Mechanics and Mechanical Engineering
Vol. 20, No. 4 (2016) 371–375

On the Expansion of the Direct Part of the Disturbing Planetary Function

Osman M. Kamel
Astronomy and Space Science Dept.,
Faculty of Sciences, Cairo University
Giza, Egypt
kamel_osman@yahoo.com

Adel S. Soliman
Theoretical Physics Dept.
National Research Center
Dokki, Giza, Egypt

Received (29 August 2016)
Revised (6 September 2016)
Accepted (11 September 2016)

We generalize the expansion of Murray–Dermott for the direct part of the disturbing function using Taylor‘s theorem. We present the values of Δ^{-s} for s = 1, 3, 5, ... which is essential for high order planetary theories. Murray – Dermott executed the expansion for s = 1 which is necessary for only first order theories.

Keywords: dynamics of the Solar System, planetary theory, Celestial mechanics.

1. Methods and Results

According to the approach of Murray–Dermott, we may write the following equalities:

$$\Delta^2 = (r^2 + r'^2 - 2rr' \cos\psi) \tag{1}$$

$$\Delta^{-s} = (r^2 + r'^2 - 2rr' \cos\psi)^{-s/2} \tag{2}$$

Let:

$$\Psi = \cos\psi - \cos(\theta - \theta') \tag{3}$$

Where: $\theta = \varpi + f$, $\theta' = \varpi' + f'$ are the true longitudes of inner and outer planet.

$$\text{i.e.} \quad \theta - \theta' = (\varpi + f) - (\theta' = \varpi' + f') \tag{4}$$

From Eqs. (2), (3):

$$\Delta^{-s} = [r^2 + r'^2 - 2rr'\{\cos(\psi - \psi') + \Psi\}]^{-s/2} \tag{5}$$

By putting $s = 1$:

$$\Delta^{-1} = [r^2 + r'^2 - 2rr'\{\cos(\psi - \psi') + \Psi\}]^{-1/2} \tag{6}$$

Then, after some algebraic calculations, we find by the application of the Binomial theorem:

$$\Delta^{-s} = \frac{1}{\Delta_0} + s(rr'\Psi)\left(\frac{1}{\Delta_0}\right)^{\frac{s+2}{s}} + s\left(\frac{s}{2}+1\right)(rr'\Psi)^2\left(\frac{1}{\Delta_0}\right)^{\frac{s+4}{s}} + ... \tag{7}$$

where:

$$\frac{1}{\Delta_0} = [r^2 + r'^2 - 2rr'\cos(\psi - \psi')]^{-s/2} \tag{8}$$

Let

$$\frac{1}{\rho_0} = [a^2 + a'^2 - 2aa'\cos(\psi - \psi')]^{-s/2} \tag{9}$$

From Eq. (7), put $s = 1$, we get:

$$\Delta^{-1} = \frac{1}{\triangle_0} + (rr'\Psi)\left(\frac{1}{\Delta_0}\right)^3 + \frac{3}{2}(rr'\Psi)^2\left(\frac{1}{\Delta_0}\right)^5 + ... \tag{10}$$

Where:

$$\frac{1}{\Delta_0} = [r^2 + r'^2 - 2rr'\cos(\psi - \psi')]^{-1/2} \tag{11}$$

$$\frac{1}{\rho_0} = [a^2 + a'^2 - 2aa'\cos(\psi - \psi')]^{-1/2} \tag{12}$$

i.e.

$$\frac{1}{\rho_0} = a'^{-1}[1 + \alpha^2 - 2\alpha\cos(\psi - \psi')]^{-1/2} \qquad \alpha = \frac{a}{a'} \tag{13}$$

Whence,

$$\left(\frac{1}{\rho_0}\right)^3 = a'^{-3}[1 + \alpha^2 - 2\alpha\cos(\psi - \psi')]^{-3/2} \tag{14}$$

$$\left(\frac{1}{\rho_0}\right)^5 = a'^{-5}[1 + \alpha^2 - 2\alpha\cos(\psi - \psi')]^{-5/2} \tag{15}$$

Applying Taylor's series expansion in ρ_0, we may write from (11), (12):

$$\frac{1}{\Delta_0}=\left(\frac{1}{\rho_0}\right)+(r-a)\frac{\partial}{\partial a}\left(\frac{1}{\rho_0}\right)+(r'-a')\frac{\partial}{\partial a'}\left(\frac{1}{\rho_0}\right)$$
$$+\left[(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}\right]^2\left(\frac{1}{\rho_0}\right)+... \tag{16}$$

...

$$\left(\frac{1}{\Delta_0}\right)^3=\left(\frac{1}{\rho_0}\right)^3+(r-a)\frac{\partial}{\partial a}\left(\frac{1}{\rho_0}\right)^3+(r'-a')\frac{\partial}{\partial a'}\left(\frac{1}{\rho_0}\right)^3$$
$$+\left[(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}\right]^2\left(\frac{1}{\rho_0}\right)^3+... \tag{17}$$

$$\left(\frac{1}{\Delta_0}\right)^5=\left(\frac{1}{\rho_0}\right)^5+(r-a)\frac{\partial}{\partial a}\left(\frac{1}{\rho_0}\right)^5+(r'-a')\frac{\partial}{\partial a'}\left(\frac{1}{\rho_0}\right)^5$$
$$+\left[(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}\right]^2\left(\frac{1}{\rho_0}\right)^5+... \tag{18}$$

