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Effect of Hall Current and Thermal Relaxation Time on Thermoelastic Materials with Double Porosity Structure by Using State Space Approach

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The present investigation is concerned with one dimensional problem in a homogeneous, isotropic thermoelastic medium with double porosity in the presence of Hall current subjected to thermomechanical sources. Thermoelastic theory with one relaxation time developed by Lord-Shulman [2] has been used to solve the problem. A state space approach has been applied to investigate the problem. As an application of the approach, normal force and thermal source have been taken to illustrate the utility of the approach. The expressions for the components of normal stress, equilibrated stress and the temperature change are obtained in the frequency domain and computed numerically. Numerical simulation is prepared for these quantities. The effect of Hall current and thermal relaxation time are depicted graphically on the resulting quantities for a specific model. Some particular cases of interest are also deduced from the present investigation.

Keywords: hall current, double porosity, thermoelasticity, state space approach, thermomechanical sources.

1. Introduction

The constitutive equations for thermoelastic material, which express the relations between the stress, the strain and the temperature change, were first introduced by Biot [1]. With Biot's theory, many solutions for thermal response caused by the change of temperature have been developed by numerous investigators. However, it involves a paradox that the thermal disturbances propagate at infinite speeds.

In recent years increasing attention has been made to remove this paradox and to develop the generalized theory of thermoelasticity, which was found to give more realistic results than the coupled or uncoupled theories of thermoelasticity, especially when short time effects or step temperature gradients are considered. The theory of generalized thermoelasticity with one relaxation time was first introduced by Lord and Shulman [2], who obtained a wave-type heat equation by postulating a new law of heat conduction instead of the classical Fourier's law. Hetnarski and Ignaczak [3] has presented a review on the generalized theories of thermoelasticity. A comprehensive work has been done in the generalized theories of thermoelasticity with one relaxation time by different investigators by considering different problems.

Porous media theories play an important role in many branches of engineering including material science, the petroleum industry, chemical engineering, biomechanics and other such fields of engineering. Representation of a fluid saturated porous medium as a single phase material has been virtually discarded. The material with the pore spaces such as concrete can be treated easily because all concrete ingredients have the same motion if the concrete body is deformed. However the situation is more complicated if the pores are filled with liquid and in that case the solid and liquid phases have different motions. Due to these different motions, the different material properties and the complicated geometry of pore structures, the mechanical behavior of a fluid saturated porous thermoelastic medium becomes very difficult. So researchers from time to time, have tried to overcome this difficulty and we see many porous media in the literature. A brief historical background of these theories is given by de Boer [4, 5].

As far as modern era is concerned Biot [6] proposed a general theory of three-dimensional deformation of fluid saturated porous salts. Biot theory is based on the assumption of compressible constituents and till recently, some of his results have been taken as standard references and basis for subsequent analysis in acoustic, geophysics and other such fields. Another interesting theory is given by Bowen [7], de Boer and Ehlers [8] in which all the constituents of a porous medium are assumed to be incompressible. The fluid saturated porous material is modeled as a two phase system composed of an incompressible solid phase and incompressible fluid phase, thus meeting the many problems in engineering practice, e.g. in soil mechanics.One important generalization of Biot's theory of poroelasticity that has been studied extensively started with the works by Barenblatt et al. [9], where the double porosity model was first proposed to express the fluid flow in hydrocarbon reservoirs and aquifers.

The double porosity model represents a new possibility for the study of important problems concerning the civil engineering. It is well–known that, under supersaturation conditions due to water of other fluid effects, the so called neutral pressures generate unbearable stress states on the solid matrix and on the fracture faces, with severe (sometimes disastrous) instability effects like landslides, rock fall or soil fluidization (typical phenomenon connected with propagation of seismic waves). In such a context it seems possible, acting suitably on the boundary pressure state, to regulate the internal pressures in order to deactivate the noxious effects related to neutral pressures; finally, a further but connected positive effect could be lightening of the solid matrix/fluid system.

Wilson and Aifanits [10] presented the theory of consolidation with the double porosity. Khaled, Beskos and Aifantis [11] employed a finite element method to consider the numerical solutions of the differential equation of the theory of consolidation with double porosity developed by Aifantis[10]. Wilson and Aifantis [12] discussed the propagation of acoustics waves in a fluid saturated porous medium. The propagation of acoustic waves in a fluid–saturated porous medium containing a continuously distributed system of fractures is discussed. The porous medium is assumed to consist of two degrees of porosity and the resulting model thus yields three types of longitudinal waves, one associated with the elastic properties of the matrix material and one each for the fluids in the pore space and the fracture space.

Beskos and Aifantis [13] presented the theory of consolidation with double porosity–II and obtained the analytical solutions to two boundary value problems. Khalili and Valliappan [14] studied the unified theory of flow and deformation in double porous media. Aifantis [15–19] introduced a multi–porous system and studied the mechanics of diffusion in solids. Moutsopoulos et al. [20] obtained the numerical simulation of transport phenomena by using the double porosity/ diffusivity continuum model. Khalili and Selvadurai [21] presented a fully coupled constitutive model for thermo–hydro–mechanical analysis in elastic media with double porosity structure. Pride and Berryman [22] studied the linear dynamics of double–porosity dual–permeability materials. Straughan [23] studied the stability and uniqueness in double porous elastic media.

Svanadze [24–28] investigated some problems on elastic solids, viscoelastic solids and thermoelastic solids with double porosity. Scarpetta et al. [29, 30] proved the uniqueness theorems in the theory of thermoelasticity for solids with double porosity and also obtained the fundamental solutions in the theory of thermoelasticity for solids with double porosity.

In recent years the state space description of linear systems has been used extensively in various areas of engineering, such as the analysis of control systems. The state space approach offers an attractive way to avoid the difficulties of the traditional linear model approach. The state–space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations. To abstract from the number of inputs, outputs and states, the variables are expressed as vectors. If the dynamical system is linear and time invariant, the differential and algebraic equations may be written in matrix form. The state-space representation provides a convenient and compact way to model and analyze systems with multiple inputs and outputs.

Bahar and Hetnarski investigated good number of problems in thermoelasticity by using state space approach [31–36]. Also Ezzat et.al. [37], Maghraby et al. [38], Youssef and Al–Lehaibi [39], Othman [40], Elisbai and Youseff [41] and Sherief and El–sayed [42] investigated different types of problems in different media by using state space approach

The foundations of magnetoelasticity were presented by Knopoff [43] and Chadwick[44] and developed by Kaliski and Petykiewicz[45]. An interesting attention is being devoted to the interaction between magnetic field and strain field in a thermoelastic solid due to its many applications in the fields of geophysics, plasma physics and related topics. In all papers quoted it was assumed that the interaction s between the two fields take place by means of the Lorentz forces appearing in the equations of motion and by means of a term entering Ohm's law and describing the electric field produced by the velocity of a material particle, moving in a magnetic field.

When the magnetic field is very strong, the conductivity will be a tensor and the effect of Hall current cannot be neglected. The conductivity normal to the magnetic field is reduced due to the free spiraling of electrons and ions about the magnetic lines of force before suffering collisions and a current is induced in a direction normal to both the electric and magnetic fields. This phenomenon is called the Hall effect. Authors like Sarkar and Lahiri [46], Salem [47], Zakaria [48–50], Attia [51] have considered the effect of Hall currents for two dimensional problems in micropolar thermoelasticity.

In the present paper, we formulate the state space approach to boundary value problem for thermoelastic material with double porosity structure in the presence of of Hall current with one relaxation time subjected to thermomechanical sources. The expressions for normal stress, equilibrated stresses and temperature distribution are obtained in closed form, computed numerically and represented graphically for normal force and thermal source. The effect of Hall currents and thermal relaxation time are shown graphically for the resulting quantities.

