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# A Semi–Analytic First Order Jupiter–Saturn Planetary Theory Part II

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In this part we present the algebraic calculus computations related to the Hamiltonian equations of motion for the Jupiter – Saturn subsystem. Also we give a comment on the methods of the solution for the system of linear and nonlinear differential equations describing the motion of this subsystem.

*Keywords*: dynamics of the Solar system, Jupiter – Saturn planetary theory.

## 1. Methods and results

### 1.1. *Equations of motion*

For the J–S subsystem, the original first order Hamiltonian equations of motion, in terms of the H. Poincare‘ variables are given by:

$$\frac{dL_u}{dt} = \frac{\partial F_1}{\partial \lambda_u} \qquad \frac{d\lambda_u}{dt} = -\frac{\partial F_1}{\partial L_u} \qquad \frac{dH_u}{dt} = \frac{\partial F_1}{\partial K_u}$$
$$\frac{dK_u}{dt} = -\frac{\partial F_1}{\partial H_u} \qquad \frac{dP_u}{dt} = \frac{\partial F_1}{\partial Q_u} \qquad \frac{dQ_u}{dt} = -\frac{\partial F_1}{\partial P_u} \tag{1}$$

$L$, $\lambda$, $H$, $K$, $P$, $Q$ are the Poincare' variables. $u = 5$, 6 for Jupiter and Saturn.

$$L_u = \beta_u \sqrt{k^2 m_0 m_{0u} a_u} \qquad \lambda_u = l_u + \varpi_u$$

$$H_u = \sqrt{2L_u \left(1 - \sqrt{1 - e_u^2}\right)} \cos \varpi_u \tag{2}$$

$$K_u = -\sqrt{2L_u \left(1 - \sqrt{1 - e_u^2}\right)} \sin \varpi_u$$

$$P_u = \sqrt{2L_u \sqrt{1 - e_u^2}\, (1 - \cos i_u)}\, \cos \Omega_u$$

$$Q_u = -\sqrt{2L_u \sqrt{1 - e_u^2}\, (1 - \cos i_u)}\, \sin \Omega_u \tag{3}$$

$$F_1 = F_{1s} + F_{1p}$$

where:

$F_{1s}$ – first order secular Hamiltonian,

$F_{1p}$ – first order periodic Hamiltonian.

So we may cite:

$$\begin{aligned} \frac{d\,(L_u, H_u, P_u)}{dt} &= \frac{\partial\,(F_{1s} + F_{1p})}{\partial\,(\lambda_u, K_u, Q_u)} \\ \frac{d\,(\lambda_u, K_u, Q_u)}{dt} &= -\frac{\partial\,(F_{1s} + F_{1p})}{\partial\,(L_u, H_u, P_u)} \qquad (u = 5, 6) \end{aligned} \tag{4}$$

## 2. Algebraic calculus computations

We have:

$$\frac{dH_u}{dt} = \frac{\partial \psi_0}{\partial K_u} + \frac{\partial \psi_1}{\partial K_u} \cos(5\lambda_6 - 2\lambda_5) + \frac{\partial \psi_2}{\partial K_u} \sin(5\lambda_6 - 2\lambda_5) \tag{5}$$

$$\frac{\partial \psi_0}{\partial K_5} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_5} f_2^{(5,6)} K_5 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} K_6\right] \tag{6}$$

$$\frac{\partial \psi_0}{\partial K_6} = \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2} \left[\frac{1}{4L_6} f_2^{(5,6)} K_6 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} K_5\right] \tag{7}$$

$$\begin{aligned}\frac{\partial\psi_1}{\partial K_5} &= -U_1\left(6H_5K_5\right) - U_2\left(2H_6K_5+2H_5K_6\right) - U_3\left(2H_6K_6\right) - U_5\left(2P_5Q_5\right)\\ &\quad -U_6\left(2P_6Q_6\right) + U_7\left(P_5Q_6+P_6Q_5\right)\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial\psi_1}{\partial K_6} &= -U_2\left(2H_5K_5\right) - U_3\left(2H_6K_5+2H_5K_6\right) - U_4\left(6H_6K_6\right) - U_8\left(2P_5Q_5\right)\\ &\quad -U_9\left(2P_6Q_6\right) + U_{10}\left(P_5Q_6+P_6Q_5\right)\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial\psi_2}{\partial K_5} &= U_{11}\left(3K_5^2-3H_5^2\right) + U_{12}\left(2K_5K_6-2H_5H_6\right) + U_{13}\left(-H_6^2+K_6^2\right)\\ &\quad +U_{15}\left(-P_5^2+Q_5^2\right) + U_{16}\left(-P_6^2+Q_6^2\right) + U_{17}\left(P_5P_6-Q_5Q_6\right))\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial\psi_2}{\partial K_6} &= U_{12}\left(-H_5^2+K_5^2\right) + U_{13}\left(-2H_5H_6+2K_5K_6\right) + U_{14}\left(-3H_6^2+3K_6^2\right)\\ &\quad ,+U_{18}\left(-P_5^2+Q_5^2\right) + U_{19}\left(-P_6^2+Q_6^2\right) + U_{20}\left(P_5P_6-Q_5Q_6\right)\end{aligned} \tag{11}$$

whence:

$$\begin{aligned}\frac{dH_5}{dt} &= \frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[\frac{1}{4L_5}f_2^{(5,6)}K_5 + \frac{1}{4\sqrt{L_5L_6}}f_9^{(5,6)}K_6\right]\\ &\quad +\{-U_1\left(6H_5K_5\right) - U_2\left(2H_6K_5+2H_5K_6\right) - U_3\left(2H_6K_6\right)\\ &\quad -U_5\left(2P_5Q_5\right) - U_6\left(2P_6Q_6\right) + U_7\left(P_5Q_6+P_6Q_5\right)\}\cos(5\lambda_6-2\lambda_5)\\ &\quad +\{U_{11}\left(3K_5^2-3H_5^2\right) + U_{12}\left(2K_5K_6-2H_5H_6\right) + U_{13}\left(-H_6^2+K_6^2\right)\\ &\quad +U_{15}\left(-P_5^2+Q_5^2\right) + U_{16}\left(-P_6^2+Q_6^2\right)\\ &\quad +U_{17}\left(P_5P_6-Q_5Q_6\right)\}\sin(5\lambda_6-2\lambda_5)\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\frac{dH_6}{dt} &= \frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2}\left[\frac{1}{4L_6} f_2^{(5,6)} K_6 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} K_5\right] \\
&+\{-U_2\left(2H_5K_5\right) - U_3\left(2H_6K_5 + 2H_5K_6\right) - U_4\left(6H_6K_6\right) \\
&-U_8\left(2P_5Q_5\right) - U_9\left(2P_6Q_6\right) + U_{10}\left(P_5Q_6 + P_6Q_5\right)\}\cos(5\lambda_6 - 2\lambda_5) \\
&+\{U_{12}\left(-H_5^2 + K_5^2\right) + U_{13}\left(-2H_5H_6 + 2K_5K_6\right) \\
&+U_{14}\left(-3H_6^2 + 3K_6^2\right) + U_{18}\left(-P_5^2 + Q_5^2\right) + U_{19}\left(-P_6^2 + Q_6^2\right) \\
&+U_{20}\left(P_5P_6 - Q_5Q_6\right)\}\sin(5\lambda_6 - 2\lambda_5)
\end{aligned} \tag{13}$$

