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Axisymmetric Torsion of an Internally Cracked Elastic Medium by Two Embedded Rigid Discs

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The present work aims to investigate a penny–shaped crack problem in the interior of a homogeneous elastic material at the symmetry plane, under an axisymmetric torsion by two circular rigid discs symmetrically located in the elastic medium. The two discs rotate with the same angle in the different direction about the axis passing through their centers. The general solution of this problem is obtained by using the Hankel transforms method. The corresponding doubly mixed boundary value problem associated with the rigid disc and the penny–shaped is reduced to a system of dual integral equations, which are transformed, to a Fredholm integral equations of the second kind. Using the quadrature rule, the resulting system is converted to a system of infnite algebraic equations. The variation in the displacement, stress and stress intensity factor are presented for some particular cases of the problem.

Keywords: elastic medium, axisymmetric torsion, penny–shaped crack, dual integral equations, Fredholm integral equations, stress intensity factor.

1. Introduction

The problems relating to the static and dynamic torsional of an elastic half–space under the action of rigid discs are of great interest in mechanical Engineering, civil engineering, geomechanics and applied mathematics. The problem is a mixed boundary–value problem with the stress and the displacement boundary conditions of the rigid disc and the crack planes. Many different studies conducted by many different researchers who have studied the problem associated with the torsional rotation of a rigid disc in bonded contact with the elastic medium. The distribution of stress in the interior of a semi-infinite elastic medium is determined when a load is applied to the surface by means of a rigid disc was first considered by Reissner and Sagoci [1]. The same problem was solved by Sneddon [2] by a different method. He used the Hankel transforms method for reduction the problem to a pair of dual integral equations. Collins [3] treated the torsional problem of an elastic half–space by supposing the displacement at any point in the half–space to be due to a distribution of wave sources over the part of the free surface in contact with the disc.

The solution of the forced vibration problem of elastic layer of finite thickness when the lower face is either stress free or rigidly clamped was given by Gladwell [4] who reduced the mixed boundary value problem to a Fredholm integral equation by Noble's method [5]. Pak and Saphores [6] provided an analytical formulation for the general torsional problem of a rigid disc embedded in an isotropic half–space. The quadrature numerical was used for solving the obtained Fredholm integral equation. Besides, Bacci and Bennati [7] employed the Hankel transforms method and the power series method with the truncation of the second term to consider the torsional of circular rigid disc adhered to the upper surface of an elastic layer fixed to an undefonnable support.

More recently, Singh et al. [8] studied the torsional of a non–homogeneous, isotropic, half–space by rotating a circular part of its boundary surface. The solution of the corresponding triple integral equations was reduced to the solution of two simultaneous integral equations. Cai and Zue [9] discussed the torsional vibration of a rigid disc bonded to a poro–elastic multilayered. They used the Hankel transforms and transferring matrix method. Yu [10] studied the forced torsional oscillations inside the multilayered solid. The elastodynamic Green's function of the center of rotation and a point load method were used to solve the problem. Pal and Mandal [11] considered the forced torsional oscillations of a transversely isotropic elastic half space under the action of an inside rigid disc. The studied problem was transformed to dual integral equations system, which was reduced to a fredholm integral equation. A similar problem with the rocking rotation was solved later on by Ahmadi and Eskandari [12]. They used an appropiate Green's function to write the mixed boundary–value problem posed as a dual integral equation.

The torsional of elastic layers with a penny shaped crack was considered by some researchers. Sih and Chen [13] studied the problem of a penny-shaped crack in layered composite under a uniform torsional stress. The displacement and stress fields throughout the composite were obtained by solving a standard Fredholm integral equation of the second kind. Low [14] investigated a problem of the effects of embedded flaws in the form of an inclusion or a crack in an elastic half space subjected to torsional deformations. The corresponding Fredholm integral equations were solved numerically by quadrature approach. The same method was used by Dhawan [15] for solving the problem of a rigid disc attached to an elastic half–space with an internal crack. By using Hankel and Laplace transforms and taking numerical inversion of Laplace transform, Basu and Mandal [16] treated the torsional load on a penny–shaped crack in an elastic layer sandwiched between two elastic half–spaces.