2. Basic equations

Following Iesan and Quintanilla [52], Lord and Shulman [2], the field equations and the constitutive relations for homogeneous thermoelastic material with double porosity structure, when the Hall current is taken into account, can be written as:

2.1. *Equation of motion*

$$\mu\Delta u_i + (\lambda+\mu)\, u_{j,ji} + b\varphi_{,i} + d\psi_{,i} - \beta T_{,i} + F_i = \rho\ddot{u}_i \tag{1}$$

2.2. *Equilibrated stress equations of motion*

$$\alpha\Delta\varphi + b_1\Delta\psi - bu_{r,r} - \alpha_1\varphi - \alpha_3\psi + \gamma_1 T = \kappa_1\ddot{\varphi} \tag{2}$$

$$b_1\Delta\varphi + \gamma\Delta\psi - du_{r,r} - \alpha_3\varphi - \alpha_2\psi + \gamma_2 T = \kappa_2\ddot{\psi} \tag{3}$$

2.3. *Equation of heat conduction*

$$(1+\tau_0\frac{\partial}{\partial t})[\rho C^*\dot{T} + \beta T_0\dot{e}_{ii} + \gamma_1 T_0\dot{\phi} + \gamma_2 T_0\dot{\psi}] = K^*\Delta \mathrm{T} \tag{4}$$

2.4. *Constitutive relations*

$$t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\delta_{ij}\varphi + d\delta_{ij}\psi - \beta\delta_{ij}T \tag{5}$$

$$\sigma_i = \alpha\varphi_{,i} + b_1\psi_{,i} \tag{6}$$

$$\zeta_i = b_1\varphi_{,i} + \gamma\psi_{,i} \tag{7}$$

The generalized Ohm's law including Hall current:

$$J_i = \sigma_0 \left(E_i + \mu_0 \varepsilon_{ijr} u_{j,t} H_r - \frac{\mu_0}{e n_e} \varepsilon_{ijr} J_j H_r \right) \tag{8}$$

where $F_i = \mu_0 \varepsilon_{ijr} J_j H_r$ is the Lorentz force; $\sigma_0 (= n_e e^2 t_e / m_e)$ is the electrical conductivity; μ_0 is the magnetic permeability; e is the charge of an electron; n_e is the number density of electrons; t_e is the electron collision time; m_e is the electron mass; E_i is the intensity tensor of the electric field; λ and μ are Lame's constants; ρ is the mass density; $\beta = (3\lambda + 2\mu)\,\alpha_t$; α_t is the coefficient of linear thermal expansion; C^* is the specific heat at constant strain; u_i are the displacement components; t_{ij} is the stress tensor; ε_{ijr} is the permutation symbol; μ_0 is the magnetic permeability; J_r is the conduction current density ; κ_1 and κ_2 are coefficients of equilibrated inertia; ν_1 is the volume fraction field corresponding to pores and ν_2 is the volume fraction field corresponding to fissures ; φ and ψ are the volume fraction fields corresponding to ν_1 and ν_2 respectively; σ_1 is the equilibrated stress corresponding to ν_1; ζ_1 is the equilibrated stress corresponding to ν_2, K^* is the coefficient of thermal conductivity; τ_0 is the relaxation time and b, d, b_1, γ, γ_1, γ_2 are constitutive coefficients; δ_{ij} is the Kronecker's delta; T is the temperature change measured form the absolute temperature $T_0\,(T_0 \neq 0)$, a superposed dot represents differentiation with respect to time variable t.

$$\nabla = \hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3} \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

are the gradient and Laplacian operators, respectively.

3. Formulation and solution of the problem

We consider a homogeneous, isotropic, perfectly conducting thermoelastic solid with double porosity occupying the region $0 \le x < \infty$. For one dimensional problem, we take $u(x_1, t)$, $\varphi(x_1, t)$, $\psi(x_1, t)$, $T(x_1, t)$. A uniform very strong magnetic field of strength H_0 is assumed to be applied in the positive y - direction and we also assume that $E = 0$. Under these assumptions, the generalized Ohm's law gives $J_1 = J_2 = 0$ everywhere in the medium.
The current density components J_3 is given by:

$$J_3 = \frac{\sigma_0 \mu_0 H_0}{1 + m^2} \left(\frac{\partial u}{\partial t} \right) \tag{9}$$

where $m = \omega_e t_e$ is the Hall parameter and $\omega_e = e\mu_0 H_0 / m_e$ is the electron frequency.
Let us introduce the following non–dimensional variables:

$$\begin{aligned}
&x_1' = \frac{\omega_1}{c_1} x_1 \qquad u' = \frac{\omega_1}{c_1} u \qquad t_{ij}' = \frac{t_{ij}}{\beta t_0} \qquad M = \frac{\sigma_0 \mu_0^2 H_0^2}{\rho \omega_1} \\
&\tau_0' = \omega_1 \tau_0 \qquad \varphi' = \frac{k_1 \omega_1^2}{\alpha_1} \varphi \qquad \psi' = \frac{k_1 \omega_1^2}{\alpha_1} \qquad T' = \frac{T}{T_0} \\
&t' = \omega_1 t \qquad \sigma_i' = \left(\frac{c_1}{\alpha \omega_1} \right) \sigma_i \qquad \zeta_i' = \left(\frac{c_1}{\alpha \omega_1} \right) \zeta_i
\end{aligned} \tag{10}$$

where $c_1^2=\frac{\lambda+2\mu}{\rho}$, $\omega_1=\frac{\rho C^* c_1^2}{K^*}$ and M is the Hartmann number or magnetic parameter.
Making use of dimensionless quantities given in (10) in equations (1)–(4) (dropping primes for convenience), we get:

$$\frac{\partial^2 u}{\partial x_1{}^2}+\delta_1\frac{\partial\varphi}{\partial x_1}+\delta_2\frac{\partial\psi}{\partial x_1}-\delta_3\frac{\partial T}{\partial x_1}-\left(\frac{M}{1+m^2}\right)\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial t^2} \tag{11}$$

$$\delta_4\frac{\partial^2\varphi}{\partial x_1{}^2}+\delta_5\frac{\partial^2\psi}{\partial x_1{}^2}-\delta_6\frac{\partial u_1}{\partial x_1}-\delta_7\varphi-\delta_8\psi+\delta_9 T=\frac{\partial^2\varphi}{\partial t^2} \tag{12}$$

$$\delta_{10}\frac{\partial^2\varphi}{\partial x_1{}^2}+\delta_{11}\frac{\partial^2\psi}{\partial x_1{}^2}-\delta_{12}\frac{\partial u}{\partial x_1}-\delta_{13}\varphi-\delta_{14}\psi+\delta_{15}T=\frac{\partial^2\psi}{\partial t^2} \tag{13}$$

$$\tau_{11}\left[\delta_{16}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_1}\right)+\delta_{17}\frac{\partial\varphi}{\partial t}+\delta_{18}\frac{\partial\psi}{\partial t}+\frac{\partial T}{\partial t}\right]=\frac{\partial^2 T}{\partial x_1{}^2} \tag{14}$$

where

$$\begin{aligned}
&\delta_1=\frac{b\alpha_1}{\rho_1^2 k_1\omega_1^2} \quad \delta_2=\frac{d\alpha_1}{\rho C_1^2 k_1\omega_1{}^2} \quad \delta_3=\frac{\beta T_0}{\rho C_1^2} \quad \delta_4=\frac{\alpha}{C_1^2 k_1}\\
&\delta_5=\frac{b_1}{C_1^2 k_1} \quad \delta_6=\frac{b}{\alpha_1} \quad \delta_7=\frac{\alpha_1}{k_1\omega_1{}^2} \quad \delta_8=\frac{\alpha_3}{k_1\omega_1{}^2}\\
&\delta_9=\frac{\gamma_1 T_0}{\alpha_1} \quad \delta_{10}=\frac{b_1}{C_1^2 k_2} \quad \delta_{11}=\frac{\gamma}{C_1^2 k_2} \quad \delta_{12}=\frac{dk_1}{\alpha_1 k_2}\\
&\delta_{13}=\frac{\alpha_3}{k_2\omega_1{}^2} \quad \delta_{14}=\frac{\alpha_2}{k_2\omega_1{}^2} \quad \delta_{15}=\frac{\gamma_2 T_0 k_1}{\alpha_1 k_2} \quad \delta_{16}=\frac{\beta C_1^2}{K^*\omega_1}\\
&\delta_{17}=\frac{\gamma_1\alpha_1 C_1^2}{K^* k_1\omega_1{}^3} \quad \delta_{18}=\frac{\gamma_2\alpha_1 C_1^2}{K^* k_1\omega_1{}^3} \quad \tau_{11}=(1+\tau_0\frac{\partial}{\partial t})
\end{aligned} \tag{15}$$

Assuming the time harmonic solution of the equations (11)–(14) as,

$$(u\,(x,t)\,,\varphi\,(x,t)\,,\psi\,(x,t)\,,T\,(x,t))=(\overline{u},\,\overline{\varphi,}\overline{\psi},\,\overline{T})e^{-i\omega t} \tag{16}$$

where ω is the frequency.
Equations (11)–(14) with the aid of equation (16) yield,

$$\overline{u}_{,11}=N_1\overline{u}+N_2\overline{\phi}_{,1}+N_3\overline{\psi}_{,1}+N_4\overline{T}_{,1} \tag{17}$$

$$\overline{\phi}_{,11}=N_5\overline{u}_{,1}+N_6\overline{\phi}+N_7\overline{\psi}+N_8\overline{T} \tag{18}$$

$$\overline{\psi}_{,11}=N_9\overline{u}_{,1}+N_{10}\overline{\phi}+N_{11}\overline{\psi}+N_{12}\overline{T} \tag{19}$$

$$\overline{T}_{,11}=N_{13}\overline{u}_{,1}+N_{14}\overline{\phi}+N_{15}\overline{\psi}+N_{16}\overline{T} \tag{20}$$

where

$$N_1=-i\omega\left(\frac{M}{1+m^2}\right)-\omega^2 \quad N_2=-\delta_1 \quad N_3=-\delta_2 \quad N_4=\delta_3, M_1=\frac{-\delta_5}{\delta_4}$$