Neglecting terms of the third degree in *H, K, P, Q*.

Similarly:

$$\frac{dK_u}{dt} = -\frac{\partial\psi_0}{\partial H_u} - \frac{\partial\psi_1}{\partial H_u}\cos\left(5\lambda_6 - 2\lambda_5\right) - \frac{\partial\psi_2}{\partial H_u}\sin\left(5\lambda_6 - 2\lambda_5\right) \tag{14}$$

$$-\frac{\partial\psi_0}{\partial H_5} = -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2}\left[\frac{1}{4L_5} f_2^{(5,6)} H_5 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} H_6\right] \tag{15}$$

$$-\frac{\partial\psi_0}{\partial H_6} = -\frac{\sigma k^4 m_0 m_{06} \beta_5 \beta_6^3}{L_6^2}\left[\frac{1}{4L_6} f_2^{(5,6)} H_6 + \frac{1}{4\sqrt{L_5 L_6}} f_9^{(5,6)} H_5\right] \tag{16}$$

$$\begin{aligned}
-\frac{\partial\psi_1}{\partial H_5} &= U_1\left(3K_5^2 - 3H_5^2\right) + U_2\left(2K_5K_6 - 2H_5H_6\right) + U_3\left(-H_6^2 + K_6^2\right) \\
&+U_5\left(-P_5^2 + Q_5^2\right) + U_6\left(-P_6^2 + Q_6^2\right) + U_7\left(P_5P_6 - Q_5Q_6\right)
\end{aligned} \tag{17}$$

$$\begin{aligned}
-\frac{\partial\psi_1}{\partial H_6} &= U_2\ \left(K_5^2 - H_5^2\right) + U_3\left(2K_5K_6 - 2H_5H_6\right) + U_4\left(-3H_6^2 + 3K_6^2\right) \\
&+U_8\left(-P_5^2 + Q_5^2\right) + U_9\left(-P_6^2 + Q_6^2\right) + U_{10}\left(P_5P_6 - Q_5Q_6\right)
\end{aligned} \tag{18}$$

$$-\frac{\partial\psi_2}{\partial H_5} = U_{11}\left(6H_5K_5\right)+U_{12}\left(2H_6K_5+2H_5K_6\right)+U_{13}\left(2H_6K_6\right)+U_{15}\left(2P_5Q_5\right) +U_{16}\left(2P_6Q_6\right)-U_{17}\left(P_5Q_6+P_6Q_5\right) \tag{19}$$

$$-\frac{\partial\psi_2}{\partial H_6} = U_{12}\left(2H_5K_5\right)+U_{13}\left(2H_6K_5+2H_5K_6\right)+U_{14}\left(6H_6K_6\right)+U_{18}\left(2P_5Q_5\right) +U_{19}\left(2P_6Q_6\right)-U_{20}\left(P_5Q_6+P_6Q_5\right) \tag{20}$$

whence:

$$\begin{aligned}\frac{dK_5}{dt} = & -\frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[\frac{1}{4L_5}f_2^{(5,6)}H_5+\frac{1}{4\sqrt{L_5L_6}}f_9^{(5,6)}H_6\right] \\ & +\{U_1\left(3K_5^2-3H_5^2\right)+U_2\left(2K_5K_6-2H_5H_6\right)+U_3\left(-H_6^2+K_6^2\right) \\ & +U_5\left(-P_5^2+Q_5^2\right)+U_6\left(-P_6^2+Q_6^2\right)+U_7\left(P_5P_6-Q_5Q_6\right)\}\cos\left(5\lambda_6-2\lambda_5\right) \\ & +\{U_{11}\left(6H_5K_5\right)+U_{12}\left(2H_6K_5+2H_5K_6\right)+U_{13}\left(2H_6K_6\right) \\ & +U_{15}\left(2P_5Q_5\right)+U_{16}\left(2P_6Q_6\right)-U_{17}\left(P_5Q_6+P_6Q_5\right)\}\sin\left(5\lambda_6-2\lambda_5\right)\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{dK_6}{dt} = & -\frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[\frac{1}{4L_6}f_2^{(5,6)}H_6+\frac{1}{4\sqrt{L_5L_6}}f_9^{(5,6)}H_5\right] \\ & +\{U_2\left(-H_5^2+K_5^2\right)+U_3\left(-2H_5H_6+2K_5K_6\right)+U_4\left(-3H_6^2+3K_6^2\right) \\ & +U_8\left(-P_5^2+Q_5^2\right)+U_9\left(-P_6^2+Q_6^2\right)+U_{10}\left(P_5P_6-Q_5Q_6\right)\}\cos(5\lambda_6-2\lambda_5) \\ & +\{U_{12}\left(2H_5K_5\right)+U_{13}\left(2H_6K_5+2H_5K_6\right)+U_{14}\left(6H_6K_6\right) \\ & +U_{18}\left(2P_5Q_5\right)+U_{19}\left(2P_6Q_6\right)-U_{20}\left(P_5Q_6+P_6Q_5\right)\}\sin(5\lambda_6-2\lambda_5)\end{aligned} \tag{22}$$

