edges increases the flexural rigidity of the plate and results in higher frequency response. In all these cases results are presented by taking rotary inertia into account.

Table 4 Effect of boundary condition [$b/a = 2$, $h/a = 0.1$]

Boundary	Modes					
Condition	1	2	3	4	5	6
FFFC=FFCF	0.856	3.546	5.272	11.434	14.505	21.613
CFFF=FCFF	3.458	5.191	9.746	14.352	18.083	20.629
FSSS=SFSS	3.964	11.373	18.080	22.920	26.154	38.033
CFCF=FCFC	4.196	8.735	17.393	21.385	26.562	30.529
FFCC	5.394	8.533	14.542	19.234	25.688	27.763
SSFS=SSSF	10.087	14.315	22.529	34.684	37.116	41.236
SSSS	12.067	19.065	30.361	39.095	45.482	45.486
SSCS=SSSC	12.593	20.613	32.879	39.312	46.249	48.762
SSCC	13.272	22.388	35.586	39.560	47.102	52.158
SCSS=CSSS	16.564	22.388	32.679	46.743	47.106	52.149
CSCS=CSSC	16.955	23.724	35.027	46.922	50.269	52.818
SCCC=CSCC	17.467	25.282	37.575	47.131	53.395	53.734
CCFF	20.727	21.629	25.126	32.092	43.250	52.262
CCCF=CCFC	21.040	23.953	30.650	41.738	53.261	56.391
CCSS	22.112	26.668	35.647	49.114	54.652	59.210
CCCS=CCSC	22.404	27.792	37.807	52.154	54.805	59.796
CCCC	22.791	29.129	40.176	54.982	55.338	60.455

Table 5. Effect of size of central cutout [SSSS rectangular plate, b/a=2, h/a=0.1].

Cut-out Size (k * k)	Modes 1	2	3	4	5	6
0.1a*0.1b	11.683	19.019	30.071	38.232	45.109	45.140
						
0.2a*0.2b	11.177	18.711	30.424	32.625	43.800	44.216
						
0.3a*0.3b	11.025	18.168	25.839	31.154	42.097	44.437
						
0.4a*0.4b	11.385	17.858	21.887	30.643	39.273	48.062
						
0.5a*0.5b	12.399	18.245	20.266	29.195	37.560	48.797
						
0.6a*0.6b	9.943	18.581	18.964	28.510	33.865	46.547
						
0.7a*0.7b	11.662	22.076	26.212	33.203	42.128	44.568
						

3.5. *Effect of central cutout*

Fig. 1 shows a rectangular plate with a central cutout. Tab. 5 presents the influence of a central cutout on the frequency of a rectangular plate with aspect ratio (b/a) = 2 and thickness ratio (h/a) = 0.1. Fundamentally frequency is the square root of the ratio of the stiffness and mass. One may argue that the presence of larger cutout will mean larger decrease in the mass and hence the frequency should increase as cutout size increases. However it is not always so. This is due to the fact that the position and size of the cutout changes the mass as well as the flexural rigidity of the plate.

4. Conclusion

The problem of free vibration in thick rectangular plates is discussed in detail. The various parameters like aspect ratio, boundary condition, size of cutout and inclusion of rotary inertia etc. are discussed here with the help of numerical examples. The convergence study proves the accuracy of the formulation. An error of less than 1% is seen with the published results. The concept regarding incorporation of mass for rotary inertia is really elegant, which may be used in other elements. The order of accuracy in the present analysis and the variety of parameters considered clearly highlights the potential of the finite element formulation used here.

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