$$M_2=\frac{\delta_6}{\delta_4} \quad M_3=\frac{\delta_7-\omega^2}{\delta_4} \quad M_4=\frac{\delta_8}{\delta_4} \quad M_5=\frac{-\delta_9}{\delta_4}$$

$$M_6 = \frac{-\delta_{10}}{\delta_{11}} \quad M_7 = \frac{\delta_{12}}{\delta_{11}} \quad M_8 = \frac{\delta_{13}}{\delta_{11}} \quad M_9 = \frac{\delta_{14} - \omega^2}{\delta_{11}}$$
$$M_{10} = \frac{-\delta_{15}}{\delta_{11}} \quad M_{11} = \frac{\delta_{17}}{\delta_{20}} \quad M_{12} = \frac{\delta_{18}}{\delta_{20}} \quad M_{13} = \frac{\delta_{19}}{\delta_{20}} \quad M_{14} = \frac{1}{\delta_{20}}$$
$$\delta_{20} = \frac{\delta_{16}}{-i\omega} \quad N_{13} = \tau_{11}\delta_{16} \quad N_{14} = \tau_{11}\delta_{17} \quad N_{15} = \tau_{11}\delta_{18} \quad N_{16} = \tau_{11}$$
$$M_{15} = 1 - M_1M_6, N_5 = \frac{M_1M_7 + M_2}{M_{15}} \quad N_6 = \frac{M_1M_8 + M_3}{M_{15}}$$
$$N_7 = \frac{M_1M_9 + M_4}{M_{15}} \quad N_8 = \frac{M_1M_{10} + M_5}{M_{15}} \quad N_9 = M_6N_5 + M_7$$
$$N_{10} = M_6N_6 + M_8 \quad N_{11} = M_6N_7 + M_9 \quad N_{12} = M_6N_8 + M_{10}$$

4. State–space formulation

Choosing as a state variable displacement $\overline{u}$, volume fraction $\overline{\varphi}$ and $\overline{\psi}$, temperature change $\overline{T}$ in the x – direction, then the equations can be written in the matrix form as:

$$\frac{dV(x,\omega)}{dx} = A(\omega)\,V(x,\omega) \tag{21}$$

and the values of $A(\omega)$, $V(x,\omega)$ are given in the appendix I.
The formal solution of system (21) can be written in the form

$$V(x,\omega) = \exp\left[A(\omega)\,x\right]V(0,\omega) \tag{22}$$

Value of $V(0,\omega)$ is given in the appendix I.
We shall use the well-known Cayley-Hamilton theorem to find the form of the matrix $\exp[A(\omega)x]$. The characteristics equation of the matrix $A(\omega)$ can be written as

$$\lambda^8 + D_1\lambda^6 + D_2\lambda^4 + D_3\lambda^2 + D_4 = 0 \tag{23}$$

where

$$\begin{aligned}
&D_1 = -N1 - N6 - N11 - N_{16} - N2N5 - N3N9 - N4N_{13}\\
&D_2 = N1N6 + N1N11 + N1N_{16} + N6N11 + N6N_{16} - N7N10 - N8N_{14}\\
&+N11N_{16} - N12N_{15} - N2N7N9 + N3N6N9 + N2N5N11 + N2N5N_{16}\\
&-N3N5N10 + N3N9N_{16} - N2N8N_{13} - N4N5N_{14} + N4N6N_{13}\\
&-N4N9N_{15} - N3N12N_{13} + N4N11N_{13}\\
&D_3 = -N_6N_{11}N_{16} + N_7N_{10}N_{16} - N1N6N11 + N1N7N10 - N_1N_6N_{16}\\
&+N1N8N_{14} - N1N11N_{16} + N1N12N_{15} + N6N12N_{15} - N7N12N_{14}\\
&-N8N10N_{15} + N8N11N_{14} + N2N7N9N_{16} - N3N6N9N_{16} - N2N8N9N_{15}\\
&+N3N8N9N_{14} + N4N6N9N_{15} - N4N7N9N_{14} - N2N5N11N_{16}\\
&+N3N5N10N_{16} + N2N5N12N_{15} - N2N7N12N_{13} + N2N8N11N_{13}\\
&-N_3N_5N_{12}N_{14} + N3N6N12N_{13} - N3N8N10N_{13} - N4N5N10N_{15}\\
&+N4N5N11N_{14} - N4N6N11N_{13} + N4N7N10N_{13}\\
&D_4 = N1N_6(N11N_{16} - N12N_{15}) + N1N7(N12N_{14} - N10N_{16})\\
&+N1N8(N10N_{15} - N11N_{14})
\end{aligned} \tag{24}$$

Equation (23) is biquadrate in λ^2, yield four roots says $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.
Now the Taylor series expansion for matrix exponential in equation (23) is given by:

$$\exp\left[A\left(\omega\right)x\right] = \int_{n=0}^{\infty} \frac{\left[A\left(\omega\right)x\right]^n}{n!} \tag{25}$$

Using Cayley–Hamilton theorem, this infinite series can be truncated as

$$\exp\left[A\left(\omega\right)x\right] = a_0 I + a_1 A + a_2 A^2 + a_3 A^3 \tag{26}$$

where a_0, a_1, a_2, a_3 are parameters depending on x and ω.
According to Cayley–Hamilton theorem the characteristic roots
$-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$ of the matrix A must satisfy equation (26). Therefore, we get:

$$\begin{aligned}
\exp\left[-\lambda_1 x\right] &= a_0 I - a_1\lambda_1 + a_2{\lambda_1}^2 - a_3{\lambda_1}^3 \\
\exp\left[-\lambda_2 x\right] &= a_0 I - a_1\lambda_2 + a_2{\lambda_2}^2 - a_3{\lambda_2}^3 \\
\exp\left[-\lambda_3 x\right] &= a_0 I - a_1\lambda_3 + a_2{\lambda_3}^2 - a_3{\lambda_3}^3 \\
\exp\left[-\lambda_4 x\right] &= a_0 I - a_1\lambda_4 + a_2{\lambda_4}^2 - a_3{\lambda_4}^3
\end{aligned} \tag{27}$$

Solving the above system of equations, we obtain the value of parameters a_0, a_1, a_2, a_3 and these values are given in appendix.
Therefore, we have

$$\exp\left[A\left(\omega\right)x\right] = L(x, \omega) \tag{28}$$

where $L(x, w)$ is a 8×8 matrix with the components:

$$\begin{aligned}
&l_{11} = a_0 + a_2 N_1 \quad l_{12} = a_3 R_1 \quad l_{13} = a_3 R_2 \quad l_{14} = a_3 R_3 \\
&l_{21} = a_3 R_5 \quad l_{22} = a_0 + a_2 N_6 \quad l_{23} = a_2 N_7 \quad l_{24} = a_2 N_8 \\
&l_{31} = a_3 R_9 \quad l_{32} = a_2 N_{10} \quad l_{33} = a_0 + a_2 N_{11} \quad l_{34} = a_2 N_{12} \\
&l_{41} = a_3 R_{13} \quad l_{42} = a_2 N_{14} \quad l_{43} = a_2 N_{15} \quad l_{44} = a_0 + a_2 N_{16} \\
&R_1 = N_2 N_6 + N_3 N_{10} + N_4 N_{14} \quad R_2 = N_2 N_7 + N_3 N_{11} + N_4 N_{15} \\
&R_3 = N_2 N_8 + N_3 N_{12} + N_4 N_{16} \\
&R_5 = N_1 N_5 \quad R_9 = N_1 N_9 \quad R_{13} = N_1 N_{13}
\end{aligned}$$

Rewriting the equation (22) with the aid of equation (28) yield,

$$V(x, \omega) = L(x, \omega) V(0, \omega) \tag{29}$$

Therefore, we obtain

$$\begin{bmatrix} \overline{u} \\ \overline{\varphi} \\ \overline{\psi} \\ \overline{T} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \tag{30}$$

5. Boundary conditions

A homogeneous isotropic thermoelastic solid with double porosity structure occupying the region $0 \leq x < \infty$ is considered. The bounding plane $x = 0$ subjected to normal force and thermal source. Mathematically these can be written as:

$$1. \quad t_{11} = -F_1 \exp[-i\omega t] \tag{31}$$

$$2. \quad \sigma_1 = -F_1 \exp[-i\omega t] \tag{32}$$

$$3. \quad \zeta_1 = -F_1 \exp[-i\omega t] \tag{33}$$

$$4. \quad T = F_2 \exp[-i\omega t] \tag{34}$$

where $F_1$and F_2 are the magnitude of the force and constant temperature applied on the boundary.

Substituting the values of $u, \varphi, \psi, T, t_{11}, \sigma_1$ and ζ_1 from the equations (5), (6), (7), (30) in the equations (31)–(34) and with the aid of equations (10) and (16), we obtain:

$$\begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ Q_5 & Q_6 & Q_7 & Q_8 \\ Q_9 & Q_{10} & Q_{11} & Q_{12} \\ Q_{13} & Q_{14} & Q_{15} & Q_{16} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} -F_1 \\ -F_1 \\ -F_1 \\ F_2 \end{bmatrix} \tag{35}$$

The values of $Q_1,\ Q_2, \ldots, Q_{16}$ are given in the appendix II.