We have also:

$$\frac{dP_u}{dt} = \frac{\partial\psi_0}{\partial Q_u}+\frac{\partial\psi_1}{\partial Q_u}\cos\left(5\lambda_6-2\lambda_5\right)+\frac{\partial\psi_2}{\partial Q_u}\sin\left(5\lambda_6-2\lambda_5\right) \tag{23}$$

$$\frac{\partial\psi_0}{\partial Q_5} = \frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[-\frac{1}{4L_5}f_3^{(5,6)}Q_5+\frac{1}{4\sqrt{L_5L_6}}f_3^{(5,6)}Q_6\right] \tag{24}$$

$$\frac{\partial\psi_0}{\partial Q_6} = \frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[-\frac{1}{4L_6}f_3^{(5,6)}Q_6+\frac{1}{4\sqrt{L_5L_6}}f_3^{(5,6)}Q_5\right] \tag{25}$$

$$\frac{\partial\psi_1}{\partial Q_5} = -U_5\left(2H_5Q_5+2K_5P_5\right)+U_7\left(K_5P_6+H_5Q_6\right)-U_8\left(2H_6Q_5+2K_6P_5\right) +U_{10}\left(K_6P_6+H_6Q_6\right) \tag{26}$$

$$\frac{\partial\psi_1}{\partial Q_6} = -U_6\left(2H_5Q_6+2K_5P_6\right)+U_7\left(K_5P_5+H_5Q_5\right)-U_9\left(2H_6Q_6+2K_6P_6\right) +U_{10}\left(K_6P_5+H_6Q_5\right) \tag{27}$$

$$\frac{\partial\psi_2}{\partial Q_5} = U_{15}\left(-2H_5P_5+2K_5Q_5\right)+U_{17}\left(-K_5Q_6+H_5P_6\right)+U_{18}\left(-2H_6P_5+2K_6Q_5\right) +U_{20}\left(-K_6Q_6+H_6P_6\right) \tag{28}$$

$$\frac{\partial\psi_2}{\partial Q_6} = U_{16}\left(-2H_5P_6+2K_5Q_6\right)+U_{17}\left(-K_5Q_5+H_5P_5\right)+U_{19}\left(-2H_6P_6+2K_6Q_6\right) +U_{20}\left(-K_6Q_5+H_6P_5\right) \tag{29}$$

whence:

$$\begin{aligned}\frac{dP_5}{dt} = {} & \frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[-\frac{1}{4L_5}f_3^{(5,6)}Q_5+\frac{1}{4\sqrt{L_5L_6}}f_3^{(5,6)}Q_6\right] \\ & +\{-U_5\left(2H_5Q_5+2K_5P_5\right)+U_7\left(K_5P_6+H_5Q_6\right) \\ & -U_8\left(2H_6Q_5+2K_6P_5\right)+U_{10}\left(K_6P_6+H_6Q_6\right)\}\cos\left(5\lambda_6-2\lambda_5\right) \\ & +\{U_{15}\left(-2H_5P_5+2K_5Q_5\right)+U_{17}\left(-K_5Q_6+H_5P_6\right) \\ & +U_{18}\left(-2H_6P_5+2K_6Q_5\right)+U_{20}\left(-K_6Q_6+H_6P_6\right)\}\sin(5\lambda_6-2\lambda_5)\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{dP_6}{dt} = {} & \frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[-\frac{1}{4L_6}f_3^{(5,6)}Q_6+\frac{1}{4\sqrt{L_5L_6}}f_3^{(5,6)}Q_5\right] \\ & +\{-U_6\left(2H_5Q_6+2K_5P_6\right)+U_7\left(K_5P_5+H_5Q_5\right) \\ & -U_9\left(2H_6Q_6+2K_6P_6\right)+U_{10}\left(K_6P_5+H_6Q_5\right)\}\cos\left(5\lambda_6-2\lambda_5\right) \\ & +\{U_{16}\left(-2H_5P_6+2K_5Q_6\right)+U_{17}\left(-K_5Q_5+H_5P_5\right) \\ & +U_{19}\left(-2H_6P_6+2K_6Q_6\right)+U_{20}\left(-K_6Q_5+H_6P_5\right)\}\sin(5\lambda_6-2\lambda_5)\end{aligned} \tag{31}$$

Similarly:

$$\frac{dQ_u}{dt} = -\frac{\partial\psi_0}{\partial P_u} - \frac{\partial\psi_1}{\partial P_u}\cos\left(5\lambda_6 - 2\lambda_5\right) - \frac{\partial\psi_2}{\partial P_u}\sin\left(5\lambda_6 - 2\lambda_5\right) \tag{32}$$

$$-\frac{\partial\psi_0}{\partial P_5} = -\frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[-\frac{1}{4L_5}f_3^{(5,6)}P_5 + \frac{1}{4\sqrt{L_5L_6}}f_3^{(5,6)}P_6\right] \tag{33}$$

$$-\frac{\partial\psi_0}{\partial P_6} = -\frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[-\frac{1}{4L_6}f_3^{(5,6)}P_6 + \frac{1}{4\sqrt{L_5L_6}}f_3^{(5,6)}P_5\right] \tag{34}$$

$$\begin{aligned} -\frac{\partial\psi_1}{\partial P_5} = {} & U_5\left(-2H_5P_5 + 2K_5Q_5\right) + U_7\left(H_5P_6 - K_5Q_6\right) + U_8\left(2K_6Q_5 - 2H_6P_5\right) \\ & + U_{10}\left(H_6P_6 - K_6Q_6\right) \end{aligned} \tag{35}$$

$$\begin{aligned} -\frac{\partial\psi_1}{\partial P_6} = {} & U_6\left(2K_5Q_6 - 2H_5P_6\right) + U_7\left(H_5P_5 - K_5Q_5\right) + U_9\left(2K_6Q_6 - 2H_6P_6\right) \\ & + U_{10}\left(H_6P_5 - K_6Q_5\right) \end{aligned} \tag{36}$$

$$\begin{aligned} -\frac{\partial\psi_2}{\partial P_5} = {} & U_{15}\left(2K_5P_5 + 2H_5Q_5\right) - U_{17}\left(K_5P_6 + H_5Q_6\right) + U_{18}\left(2H_6Q_5 + 2K_6P_5\right) \\ & - U_{20}\left(K_6P_6 + H_6Q_6\right) \end{aligned} \tag{37}$$