Then, from Eqs. (10), (16), (17), (18):

$$\Delta^{-1}=\left[1+(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}+...\right]$$
$$\times\left[\frac{1}{\rho_0}+rr'\Psi\left(\frac{1}{\rho_0}\right)^3+\frac{3}{2}(rr'\Psi)^2\left(\frac{1}{\rho_0}\right)^5+...\right] \tag{19}$$

From Eqs. (13), (14), (15) and the definition of Laplace coefficients:

$$\Delta^{-1}=\left[1+(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}+\left\{(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}\right\}^2+...\right]$$
$$\times\left[a'^{-1}\left\{\frac{1}{2}\sum_j b_{1/2}^{(j)}(\alpha)\cos j(\theta-\theta')\right\}\right.$$
$$+(rr'\Psi)a'^{-3}\left\{\frac{1}{2}\sum_j b_{3/2}^{(j)}(\alpha)\cos j(\theta-\theta')\right\}$$
$$\left.+\frac{3}{2}(rr'\Psi)a'^{-5}\left\{\frac{1}{2}\sum_j b_{5/2}^{(j)}(\alpha)\cos j(\theta-\theta')\right\}+...\right] \tag{20}$$

From Eq. (7), set $s = 3,\, 5$, we get:

$$\Delta^{-3} = \frac{1}{\Delta_0} + 3(rr'\Psi)\left(\frac{1}{\Delta_0}\right)^{5/3} + \frac{15}{2}(rr'\Psi)^2\left(\frac{1}{\Delta_0}\right)^{7/3} + ... \tag{21}$$

$$\Delta^{-5} = \frac{1}{\Delta_0} + 5(rr'\Psi)\left(\frac{1}{\Delta_0}\right)^{7/5} + \frac{35}{2}(rr'\Psi)^2\left(\frac{1}{\Delta_0}\right)^{9/5} + ... \tag{22}$$

By exactly the same procedure, we find Δ^{-3}, Δ^{-5}, Δ^{-7}, ... which is necessary for expansions of high order theories.

$$\begin{aligned}\Delta^{-3} = &\left[1 + (r-a)\frac{\partial}{\partial a} + (r'-a')\frac{\partial}{\partial a'} + \left\{(r-a)\frac{\partial}{\partial a} + (r'-a')\frac{\partial}{\partial a'}\right\}^2 + ...\right]\\ &\times \left[a'^{-3}\left\{\frac{1}{2}\sum_j b^{(j)}_{3/2}(\alpha)\cos j(\theta-\theta')\right\}\right.\\ &+3(rr'\Psi)a'^{-5}\left\{\frac{1}{2}\sum_j b^{(j)}_{5/2}(\alpha)\cos j(\theta-\theta')\right\}\\ &\left.+\frac{15}{2}(rr'\Psi)a'^{-7}\left\{\frac{1}{2}\sum_j b^{(j)}_{7/2}(\alpha)\cos j(\theta-\theta')\right\} + ...\right]\end{aligned} \tag{23}$$

and

$$\begin{aligned}\Delta^{-5} = &\left[1 + (r-a)\frac{\partial}{\partial a} + (r'-a')\frac{\partial}{\partial a'} + \left\{(r-a)\frac{\partial}{\partial a} + (r'-a')\frac{\partial}{\partial a'}\right\}^2 + ...\right]\\ &\times \left[a'^{-5}\left\{\frac{1}{2}\sum_j b^{(j)}_{5/2}(\alpha)\cos j(\theta-\theta')\right\}\right.\\ &+5(rr'\Psi)a'^{-7}\left\{\frac{1}{2}\sum_j b^{(j)}_{7/2}(\alpha)\cos j(\theta-\theta')\right\}\\ &\left.+\frac{35}{2}(rr'\Psi)a'^{-9}\left\{\frac{1}{2}\sum_j b^{(j)}_{9/2}(\alpha)\cos j(\theta-\theta')\right\} + ...\right]\end{aligned} \tag{24}$$

For the construction of an analytical order by order planetary theory, we must have at our disposal the expansions of Δ^{-3}, Δ^{-5}, ... i.e. the mutual distance of the two planets raised to any real negative odd integer. The literal expansion of where $s = 1,\, 3,\, 5, ...$ is acquired from:

1. The expansion of $(r - a)$ in terms of the mean anomaly up to the desired power of e, $\gamma = \sin\frac{I}{2}$.

2. The expansion of $\cos j(\theta - \theta')$ and Ψ.

3. The expression of the Laplacian coefficient in terms of α.

4. The conversion of the partial derivatives w,r.t. a and a‘ to be w.r.t. $\alpha + \frac{a}{a'}$.

References

[1] **Brouwer, D. and Clemence, G, M.**: Methods of Celestial Mechanics, *Academic Press*, XV, **1965**.

[2] **Abu–El–Ata, N. and Chapront, J.**: Astronomy and Astrophysics, 38, 57, **1975**.

[3] **Kamel, O. M. and Bakry, A. A.**: Astrophysics and Space science, 78, 3–26, **1981**.

[4] **Kamel, O. M.**: The Moon and the Planets, 26, 239–277, **1982**.

[5] **Murray, C. D. and Dermott, S. F.**: Solar System Dynamics, *Cambridge University Press*, 6, **1999**.