Solving (35) for A_1, A_2, A_3, A_4 and substituting the resulting values in equation (30) yield the value of normal stress, equilibrated stresses and temperature distribution as,

$$t_{11} = (S_1 \frac{\Gamma_1}{\Gamma} + S_2 \frac{\Gamma_2}{\Gamma} + S_3 \frac{\Gamma_3}{\Gamma} + S_4 \frac{\Gamma_4}{\Gamma}) e^{-i\omega t} \tag{36}$$

$$\sigma_1 = (S_5 \frac{\Gamma_1}{\Gamma} + S_6 \frac{\Gamma_2}{\Gamma} + S_7 \frac{\Gamma_3}{\Gamma} + S_8 \frac{\Gamma_4}{\Gamma}) e^{-i\omega t} \tag{37}$$

$$\zeta_1 = (S_9 \frac{\Gamma_1}{\Gamma} + S_{10} \frac{\Gamma_2}{\Gamma} + S_{11} \frac{\Gamma_3}{\Gamma} + S_{12} \frac{\Gamma_4}{\Gamma}) e^{-i\omega t} \tag{38}$$

$$T = (l_{41} \frac{\Gamma_1}{\Gamma} + l_{42} \frac{\Gamma_2}{\Gamma} + l_{43} \frac{\Gamma_3}{\Gamma} + l_{44} \frac{\Gamma_4}{\Gamma}) e^{-i\omega t} \tag{39}$$

6. Particular cases

6.1. If $F_2 = 0$ in equation (36)–(39), we obtain the corresponding expressions for normal force.

6.2. If $F_1 = 0$ in equation (36)–(39), yields the corresponding expressions for thermal source.

6.3. If $m = 0$ in equation (36)–(39), we obtain the corresponding expressions without the effect of Hall currents.

7. Numerical results and discussion

The material chosen for the purpose of numerical computation is copper, whose physical data is given by Sherief and Saleh [53] as:

$\lambda = 7.76 \times 10^{10} Nm^{-2}$ $c^* = 3.831 \times 10^3 m^2 s^{-2} K^{-1}$ $\mu = 3.86 \times 10^{10} Nm^{-2}$
$k = 3.86 \times 10^3 Ns^{-1} K^{-1}$ $\omega_1 = 1 \times 10^{11} s^{-1}$ $T_0 = 0.293 \times 10^3 K$
$\alpha_t = 1.78 \times 10^{-5} K^{-1}$ $t = 0.1s$, $\rho = 8.954 \times 10^3 Kgm^{-3}$

Following Khalili [54], the double porous parameters are taken as:

$\alpha_2 = 2.4 \times 10^{10} Nm^{-2}$ $\alpha_3 = 2.5 \times 10^{10} Nm^{-2}$ $\gamma = 1.1 \times 10^{-5} N$
$\alpha = 1.3 \times 10^{-5} N \gamma_1 = 0.16 \times 10^5 Nm^{-2}$ $b_1 = 0.12 \times 10^{-5} N$ $d = 0.1 \times 10^{10} Nm^{-2}$
$\gamma_2 = 0.219 \times 10^5 Nm^{-2}$ $k_1 = 0.1456 \times 10^{-12} Nm^{-2} s^2$ $b = 0.9 \times 10^{10} Nm^{-2}$
$\alpha_1 = 2.3 \times 10^{10}\ Nm^{-2}$ $k_2 = 0.1546 \times 10^{-12} Nm^{-2} s^2$

Following Zakaria [49], the electric constants are taken as:

$\sigma_0 = 9.36 \times 10^5\ Col^2/Cl.cm.s$ $H_0 = 10^8 Col/cm.s$

The software MATLAB has been used to determine the values of normal stress, equilibrated stresses and temperature distribution. The variation of these values with respect to distance x have been shown in Figs. 1–8 and Figs. 9–16 for normal force and thermal source respectively. Figs. 1–4, Figs. 9–12 and Figs. 5–8, Figs. 13–16 depicts the behavior of variation of normal stress, equilibrated stresses and temperature distribution with Hall parameter (m) and relaxation time (τ_0) with respect to distance x respectively. In Figs. 1–4 and Figs. 9–12, the solid lines corresponds to thermoelastic material with double porous structure without the effect of Hall currents(TWOH) and small dashes line corresponds to thermoelastic material with double porous structure with Hall current effect (TWH) whereas in Figs. 5–8 and Figs. 13–16, solid line corresponds to the value when relaxation time $\tau_0 = 0.02$ and small dashes line corresponds to $\tau_0 = 0.03$.

7.1. *Normal force*

Fig. 1 shows the variation of normal stress t_{11} w.r.t distance x. The behavior of variation is oscillatory in nature for both TWOH and TWH. It is noticed that the magnitude values are smaller in case of TWH in comparison with TWOH.

Figs. 2 and 3 depict the variations of equilibrated stresses σ_1 and ζ_1 w.r.t. distance x respectively. The behavior is similar for both TWOH and TWH with the difference in magnitude value. The magnitude values of equilibrated stresses are more in case of TWH as compared to TWOH.

Fig. 4 represents the variation of temperature distribution T w.r.t distance x. It is evident that as away from the source, for TWOH, the magnitude values of T decreases whereas reverse behaviour is noticed for TDWH.

Fig. 5 shows the variation of normal stress t_{11} w.r.t. distance x. The behavior of variation is oscillatory in nature for both the values of relaxation time. It is noticed that the magnitude values increase with increase in the value of relaxation time.

Fig. 6 and 7 depict the variation of equilibrated stresses σ_1 and ζ_1 w.r.t. distance x respectively. The behavior is similar for both the cases with the difference in magnitude value. It is found that the magnitude values increase with increase in the value of relaxation times.

Fig. 8 illustrates the variation of temperature distribution T w.r.t. distance x. For $\tau_0 = 0.02$ the magnitude value of T decreases for $0 \le x < 2.6$ and again increases

as moving away from the source wheras for $\tau_0 = 0.03$, as moving away from the source, the magnitude value decreases. To depict the comparison, the values of T for $\tau_0 = 0.02$ have been magnified by multiplying 10.
To its original values.

7.2. *Thermal source*

Fig. 9 shows the variation of normal stress t_{11} w.r.t. distance x. The behavior of variation is oscillatory in nature for both TWOH and TWH. Also the magnitude value is smaller in case of TWOH in comparison to TWH near the application of the source while the reverse pattern is observed away from the source.
Fig. 10 and 11 depict the variation of equilibrated stresses σ_1 and ζ_1 w.r.t. distance x respectively. The variation is of oscillatory nature for both the cases while the magnitude value shows opposite behavior for all values of x.
Fig. 12 represents the variation of temperature distribution T w.r.t. distance x. For TWOH, the magnitude value of T increases as moving away from the source while a reverse behavior is noticed in case of TWH.
Fig. 13 shows the variation of normal stress t_{11} w.r.t. distance x. The variation is of oscillatory nature for both the values of relaxation time. The magnitude values also increases with increase in the time of relaxation.
Fig. 14 and 15 depict the variation of equilibrated stresses σ_1 and ζ_1 w.r.t. distance x respectively. The behavior is similar for both the cases with the difference in magnitude value. It is found that the magnitude values increases with increase in the value of relaxation time.

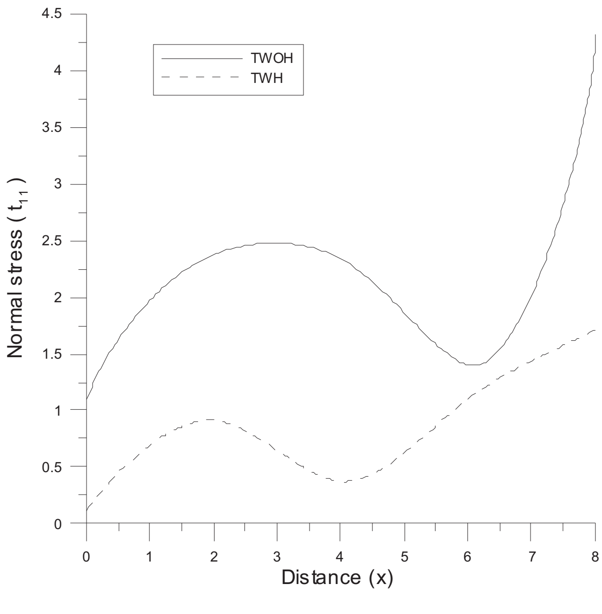

Figure 1 Variation of normal stress $t_1 1$ with distance x (Normal force; Effect of Hall current)