$$\begin{aligned} -\frac{\partial\psi_2}{\partial P_6} = {} & U_{16}\left(2K_5P_6 + 2H_5Q_6\right) - U_{17}\left(K_5P_5 + H_5Q_5\right) + U_{19}\left(2K_6P_6 + 2H_6Q_6\right) \\ & - U_{20}\left(K_6P_5 + H_6Q_5\right) \end{aligned} \tag{38}$$

whence:

$$\begin{aligned} \frac{dQ_5}{dt} = {} & -\frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[-\frac{1}{4L_5}f_3^{(5,6)}P_5 + \frac{1}{4\sqrt{L_5L_6}}f_3^{(5,6)}P_6\right] \\ & + \{U_5\left(2K_5Q_5 - 2H_5P_5\right) + U_7\left(H_5P_6 - K_5Q_6\right) \\ & + U_8\left(2K_6Q_5 - 2H_6P_5\right) + U_{10}\left(H_6P_6 - K_6Q_6\right)\}\cos\left(5\lambda_6 - 2\lambda_5\right) \\ & + \{U_{15}\left(2K_5P_5 + 2H_5Q_5\right) - U_{17}\left(K_5P_6 + H_5Q_6\right) \\ & + U_{18}\left(2K_6P_5 + 2H_6Q_5\right) - U_{20}\left(K_6P_6 + H_6Q_6\right)\}\sin(5\lambda_6 - 2\lambda_5) \end{aligned} \tag{39}$$

$$
\begin{aligned}
\frac{dQ_6}{dt} &= -\frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[-\frac{1}{4L_6}f_3^{(5,6)}P_6+\frac{1}{4\sqrt{L_5L_6}}f_3^{(5,6)}P_5\right] \\
&+\{U_6\left(2K_5Q_6-2H_5P_6\right)+U_7\left(H_5P_5-K_5Q_5\right) \\
&+U_9\left(2K_6Q_6-2H_6P_6\right)+U_{10}\left(H_6P_5-K_6Q_5\right)\}\cos\left(5\lambda_6-2\lambda_5\right) \\
&+\{U_{16}\left(2K_5P_6+2H_5Q_6\right)-U_{17}\left(K_5P_5+H_5Q_5\right) \\
&+U_{19}\left(2K_6P_6+2H_6Q_6\right)-U_{20}\left(K_6P_5+H_6Q_5\right)\}\sin(5\lambda_6-2\lambda_5)
\end{aligned} \tag{40}
$$

The $U$'s and the $f$'s are functions in $L_5$, $L_6$.

We have for the variables $L_u$, $\lambda_u$; $u = 5$, 6.

$$
\frac{dL_u}{dt} = \frac{\partial\psi_0}{\partial\lambda_u}+\frac{\partial}{\partial\lambda_u}\left[\{\psi_1\cos\left(5\lambda_6-2\lambda_5\right)+\psi_2\sin\left(5\lambda_6-2\lambda_5\right)\}\right] \tag{41}
$$

$$
\begin{aligned}
\frac{dL_5}{dt} &= \frac{2\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}[\{U_1(H_5^3-3H_5K_5^2)+U_2(H_6H_5^2-H_6K_5^2-2H_5K_5K_6) \\
&+U_3(H_5H_6^2-H_5K_6^2-2H_6K_5K_6)+U_4(H_6^3-3H_6K_6^2) \\
&+U_5\left(H_5P_5^2-H_5Q_5^2-2K_5P_5Q_5\right)+U_6\left(H_5P_6^2-H_5Q_6^2-2K_5P_6Q_6\right)] \\
&+U_7\left(K_5P_6Q_5+K_5P_5Q_6-H_5P_5P_6+H_5Q_5Q_6\right) \\
&+U_8\left(H_6P_5^2-H_6Q_5^2-2K_6P_5Q_5\right)+U_9\left(H_6P_6^2-H_6Q_6^2-2K_6P_6Q_6\right) \\
&+U_{10}\left(K_6P_6Q_5+K_6P_5Q_6-H_6P_5P_6+H_6Q_5Q_6\right)\}\sin\left(5\lambda_6-2\lambda_5\right) \qquad (42) \\
&-\{U_{11}\left(K_5^3-3K_5H_5^2\right)+U_{12}\left(-K_6H_5^2+K_6K_5^2-2H_5H_6K_5\right) \\
&+U_{13}\left(-K_5H_6^2+K_5K_6^2-2H_5H_6K_6\right)+U_{14}\left(K_6^3-3K_6H_6^2\right) \\
&+U_{15}\left(-K_5P_5^2+K_5Q_5^2-2H_5P_5Q_5\right)+U_{16}\left(K_5Q_6^2-K_5P_6^2-2H_5P_6Q_6\right) \\
&+U_{17}\left(K_5P_5P_6-K_5Q_5Q_6+H_5P_6Q_5+H_5P_5Q_6\right) \\
&+U_{18}\left(K_6Q_5^2-K_6P_5^2-2H_6P_5Q_5\right)+U_{19}\left(K_6Q_6^2-K_6P_6^2-2H_6P_6Q_6\right) \\
&+U_{20}\left(K_6P_5P_6-K_6Q_5Q_6+H_6Q_5P_6+H_6P_5Q_6\right)\}\cos\left(5\lambda_6-2\lambda_5\right)
\end{aligned}
$$