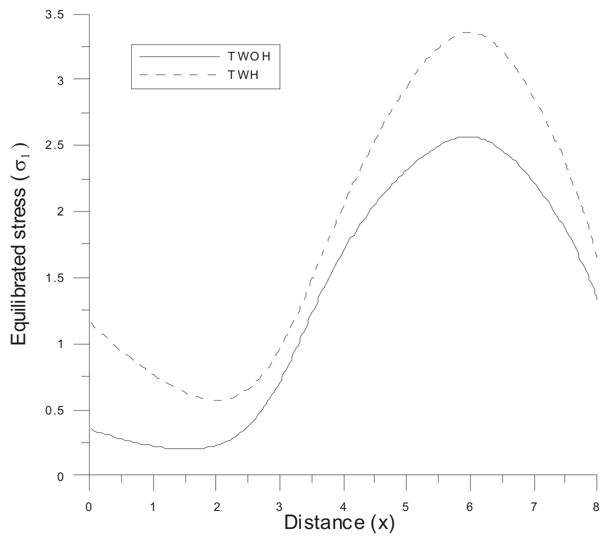

Figure 2 Variation of equilibrated stress σ_1 with distance x (Normal force; Effect of Hall current)

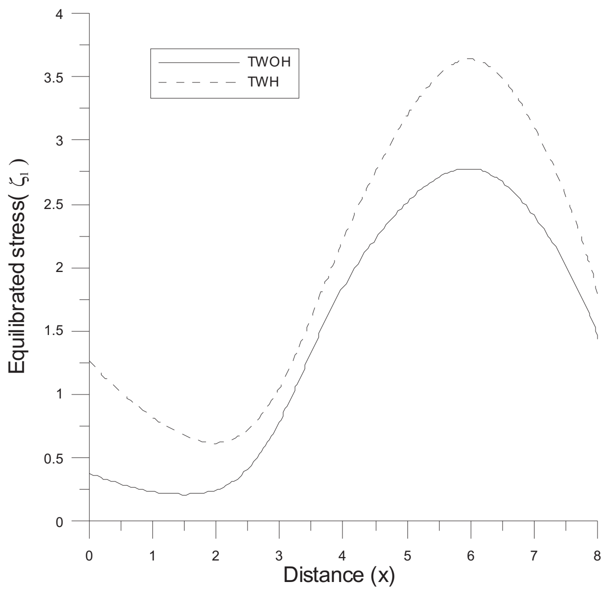

Figure 3 Variation of equilibrated stress ζ_1 with distance x (Normal force; Effect of Hall current)

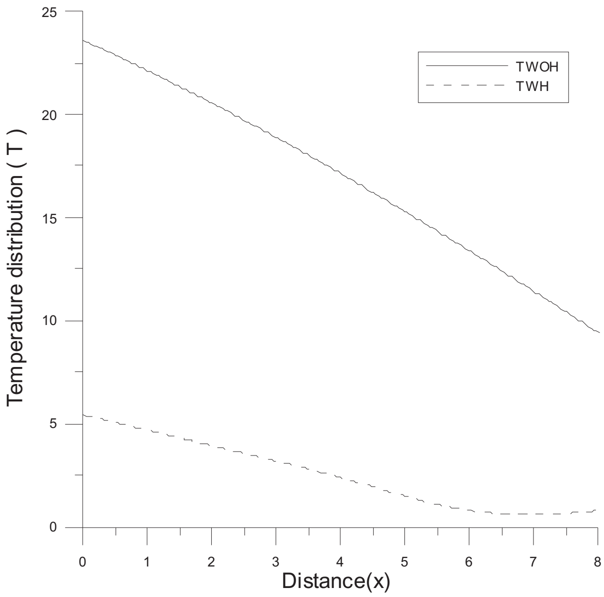

Figure 4 Variation of temperature distribution T with distance x (Normal force; Effect of Hall current)

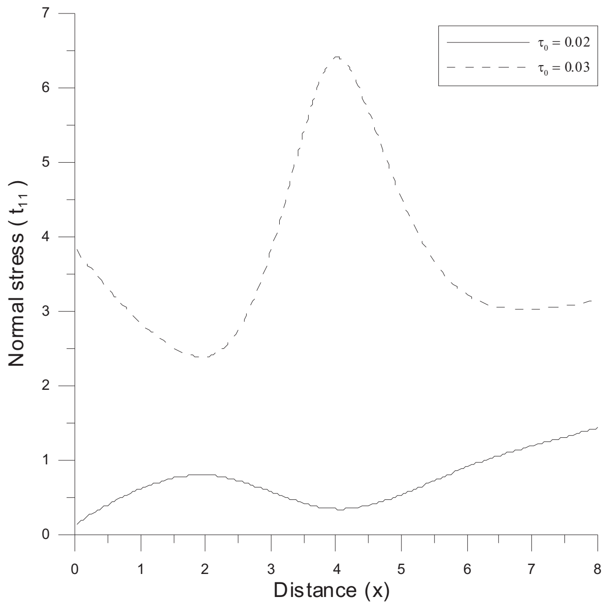

Figure 5 Variation of normal stress t_{11} with distance x (Normal force; Effect of relaxation time)

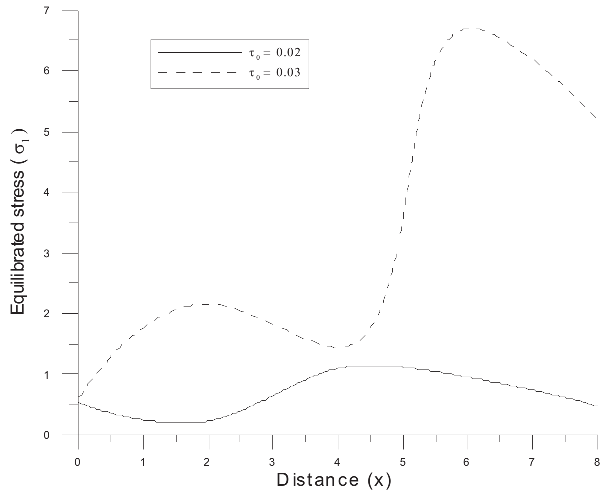

Figure 6 Variation of equilibrated stress σ_1 with distance x (Normal force; Effect of relaxation time)

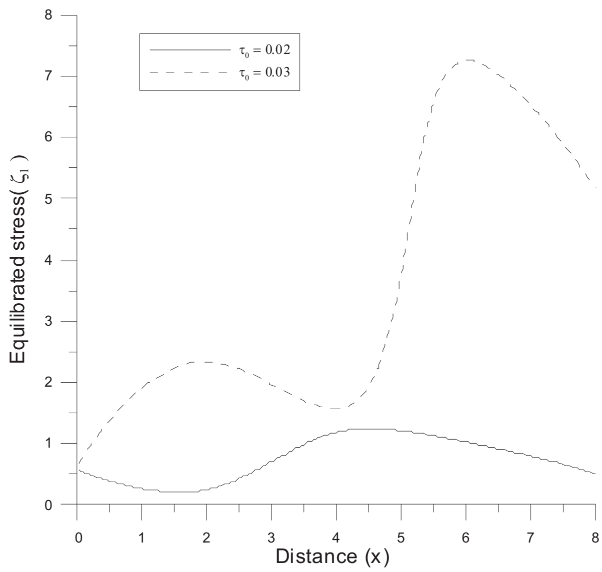

Figure 7 Variation of equilibrated stress ζ_1 with distance x (Normal force; Effect of relaxation time)

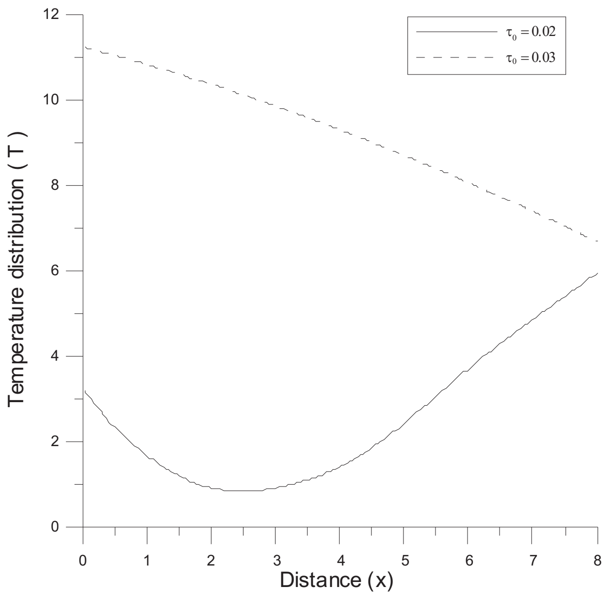

Figure 8 Variation of temperature distribution T with distance x (Normal force; Effect of relaxation time)

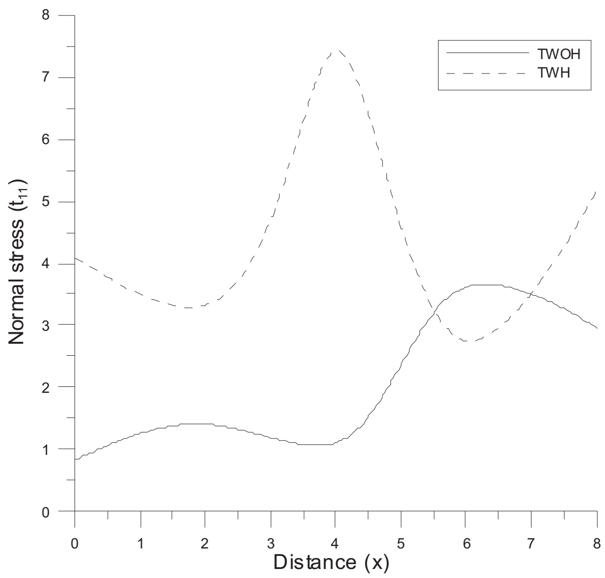

Figure 9 Variation of normal stress t_{11} with distance x(Thermal source; Effect of Hall current)

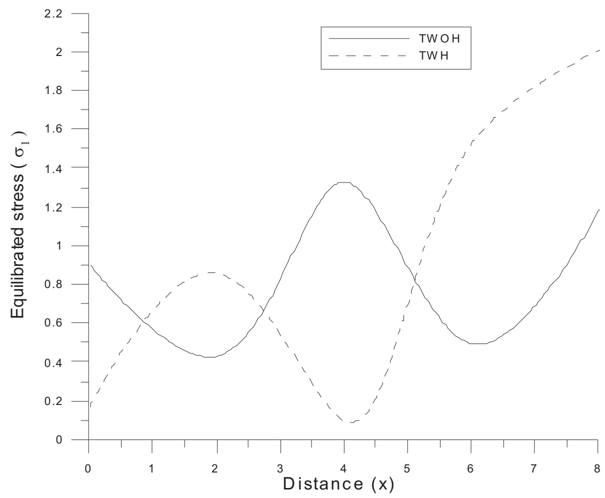

Figure 10 Variation of equilibrated stress σ_1 with distance x (Thermal source; Effect of Hall current)

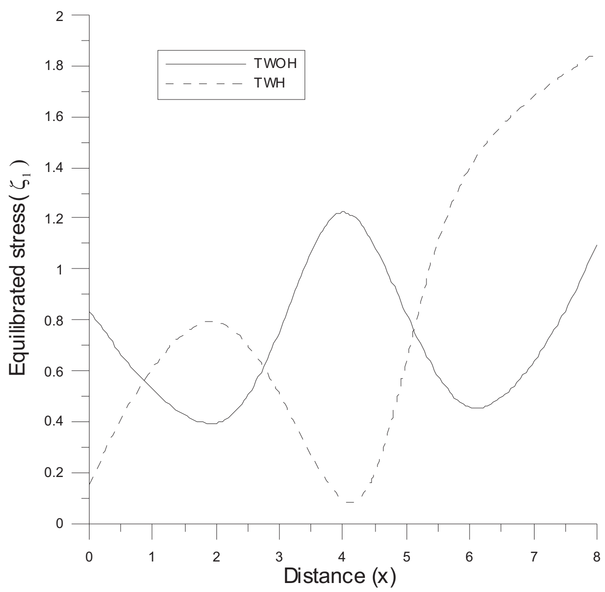

Figure 11 Variation of equilibrated stress ζ_1 with distance x (Thermal source; Effect of Hall current)

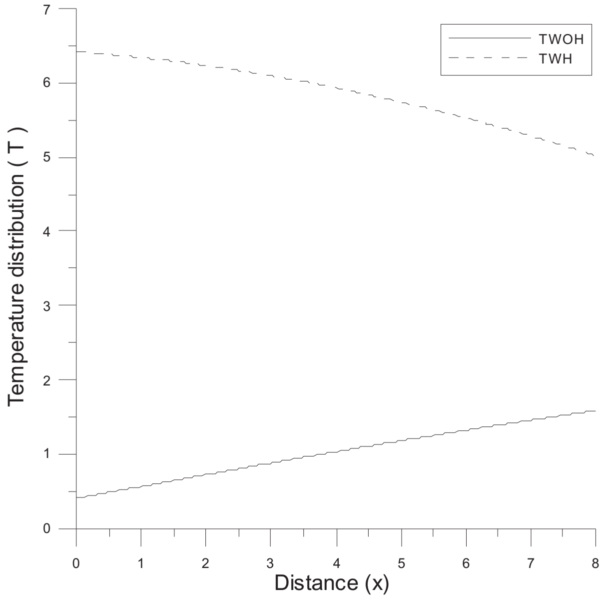

Figure 12 Variation of temperature distribution T with distance x (Thermal source; Effect of Hall current)

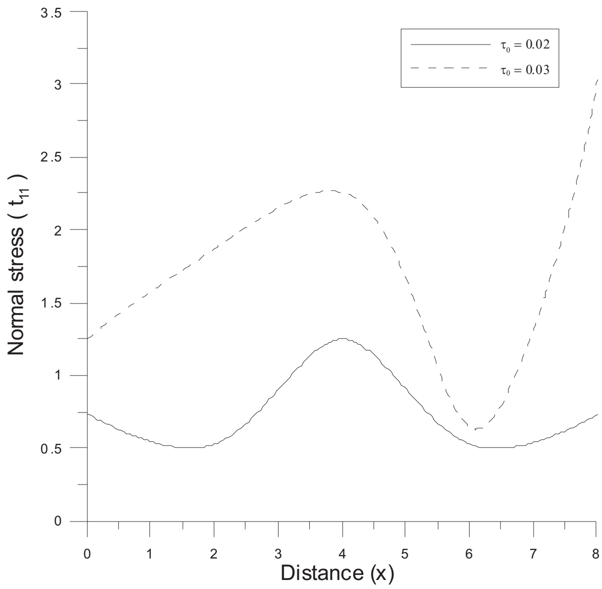

Figure 13 Variation of normal stress t_{11} with distance x (Thermal source; Effect of relaxation time)

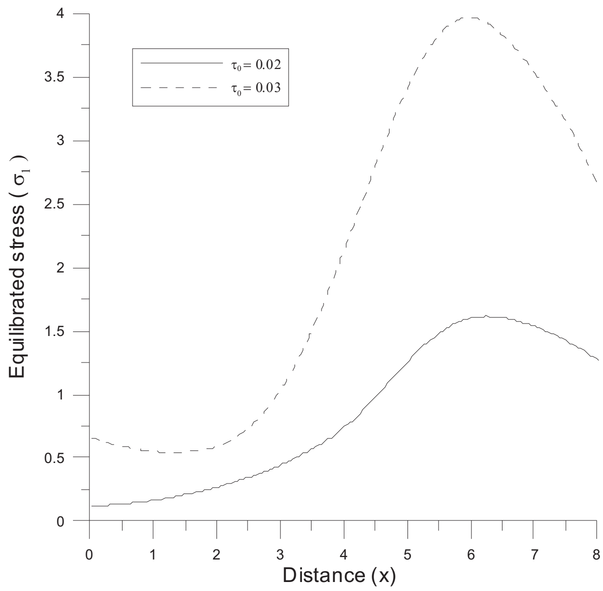

Figure 14 Variation of equilibrated stress σ_1 with distance x (Thermal source; Effect of relaxation time)

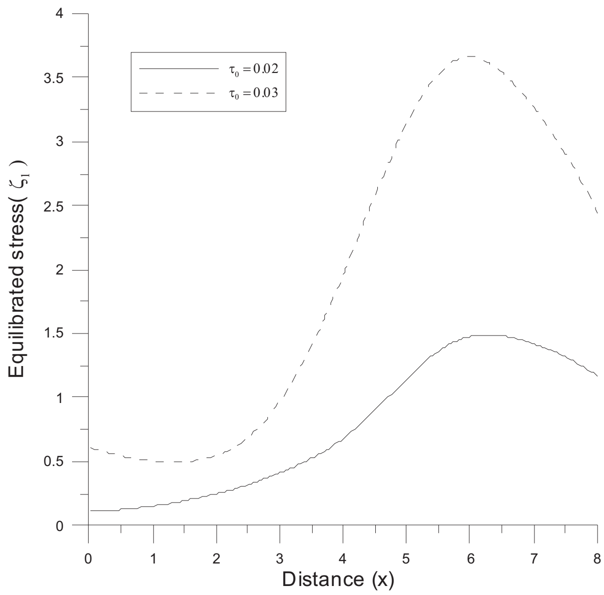

Figure 15 Variation of equilibrated stress ζ_1 with distance x (Thermal source; Effect of relaxation time)

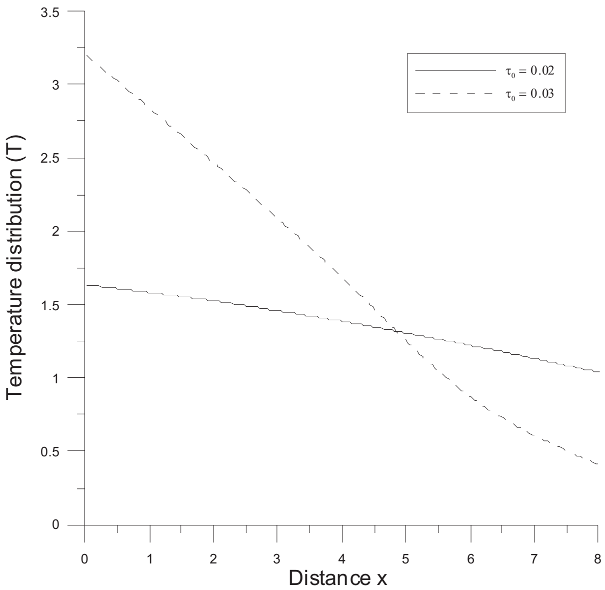

Figure 16 Variation of temperature distribution T with distance x (Thermal source; Effect of relaxation time)

Fig. 16 illustrates the variation of temperature distribution T w.r.t. distance x. For $\tau_0 = 0.02$, the magnitude value of T is smaller in comparison to the values when $\tau_0 = 0.03$ near the application of the source while the reverse pattern is noticed away from the source.