$$\begin{aligned}
\frac{dL_6}{dt} \quad = \quad & \frac{5\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}[-\{U_1\left(H_5^3-3H_5K_5^2\right) \\
& +U_2\left(H_6H_5^2-H_6K_5^2-2H_5K_5K_6\right) \\
& +U_3\left(H_5H_6^2-H_5K_6^2-2H_6K_5K_6\right)+U_4\left(H_6^3-3H_6K_6^2\right) \\
& +U_5\left(H_5P_5^2-H_5Q_5^2-2K_5P_5Q_5\right)+U_6\left(H_5P_6^2-H_5Q_6^2-2K_5P_6Q_6\right) \\
& +U_7\left(K_5P_6Q_5+K_5P_5Q_6-H_5P_5P_6+H_5Q_5Q_6\right) \\
& +U_8\left(H_6P_5^2-H_6Q_5^2-2K_6P_5Q_5\right) \\
& +U_9\left(H_6P_6^2-H_6Q_6^2-2K_6P_6Q_6\right) \\
& +U_{10}\left(K_6P_6Q_5+K_6P_5Q_6-H_6P_5P_6+H_6Q_5Q_6\right)\}\sin\left(5\lambda_6-2\lambda_5\right) \\
& +\{U_{11}\left(K_5^3-3K_5H_5^2\right)+U_{12}\left(-K_6H_5^2+K_6K_5^2-2H_5H_6K_5\right) \\
& +U_{13}\left(-K_5H_6^2+K_5K_6^2-2H_5H_6K_6\right)+U_{14}\left(K_6^3-3K_6H_6^2\right) \\
& +U_{15}\left(-K_5P_5^2+K_5Q_5^2-2H_5P_5Q_5\right) \\
& +U_{16}\left(K_5Q_6^2-K_5P_6^2-2H_5P_6Q_6\right) \\
& +U_{17}\left(K_5P_5P_6-K_5Q_5Q_6+H_5P_6Q_5+H_5P_5Q_6\right) \\
& +U_{18}\left(K_6Q_5^2-K_6P_5^2-2H_6P_5Q_5\right) \\
& +U_{19}\left(K_6Q_6^2-K_6P_6^2-2H_6P_6Q_6\right) \\
& +U_{20}\left(K_6P_5P_6-K_6Q_5Q_6+H_6Q_5P_6+H_6P_5Q_6\right)\}\cos\left(5\lambda_6-2\lambda_5\right)]
\end{aligned} \tag{43}$$

and

$$\frac{d\lambda_u}{dt}=-\frac{\partial\psi_0}{\partial L_u}\;-\;\frac{\partial\psi_1}{\partial L_u}\cos\left(5\lambda_6-2\lambda_5\right)-\frac{\partial\psi_2}{\partial L_u}\,\sin\left(5\lambda_6-2\lambda_5\right) \tag{44}$$

We find that:

$$
\begin{aligned}
\frac{\partial\psi_0}{\partial L_5} = & -\frac{k^4 m_0^2 m_{05}\beta_5^3}{L_5^3} + \frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[\frac{1}{2}\frac{\partial f_1}{\partial L_5} + \frac{\left(L_5\frac{\partial f_2}{\partial L_5} - f_2\right)}{8L_5^2}\left(H_5^2 + K_5^2\right)\right. \\
& -\frac{\left(L_5\frac{\partial f_3}{\partial L_5} - f_3\right)}{8L_5^2}\left(P_5^2 + Q_5^2\right) + \frac{1}{8L_6}\frac{\partial f_2}{\partial L_5}\left(H_6^2 + K_6^2\right) \\
& -\frac{1}{8L_6}\frac{\partial f_3}{\partial L_5}\left(P_6^2 + Q_6^2\right) + \frac{\left(2L_5\frac{\partial f_3}{\partial L_5} - f_3\right)}{8L_5^{3/2}L_6^{1/2}}\left(P_5P_6 + Q_5Q_6\right) \\
& \left.+\frac{\left(2L_5\frac{\partial f_9}{\partial L_5} - f_9\right)}{8L_5^{3/2}L_6^{1/2}}\left(H_5H_6 + K_5K_6\right)\right]
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\frac{\partial\psi_0}{\partial L_6} = & -\frac{k^4 m_0^2 m_{06}\beta_6^3}{L_6^3} - \frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^3}\left[f_1 + \frac{f_2}{4L_5}\left(H_5^2 + K_5^2\right)\right. \\
& -\frac{f_3}{4L_5}\left(P_5^2 + Q_5^2\right) + \frac{f_2}{4L_6}\left(H_6^2 + K_6^2\right) - \frac{f_3}{4L_6}\left(P_6^2 + Q_6^2\right) \\
& \left.+\frac{f_3}{2\left(L_5L_6\right)^{1/2}}\left(P_5P_6 + Q_5Q_6\right) + \frac{f_9}{2\left(L_5L_6\right)^{1/2}}\left(H_5H_6 + K_5K_6\right)\right] \\
& +\frac{\sigma k^4 m_0 m_{06}\beta_5\beta_6^3}{L_6^2}\left[\frac{1}{2}\frac{\partial f_1}{\partial L_6} + \frac{1}{8L_5}\frac{\partial f_2}{\partial L_6}\left(H_5^2 + K_5^2\right)\right. \\
& -\frac{1}{8L_5}\frac{\partial f_3}{\partial L_6}\left(P_5^2 + Q_5^2\right) + \frac{\left(L_6\frac{\partial f_2}{\partial L_6} - f_2\right)}{8L_6^2}\left(H_6^2 + K_6^2\right) \\
& -\frac{\left(L_6\frac{\partial f_3}{\partial L_6} - f_3\right)}{8L_6^2}\left(P_6^2 + Q_6^2\right) + \frac{\left(2L_6\frac{\partial f_3}{\partial L_6} - f_3\right)}{8L_5^{1/2}L_6^{3/2}}\left(P_5P_6 + Q_5Q_6\right) \\
& \left.+\frac{\left(2L_6\frac{\partial f_9}{\partial L_6} - f_9\right)}{8L_5^{1/2}L_6^{3/2}}\left(H_5H_6 + K_5K_6\right)\right]
\end{aligned} \tag{46}
$$