8. Conclusions

The behaviour of normal stress, equilibrated stresses and temperature distribution in an isotropic homogeneous thermoelastic material with double porosity structure for L-S theory under the effect of Hall currents has been investigated due to normal force and thermal source by using state space approach. It is observed that

1. For normal force, the behavior of normal stress is similar with and without the effect of Hall current for all values of x but the magnitude value is larger in case where Hall current is not taken into consideration i.e. $m = 0$ while for thermal source, the variation is similar for both the cases near the application of the source but an opposite pattern is observed as moving away from the source.

2. In case of normal force, the variation of equilibrated stresses is identical with and without the effect of Hall current for all values of x. It is observed that the magnitude value of the equilibrated stresses increases due to effect of Hall

current. For thermal source, the behavior of variation is oscillatory in nature with and without the Hall current effect but the magnitude values shows an opposite pattern.

3. The magnitude values of temperature distribution is larger under the effect of Hall currents in case of normal force wheras a reverse trend is observed for thermal source.

4. The magnitude values of normal stress and equilibrated stresses increases with the increase in relaxation time for both normal force and thermal source while the pattern of variation remains same.

5. The magnitude values of temperature distribution is large for greater value of relaxation time near the application of the source while reverse behavior is observed as moving away from the source.

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Appendix I

$$A(x,w)=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ N_1 & 0 & 0 & 0 & 0 & N_2 & N_3 & N_4 \\ 0 & N_6 & N_7 & N_8 & N_5 & 0 & 0 & 0 \\ 0 & N_{10} & N_{11} & N_{12} & N_9 & 0 & 0 & 0 \\ 0 & N_{14} & N_{15} & N_{16} & N_{13} & 0 & 0 & 0 \end{bmatrix}$$

$$V(x,w)=\begin{bmatrix} \overline{u}(x,w) \\ \overline{\varphi}(x,w) \\ \overline{\psi}(x,w) \\ \overline{T}(x,w) \\ (\overline{u}(x,w))_{,1} \\ (\overline{\varphi}(x,w))_{,1} \\ (\overline{\psi}(x,w))_{,1} \\ (\overline{T}(x,w))_{,1} \end{bmatrix} \qquad V(0,w)=\begin{bmatrix} \overline{u}(0w) \\ \overline{\varphi}(0,w) \\ \overline{\psi}(0,w) \\ \overline{T}(0,w) \\ (\overline{u}(0,w))_{,1} \\ (\overline{\varphi}(0,w))_{,1} \\ (\overline{\psi}(0,w))_{,1} \\ (\overline{T}(0,w))_{,1} \end{bmatrix}$$

$$\begin{aligned} a_0 &= e^{-\lambda_1 x}[1-\frac{\lambda_1\lambda_2(\lambda_3+\lambda_4)+\lambda_1\lambda_3\lambda_4}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}+\frac{\lambda_1{}^2(\lambda_2+\lambda_3+\lambda_4)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} \\ &-\frac{\lambda_1{}^3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}]-e^{-\lambda_2 x}[\frac{\lambda_1{}^2(\lambda_3+\lambda_4)+\lambda_1\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ &-\frac{\lambda_1{}^2(\lambda_1+\lambda_3+\lambda_4)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4))}+\frac{\lambda_1{}^3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}] \\ &-e^{-\lambda_3 x}[\frac{\lambda_1{}^2(\lambda_2+\lambda_4)+\lambda_1\lambda_2\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}-\frac{\lambda_1{}^2(\lambda_1+\lambda_2+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4))} \\ &+\frac{\lambda_1{}^3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_{3s}-\lambda_4)}]-e^{-\lambda_4 x}[\frac{\lambda_1{}^2(\lambda_2+\lambda_3)+\lambda_1\lambda_2\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \\ &-\frac{\lambda_1{}^2(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}+\frac{\lambda_1{}^3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}] \end{aligned}$$

$$\begin{aligned} a_1 &= -e^{-\lambda_1 x}[\frac{\lambda_2(\lambda_3+\lambda_4)+\lambda_3\lambda_4}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}]-e^{-\lambda_2 x}[\frac{\lambda_1(\lambda_3+\lambda_4)+\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}] \\ &-e^{-\lambda_3 x}[\frac{\lambda_1(\lambda_2+\lambda_4)+\lambda_2\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}]-e^{-\lambda_4 x}[\frac{\lambda_1(\lambda_2+\lambda_3)+\lambda_2\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}] \end{aligned}$$

$$\begin{aligned} a_2 &= -e^{-\lambda_1 x}[\frac{(\lambda_2+\lambda_3+\lambda_4)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}]-e^{-\lambda_2 x}[\frac{(\lambda_1+\lambda_3+\lambda_4)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}] \\ &-e^{-\lambda_3 x}[\frac{(\lambda_1+\lambda_2+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}]-e^{-\lambda_4 x}[\frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}] \end{aligned}$$

$$\begin{aligned} a_3 &= -e^{-\lambda_1 x}[\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}]-e^{-\lambda_2 x}[\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}] \\ &-e^{-\lambda_3 x}[\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}]-e^{-\lambda_4 x}[\frac{1}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}] \end{aligned}$$

Appendix II

$$
\begin{aligned}
&Q_1 = P_1(Z_1 + N_1 Z_3) + P_2\left(a_3^0 R_5\right) + P_3\left(a_3^0 R_9\right) - a_3^0 R_{13}\\
&Q_2 = P_1 R_1 Z_4 + P_2\left(a_0^0 + a_2^0 N_6\right) + P_3\left(a_0^0 + a_2^0 N_{11}\right) - a_2^0 N_{14}\\
&Q_3 = P_1 R_2 Z_4 + P_2\left(a_2^0 N_7\right) + P_3\left(a_0^0 + a_2^0 N_{11}\right) - a_2^0 N_{15}\\
&Q_4 = P_1 R_3 Z_4 + P_2\left(a_2^0 N_8\right) + P_3\left(a_2^0 N_{12}\right) - \left(a_0^0 + a_2^0 N_{16}\right)\\
&Q_5 = P_4 R_5 Z_4 + P_5 R_9 Z_4, Q_6 = P_4\left(Z_1 + N_6 Z_3\right) + P_5 N_{10} Z_3\\
&Q_7 = P_4 N_7 Z_3 + P_5\left(Z_1 + N_{11} Z_3\right) \qquad Q_8 = P_4 N_8 Z_3 + P_5 N_{12} Z_3\\
&Q_9 = P_5 R_5 Z_4 + P_6 R_9 Z_4, Q_{10} = P_5\left(Z_1 + N_6 Z_3\right) + P_6 N_{10} Z_3\\
&Q_{11} = P_5 N_7 Z_3 + P_6\left(Z_1 + N_{11} Z_3\right) \qquad Q_{12} = P_5 N_8 Z_3 + P_6 N_{12} Z_3\\
&Q_{13} = a_3^0 R_{13} \qquad Q_{14} = a_2^0 N_{14} \qquad Q_{15} = a_2^0 N_{15} \qquad Q_{16} = a_0^0 + a_2^0 N_{16}
\end{aligned}
$$

where:

$$
\begin{aligned}
&P_1 = \frac{\lambda + 2\mu}{\beta T_0} \qquad P_2 = \frac{b\alpha_1}{k_1\omega^2\beta T_0} \qquad P_3 = \frac{d\alpha_1}{k_1\omega^2\beta T_0}\\
&P_4 = \frac{\alpha_1}{k_1\omega^2} \qquad P_5 = \frac{b_1\alpha_1}{\alpha k_1\omega^2} \qquad P_6 = \frac{\gamma\alpha_1}{\alpha k_1\omega^2}\\
&Z_1 = -\lambda_1 D_{11} - \lambda_2 D_{12} - \lambda_3 D_{13} - \lambda_4 D_{14}\\
&Z_2 = -\lambda_1 D_{21} - \lambda_2 D_{22} - \lambda_3 D_{23} - \lambda_4 D_{24}\\
&Z_3 = -\lambda_1 D_{31} - \lambda_2 D_{32} - \lambda_3 D_{33} - \lambda_4 D_{34}\\
&Z_4 = -\lambda_1 D_{41} - \lambda_2 D_{42} - \lambda_3 D_{43} - \lambda_4 D_{44}\\
&Y_1 = -\lambda_1 D_{11} e^{-\lambda_1 x} - \lambda_2 D_{12} e^{-\lambda_2 x} - \lambda_3 D_{13} e^{-\lambda_3 x} - \lambda_4 D_{14} e^{-\lambda_4 x}\\
&Y_2 = -\lambda_1 D_{11} e^{-\lambda_1 x} - \lambda_2 D_{12} e^{-\lambda_2 x} - \lambda_3 D_{13} e^{-\lambda_3 x} - \lambda_4 D_{14} e^{-\lambda_4 x}\\
&Y_3 = -\lambda_1 D_{21} e^{-\lambda_1 x} - \lambda_2 D_{22} e^{-\lambda_2 x} - \lambda_3 D_{23} e^{-\lambda_3 x} - \lambda_4 D_{24} e^{-\lambda_4 x}\\
&Y_4 = -\lambda_1 D_{31} e^{-\lambda_1 x} - \lambda_2 D_{32} e^{-\lambda_2 x} - \lambda_3 D_{33} e^{-\lambda_3 x} - \lambda_4 D_{34} e^{-\lambda_4 x}\\
&S_1 = P_1(Y_1 + N_1 Y_3) + P_2\left(l_{21}\right) + P_3 l_{31} - l_{41}\\
&S_2 = P_1(R_1 Y_4) + P_2\left(l_{22}\right) + P_3\left(l_{32}\right) - l_{42}\\
&S_3 = P_1(R_2 Y_4) + P_2\left(l_{23}\right) + P_3\left(l_{33}\right) - l_{43}\\
&S_4 = P_1 R_3 Y_4 + P_2\left(l_{24}\right) + P_3\left(l_{34}\right) - \left(l_{44}\right)\\
&S_5 = P_4 R_5 Y_4 + P_5 R_9 Y_4, S_6 = P_4(Y_4 + N_6 Y_3) + P_5 N_{10} Y_3\\
&S_7 = P_4 N_7 Y_3 + P_5(Y_1 + N_{11} Y_3)\\
&S_8 = P_4 N_8 Y_3 + P_5 N_{12} Y_3\\
&S_9 = P_5 R_5 Y_4 + P_6 R_9 Y_4, S_{10} = P_5(Y_1 + N_6 Y_3) + P_6 N_{10} Y_3\\
&S_{11} = P_5 N_7 Y_3 + P_6\left(Y_1 + N_{11} Y_3\right)\\
&S_{12} = P_5 N_8 Y_3 + P_6 N_{12} Y_3
\end{aligned}
$$

$$D_{11}=1-\frac{\lambda_1\lambda_2\left(\lambda_3+\lambda_4\right)+\lambda_1\lambda_3\lambda_4}{\left(\lambda_1-\lambda_2\right)\left(\lambda_1-\lambda_3\right)\left(\lambda_1-\lambda_4\right)}+\frac{{\lambda_1}^2(\lambda_2+\lambda_3+\lambda_4)}{\left(\lambda_1-\lambda_2\right)\left(\lambda_1-\lambda_3\right)\left(\lambda_1-\lambda_4\right)}$$
$$-\frac{{\lambda_1}^3}{\left(\lambda_1-\lambda_2\right)\left(\lambda_1-\lambda_3\right)\left(\lambda_1-\lambda_4\right)}$$
$$D_{12}=-[\frac{{\lambda_1}^2\left(\lambda_3+\lambda_4\right)+\lambda_1\lambda_3\lambda_4}{\left(\lambda_2-\lambda_1\right)\left(\lambda_2-\lambda_3\right)\left(\lambda_2-\lambda_4\right)}-\frac{{\lambda_1}^2(\lambda_1+\lambda_3+\lambda_4)}{\left(\lambda_2-\lambda_1\right)\left(\lambda_2-\lambda_3\right)\left(\lambda_2-\lambda_4\right))}$$
$$+\frac{{\lambda_1}^3}{\left(\lambda_2-\lambda_1\right)\left(\lambda_2-\lambda_3\right)\left(\lambda_2-\lambda_4\right)}]$$
$$D_{13}=-[\frac{{\lambda_1}^2\left(\lambda_2+\lambda_4\right)+\lambda_1\lambda_2\lambda_4}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)\left(\lambda_3-\lambda_4\right)}-\frac{{\lambda_1}^2\left(\lambda_1+\lambda_2+\lambda_4\right)}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)\left(\lambda_3-\lambda_4\right)}$$
$$+\frac{{\lambda_1}^3}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)\left(\lambda_3-\lambda_4\right)}]$$
$$D_{14}=-[\frac{{\lambda_1}^2\left(\lambda_2+\lambda_3\right)+\lambda_1\lambda_2\lambda_3}{\left(\lambda_4-\lambda_1\right)\left(\lambda_4-\lambda_2\right)\left(\lambda_4-\lambda_3\right)}-\frac{{\lambda_1}^2\left(\lambda_1+\lambda_2+\lambda_3\right)}{\left(\lambda_4-\lambda_1\right)\left(\lambda_4-\lambda_2\right)\left(\lambda_4-\lambda_3\right)}$$
$$+\frac{{\lambda_1}^3}{\left(\lambda_4-\lambda_1\right)\left(\lambda_4-\lambda_2\right)\left(\lambda_4-\lambda_3\right)}]$$
$$D_{31}=-\frac{(\lambda_2+\lambda_3+\lambda_4)}{\left(\lambda_1-\lambda_2\right)\left(\lambda_1-\lambda_3\right)\left(\lambda_1-\lambda_4\right)}\qquad D_{32}=-\frac{(\lambda_1+\lambda_3+\lambda_4)}{\left(\lambda_2-\lambda_1\right)\left(\lambda_2-\lambda_3\right)\left(\lambda_2-\lambda_4\right)}$$
$$D_{33}=-\frac{(\lambda_1+\lambda_2+\lambda_4)}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)\left(\lambda_3-\lambda_4\right)}\qquad D_{34}=-\frac{(\lambda_1+\lambda_2+\lambda_3)}{\left(\lambda_4-\lambda_1\right)\left(\lambda_4-\lambda_2\right)\left(\lambda_4-\lambda_3\right)}$$
$$D_{41}=-\frac{1}{\left(\lambda_1-\lambda_2\right)\left(\lambda_1-\lambda_3\right)\left(\lambda_1-\lambda_4\right)}\qquad D_{42}=-\frac{1}{\left(\lambda_2-\lambda_1\right)\left(\lambda_2-\lambda_3\right)\left(\lambda_2-\lambda_4\right)}$$
$$D_{43}=-\frac{1}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)\left(\lambda_3-\lambda_4\right)}\qquad D_{44}=-\frac{1}{\left(\lambda_4-\lambda_1\right)\left(\lambda_4-\lambda_2\right)\left(\lambda_4-\lambda_3\right)}$$

$$\Gamma=\begin{vmatrix} Q_1 & Q_2 & Q_3 & Q_4\\ Q_5 & Q_6 & Q_7 & Q_8\\ Q_9 & Q_{10} & Q_{11} & Q_{12}\\ Q_{13} & Q_{14} & Q_{15} & Q_{16}\end{vmatrix}\qquad \Gamma_1=\begin{vmatrix} -F_1 & Q_2 & Q_3 & Q_4\\ -F_1 & Q_6 & Q_7 & Q_8\\ -F_1 & Q_{10} & Q_{11} & Q_{12}\\ F_2 & Q_{14} & Q_{15} & Q_{16}\end{vmatrix}$$

$$\Gamma_2=\begin{vmatrix} Q_1 & -F_1 & Q_3 & Q_4\\ Q_5 & -F_1 & Q_7 & Q_8\\ Q_9 & -F_1 & Q_{11} & Q_{12}\\ Q_{13} & F_2 & Q_{15} & Q_{16}\end{vmatrix}\qquad \Gamma_3=\begin{vmatrix} Q_1 & Q_2 & -F_1 & Q_4\\ Q_5 & Q_6 & -F_1 & Q_8\\ Q_9 & Q_{10} & -F_1 & Q_{12}\\ Q_{13} & Q_{14} & F_2 & Q_{16}\end{vmatrix}$$

$$\Gamma_4=\begin{vmatrix} Q_1 & Q_2 & Q_3 & -F_1\\ Q_5 & Q_6 & Q_7 & -F_1\\ Q_9 & Q_{10} & Q_11 & -F_1\\ Q_{13} & Q_{14} & Q_{15} & F_2\end{vmatrix}$$

and $a_0^0=a_0$, $a_2^0=a_2$, $a_3^0=a_3$ at $x=0$, $A_1=\frac{\Gamma_1}{\Gamma}$, $A_2=\frac{\Gamma_2}{\Gamma}$, $A_3=\frac{\Gamma_3}{\Gamma}$, $A_4=\frac{\Gamma_4}{\Gamma}$