$$\begin{aligned}
\frac{\partial \psi_1}{\partial L_5} &= \frac{\partial U_1}{\partial L_5}\left(H_5^3 - 3H_5K_5^2\right) + \frac{\partial U_2}{\partial L_5}\left(H_6H_5^2 - H_6K_5^2 - 2K_6H_5K_5\right) \\
&+ \frac{\partial U_3}{\partial L_5}\left(H_5H_6^2 - H_5K_6^2 - 2K_5H_6K_6\right) + \frac{\partial U_4}{\partial L_5}\left(H_6^3 - 3H_6K_6^2\right) \\
&+ \frac{\partial U_5}{\partial L_5}\left(H_5P_5^2 - H_5Q_5^2 - 2K_5P_5Q_5\right) + \frac{\partial U_6}{\partial L_5}\left(H_5P_6^2 - H_5Q_6^2 - 2K_5P_6Q_6\right) \\
&+ \frac{\partial U_7}{\partial L_5}\left(K_5Q_5P_6 + K_5P_5Q_6 - H_5P_5P_6 + H_5Q_5Q_6\right) \qquad (47) \\
&+ \frac{\partial U_8}{\partial L_5}\left(H_6P_5^2 - H_6Q_5^2 - 2K_6P_5Q_5\right) + \frac{\partial U_9}{\partial L_5}\left(H_6P_6^2 - H_6Q_6^2 - 2K_6P_6Q_6\right) \\
&+ \frac{\partial U_{10}}{\partial L_5}\left(K_6Q_5P_6 + K_6P_5Q_6 - H_6P_5P_6 + H_6Q_5Q_6\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi_1}{\partial L_6} &= \frac{\partial U_1}{\partial L_6}\left(H_5^3 - 3H_5K_5^2\right) + \frac{\partial U_2}{\partial L_6}\left(H_6H_5^2 - H_6K_5^2 - 2K_6H_5K_5\right) \\
&+ \frac{\partial U_3}{\partial L_6}\left(H_5H_6^2 - H_5K_6^2 - 2K_5H_6K_6\right) + \frac{\partial U_4}{\partial L_6}\left(H_6^3 - 3H_6K_6^2\right) \\
&+ \frac{\partial U_5}{\partial L_6}\left(H_5P_5^2 - H_5Q_5^2 - 2K_5P_5Q_5\right) + \frac{\partial U_6}{\partial L_6}\left(H_5P_6^2 - H_5Q_6^2 - 2K_5P_6Q_6\right) \\
&+ \frac{\partial U_7}{\partial L_6}\left(K_5Q_5P_6 + K_5P_5Q_6 - H_5P_5P_6 + H_5Q_5Q_6\right) \qquad (48) \\
&+ \frac{\partial U_8}{\partial L_6}\left(H_6P_5^2 - H_6Q_5^2 - 2K_6P_5Q_5\right) + \frac{\partial U_9}{\partial L_6}\left(H_6P_6^2 - H_6Q_6^2 - 2K_6P_6Q_6\right) \\
&+ \frac{\partial U_{10}}{\partial L_6}\left(K_6Q_5P_6 + K_6P_5Q_6 - H_6P_5P_6 + H_6Q_5Q_6\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi_2}{\partial L_5} &= \frac{\partial U_{11}}{\partial L_5}\left(K_5^3 - 3K_5H_5^2\right) + \frac{\partial U_{12}}{\partial L_5}\left(K_6K_5^2 - K_6H_5^2 - 2H_6H_5K_5\right) \\
&+ \frac{\partial U_{13}}{\partial L_5}\left(K_5K_6^2 - K_5H_6^2 - 2H_5H_6K_6\right) + \frac{\partial U_{14}}{\partial L_5}\left(K_6^3 - 3K_6H_6^2\right) \\
&+ \frac{\partial U_{15}}{\partial L_5}\left(K_5Q_5^2 - K_5P_5^2 - 2H_5P_5Q_5\right) + \frac{\partial U_{16}}{\partial L_5}\left(K_5Q_6^2 - K_5P_6^2 - 2H_5P_6Q_6\right) \\
&+ \frac{\partial U_{17}}{\partial L_5}\left(K_5P_5P_6 - K_5Q_5Q_6 + H_5Q_5P_6 + H_5P_5Q_6\right) \qquad (49) \\
&+ \frac{\partial U_{18}}{\partial L_5}\left(K_6Q_5^2 - K_6P_5^2 - 2H_6P_5Q_5\right) + \frac{\partial U_{19}}{\partial L_5}\left(K_6Q_6^2 - K_6P_6^2 - 2H_6P_6Q_6\right) \\
&+ \frac{\partial U_{20}}{\partial L_5}\left(K_6P_5P_6 + H_6Q_5P_6 + H_6P_5Q_6 - K_6Q_5Q_6\right)
\end{aligned}$$

$$\begin{aligned}\frac{\partial\psi_2}{\partial L_6} &= \frac{\partial U_{11}}{\partial L_6}\left(K_5^3-3K_5H_5^2\right)+\frac{\partial U_{12}}{\partial L_6}\left(K_6K_5^2-K_6H_5^2-2H_6H_5K_5\right)\\
&+\frac{\partial U_{13}}{\partial L_6}\left(K_5K_6^2-K_5H_6^2-2H_5H_6K_6\right)+\frac{\partial U_{14}}{\partial L_6}\left(K_6^3-3K_6H_6^2\right)\\
&+\frac{\partial U_{15}}{\partial L_6}\left(K_5Q_5^2-K_5P_5^2-2H_5P_5Q_5\right)+\frac{\partial U_{16}}{\partial L_6}\left(K_5Q_6^2-K_5P_6^2-2H_5P_6Q_6\right)\\
&+\frac{\partial U_{17}}{\partial L_6}\left(K_5P_5P_6-K_5Q_5Q_6+H_5Q_5P_6+H_5P_5Q_6\right) \qquad (50)\\
&+\frac{\partial U_{18}}{\partial L_6}\left(K_6Q_5^2-K_6P_5^2-2H_6P_5Q_5\right)+\frac{\partial U_{19}}{\partial L_6}\left(K_6Q_6^2-K_6P_6^2-2H_6P_6Q_6\right)\\
&+\frac{\partial U_{20}}{\partial L_6}\left(K_6P_5P_6+H_6Q_5P_6+H_6P_5Q_6-K_6Q_5Q_6\right)\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{d\lambda_5}{dt} &= \frac{k^4m_0^2m_{05}\beta_5^3}{L_5^3}--\frac{\sigma k^4m_0m_{06}\beta_5\beta_6^3}{L_6^2}\left[\frac{1}{2}\frac{\partial f_1}{\partial L_5}+\frac{\left(L_5\frac{\partial f_2}{\partial L_5}-f_2\right)}{8L_5^2}\left(H_5^2+K_5^2\right)\right.\\
&-\frac{\left(L_5\frac{\partial f_3}{\partial L_5}-f_3\right)}{8L_5^2}\left(P_5^2+Q_5^2\right)+\frac{1}{8L_6}\frac{\partial f_2}{\partial L_5}\left(H_6^2+K_6^2\right)\\
&+-\frac{1}{8L_6}\frac{\partial f_3}{\partial L_5}\left(P_6^2+Q_6^2\right)+\frac{\left(2L_5\frac{\partial f_3}{\partial L_5}-f_3\right)}{8L_6^{1/2}L_5^{3/2}}\left(P_5P_6+Q_5Q_6\right)\\
&\left.+\frac{\left(2L_5\frac{\partial f_9}{\partial L_5}-f_9\right)}{8L_6^{1/2}L_5^{3/2}}\left(H_5H_6+K_5K_6\right)\right]\\
&-\left[\frac{\partial U_1}{\partial L_5}\left(H_5^3-3H_5K_5^2\right)+\frac{\partial U_2}{\partial L_5}\left(H_6H_5^2-H_6K_5^2-2K_6H_5K_5\right)\right.\\
&+\frac{\partial U_3}{\partial L_5}\left(H_5H_6^2-H_5K_6^2-2K_5H_6K_6\right)+\frac{\partial U_4}{\partial L_5}\left(H_6^3-3H_6K_6^2\right)\\
&+\frac{\partial U_5}{\partial L_5}\left(H_5P_5^2-H_5Q_5^2-2K_5P_5Q_5\right)+\frac{\partial U_6}{\partial L_5}\left(H_5P_6^2-H_5Q_6^2-2K_5P_6Q_6\right)\\
&+\frac{\partial U_7}{\partial L_5}\left(K_5Q_5P_6+K_5P_5Q_6-H_5P_5P_6+H_5Q_5Q_6\right) \qquad (51)\\
&+\frac{\partial U_8}{\partial L_5}\left(H_6P_5^2-H_6Q_5^2-2K_6P_5Q_5\right)+\frac{\partial U_9}{\partial L_5}\left(H_6P_6^2-H_6Q_6^2-2K_6P_6Q_6\right)\\
&\left.+\frac{\partial U_{10}}{\partial L_5}\left(K_6Q_5P_6+K_6P_5Q_6-H_6P_5P_6+H_6Q_5Q_6\right)\right]\cos\left(5\lambda_6-2\lambda_5\right)\end{aligned}$$

$$
\begin{aligned}
&-\left[\frac{\partial U_{11}}{\partial L_5}\left(K_5^3-3K_5H_5^2\right)+\frac{\partial U_{12}}{\partial L_5}\left(K_6K_5^2-K_6H_5^2-2H_6H_5K_5\right)\right.\\
&+\frac{\partial U_{13}}{\partial L_5}\left(K_5K_6^2-K_5H_6^2-2H_5H_6K_6\right)+\frac{\partial U_{14}}{\partial L_5}\left(K_6^3-3K_6H_6^2\right)\\
&+\frac{\partial U_{15}}{\partial L_5}\left(K_5Q_5^2-K_5P_5^2-2H_5P_5Q_5\right)+\frac{\partial U_{16}}{\partial L_5}\left(K_5Q_6^2-K_5P_6^2-2H_5P_6Q_6\right)\\
&+\frac{\partial U_{17}}{\partial L_5}\left(K_5P_5P_6-K_5Q_5Q_6+H_5Q_5P_6+H_5P_5Q_6\right)\\
&+\frac{\partial U_{18}}{\partial L_5}\left(K_6Q_5^2-K_6P_5^2-2H_6P_5Q_5\right)+\frac{\partial U_{19}}{\partial L_5}\left(K_6Q_6^2-K_6P_6^2-2H_6P_6Q_6\right)\\
&\left.+\frac{\partial U_{20}}{\partial L_5}\left(K_6P_5P_6+H_6Q_5P_6+H_6P_5Q_6-K_6Q_5Q_6\right)\right]\sin\left(5\lambda_6-2\lambda_5\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_6}{dt} \quad = \quad & \frac{k^4m_0^2m_{06}\beta_6^3}{L_6^3}+\frac{\sigma k^4m_0m_{06}\beta_5\beta_6^3}{L_6^3}\left[f_1+\frac{f_2}{4L_5}\left(H_5^2+K_5^2\right)-\frac{f_3}{4L_5}\left(P_5^2+Q_5^2\right)\right.\\
&+\frac{f_2}{4L_6}\left(H_6^2+K_6^2\right)-\frac{f_3}{4L_6}\left(P_6^2+Q_6^2\right)\\
&\left.+\frac{f_3}{2\left(L_5L_6\right)^{1/2}}\left(P_5P_6+Q_5Q_6\right)+\frac{f_9}{2\left(L_5L_6\right)^{1/2}}\left(H_5H_6+K_5K_6\right)\right]\\
&-\frac{\sigma k^4m_0m_{06}\beta_5\beta_6^3}{L_6^2}\left[\frac{1}{2}\frac{\partial f_1}{\partial L_6}+\frac{1}{8L_5}\frac{\partial f_2}{\partial L_6}\left(H_5^2+K_5^2\right)\right.\\
&-\frac{1}{8L_5}\frac{\partial f_3}{\partial L_6}\left(P_5^2+Q_5^2\right)+\frac{\left(L_6\frac{\partial f_2}{\partial L_6}-f_2\right)}{8L_6^2}\left(H_6^2+K_6^2\right)\\
&-\frac{\left(L_6\frac{\partial f_3}{\partial L_6}-f_3\right)}{8L_6^2}\left(P_6^2+Q_6^2\right)+\frac{\left(2L_6\frac{\partial f_3}{\partial L_6}-f_3\right)}{8L_5^{1/2}L_6^{3/2}}\left(P_5P_6+Q_5Q_6\right)\\
&\left.+\frac{\left(2L_6\frac{\partial f_9}{\partial L_6}-f_9\right)}{8L_5^{1/2}L_6^{3/2}}\left(H_5H_6+K_5K_6\right)\right] \qquad (52)\\
&-\left[\frac{\partial U_1}{\partial L_6}\left(H_5^3-3H_5K_5^2\right)+\frac{\partial U_2}{\partial L_6}\left(H_6H_5^2-H_6K_5^2-2K_6H_5K_5\right)\right.\\
&+\frac{\partial U_3}{\partial L_6}\left(H_5H_6^2-H_5K_6^2-2K_5H_6K_6\right)+\frac{\partial U_4}{\partial L_6}\left(H_6^3-3H_6K_6^2\right)\\
&+\frac{\partial U_5}{\partial L_6}\left(H_5P_5^2-H_5Q_5^2-2K_5P_5Q_5\right)+\frac{\partial U_6}{\partial L_6}\left(H_5P_6^2-H_5Q_6^2-2K_5P_6Q_6\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\partial U_7}{\partial L_6}\left(K_5Q_5P_6+K_5P_5Q_6-H_5P_5P_6+H_5Q_5Q_6\right)\\
&+\frac{\partial U_8}{\partial L_6}\left(H_6P_5^2-H_6Q_5^2-2K_6P_5Q_5\right)\\
&+\frac{\partial U_9}{\partial L_6}\left(H_6P_6^2-H_6Q_6^2-2K_6P_6Q_6\right)\\
&+\left.\frac{\partial U_{10}}{\partial L_6}\left(K_6Q_5P_6+K_6P_5Q_6-H_6P_5P_6+H_6Q_5Q_6\right)\right]\cos\left(5\lambda_6-2\lambda_5\right)\\
&-\left[\frac{\partial U_{11}}{\partial L_6}\left(K_5^3-3K_5H_5^2\right)+\frac{\partial U_{12}}{\partial L_6}\left(K_6K_5^2-K_6H_5^2-2H_6H_5K_5\right)\right.\\
&+\frac{\partial U_{13}}{\partial L_6}\left(K_5K_6^2-K_5H_6^2-2H_5H_6K_6\right)+\frac{\partial U_{14}}{\partial L_6}\left(K_6^3-3K_6H_6^2\right)\\
&+\frac{\partial U_{15}}{\partial L_6}\left(K_5Q_5^2-K_5P_5^2-2H_5P_5Q_5\right)+\frac{\partial U_{16}}{\partial L_6}\left(K_5Q_6^2-K_5P_6^2-2H_5P_6Q_6\right)\\
&+\frac{\partial U_{17}}{\partial L_6}\left(K_5P_5P_6-K_5Q_5Q_6+H_5Q_5P_6+H_5P_5Q_6\right)\\
&+\frac{\partial U_{18}}{\partial L_6}\left(K_6Q_5^2-K_6P_5^2-2H_6P_5Q_5\right)+\frac{\partial U_{19}}{\partial L_6}\left(K_6Q_6^2-K_6P_6^2-2H_6P_6Q_6\right)\\
&+\left.\frac{\partial U_{20}}{\partial L_6}\left(K_6P_5P_6+H_6Q_5P_6+H_6P_5Q_6-K_6Q_5Q_6\right)\right]\sin\left(5\lambda_6-2\lambda_5\right)
\end{aligned}
$$

The $f$'s and the $U$'s are functions in $L_5$, $L_6$.

## 3. Methods for the solution of the equations of motion

The system of differential equations of motion is reduced to the following secular linear set; when we neglect terms of degree higher than the second in the Poincare' variables $H_u$, $K_u$, $P_u$, $Q_u$, $u = 5, 6$.

$$
\begin{aligned}
\frac{dH_5}{dt}&=\rho_1K_5+\rho_2K_6 \qquad & \frac{dH_6}{dt}&=\rho_3K_5+\rho_4K_6\\
\frac{dK_5}{dt}&=\rho_5H_5+\rho_6H_6 \qquad & \frac{dK_6}{dt}&=\rho_7H_5+\rho_8H_6
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
\frac{dP_5}{dt}&=\rho_9Q_5+\rho_{10}Q_6 \qquad & \frac{dP_6}{dt}&=\rho_{11}Q_5+\rho_{12}Q_6\\
\frac{dQ_5}{dt}&=\rho_{13}P_5+\rho_{14}P_6 \qquad & \frac{dQ_6}{dt}&=\rho_{15}P_5+\rho_{16}P_6
\end{aligned}
\tag{54}
$$

The $\rho$'s are functions of $L_5$, $L_6$. Evidently there exists for this set (53), (54) the analytical, well known solution of Laplace – Lagrange for secular perturbations.

But when we take into consideration the non – linear terms of the R.H.S. of Eqs(4), we do not have a general analytical solution, it depends on the expression of the system. We can make some approximations to simplify the system of differential equations. We can from time to time find some solution or some approximation of the solution. By adding some hypothesis on the parameters of the system, we can have some solution as a series. There are numerical methods to resolve non linear systems (you can see it in many numerical analysis books). For every collection of numerical values of the parameters, we can search a numerical value of the solution, but we must be sure in advance that the solution is unique, near the particular values of the parameters, we have fixed. So, we can construct point by point the numerical solution. Alternatively, we may substitute the values of $H_u$, $K_u$, $P_u$, $Q_u$ obtained from the secular perturbation equations (53), (54) in the R.H.S. of Eqs. (12), (13), (21), (22), (30), (31), (39), (40), (51), (52), as an approximation, to get the values of d($H_{u,}$, $K_u$, $P_u$, $Q_u$)/dt and repeat the process several times till we acquire the same numerical results, namely the successive approximation method. Afterwards we integrate to find the values of $H_u$(t), $K_u$(t), $P_u$(t), $Q_u$(t) at any epoch. For the variables $\lambda_u$, we may use the formula $\lambda = n\,(t\,-\,t_0) +\, ta\,n^{-1}\left(\frac{-K}{H}\right)$, where $\lambda$ is the mean longitude, $t_0$ refer to the initial time and $n\,=\,\sqrt{\mu}\;a^{-3/2}$.

## References


[1] **Kamel, O. S. M.**: M.Sc. Thesis submitted to the Faculty of Sciences, Cairo University, Egypt, **1970**.

[2] **Kamel, O. S. M.**: Ph.D. Thesis submitted to the Faculty of Sciences, Cairo University, Egypt, **1973**.

[3] **Kamel, O. S. M. and Bakry, A.A.**: Astrophysics and Space Science 78, 3–26 **1981**.

[4] **Kamel, O. S. M.**: The Moon and the Planets , 26, 239–277, **1982**.

[5] **Brouwer, D. and Clemence, G. M.**: Methods of Celestial Mechanics, Academic Press, **1965**.

[6] **Murray, C. D. and Dermott, S. F.**: Solar System Dynamics, *Cambridge University Press*, **1999**.

[7] **Moursund, D. G. and Duris, C. S.**: Elementary Theory & Application of Numerical Analysis, McGraw – Hill, Inc., **1967**.

[8] **Braun, M.**: Differential Equations and Their Applications, Springer – Verlag, New York, Inc., **1983**.

[9] **Stark, P. A.**: Introduction to numerical methods, MacMillan Company, London, **1970**.

[10] **Kamel, O. K.**: A semi – analytic first order Jupiter – Saturn planetary theory. (Part I: outline), *Mechanics and Mechanical Engineering*, 20(1), **2016**.

[11] **El Mabsout, B.**: Private communication, **2012**.