In this paper, we investigate the problem of a penny–shaped crack in the interior of a homogeneous elastic medium under an axisymmetric torsion applied to two symmetrical rigid discs. With the aid of the Hankel integral transformation method. The mixed boundary–value problem is written as a system dual of integral equations. The corresponding system of Fredholm integral equations was approached by sets of linear equations. After getting the unknown coefficients of this system we obtain numerical results and display curves according to certain pertinent parameters.

2. Formulation of the problem

We consider the axisymmetric torsion of two rigid coaxial discs of a radius b symmetrically located at $z = \pm h$ planes in an infinite, isotropic and homogeneous elastic medium, containing a penny–shaped crack at the symmetry plane $z = 0$. The faces of the crack are supposed stress free while the dics rotate with an equal angle ω, but of opposite sign, about the z axis passing through their centers as shown in Fig. 1. Due to symmetry about the $z = 0$ plane, it is sufficient to consider the

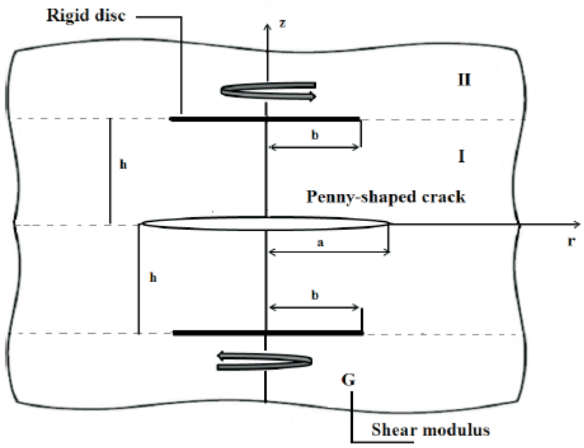

Figure 1 Geometry and coordinate system

problem in the upper half–space only $z \geq 0$. For the static axisymmetric torsion of a homogeneous isotropic material and linear elastic behaviour, the only nonzero displacement component is $u_\theta(r, z)$. Then the only non-zero components stresses are related to the displacement component by:

$$\tau_{\theta z} = G\frac{\partial u_\theta}{\partial z}, \qquad \tau_{\theta r} = Gr\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) \tag{1}$$

where G is the shear modulus of the material. The displacement satisfies the following differential equation:

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial u_\theta}{r\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = 0 \tag{2}$$

By means of the Hankel's transformation integral and its inverse given by [19]:

$$F(\lambda, z) = \int_0^\infty f(r, z) r J_1(\lambda r) dr$$

and

$$f(r,z)=\int_0^\infty F(\lambda,z)\lambda J_1(\lambda r)d\lambda$$

where J_1 is the Bessel function of the first kind of order one. The general solution of the above differential equation is furnished for the regions $I(0\le z\le h)$ and $II(z\ge h)$ as shown in fig.1 by

$$u_\theta^{(i)}(r,z)=\int_0^\infty \left[A_i(\lambda)e^{-\lambda z}+B_i(\lambda)e^{\lambda z}\right]J_1(\lambda r)d\lambda \qquad i=1,2 \tag{3}$$

where A_i and B_i are unknown functions.

3. Boundary and continuity conditions

Let us assume the contact between the disc and the elastic half–space is perfectly bonded all along their common interface. We consider the symmetry plane condition, the regularity conditions at infinity, the boundary and continuity conditions at $z=0$ and $z=h$, as shown in the following. The regularity conditions at infinity are given by:

$$\lim_{r,z\to\infty} u_\theta^{(i)}(r,z)=0 \qquad \lim_{r,z\to\infty} \tau_{\theta z}^{(i)}(r,z)=0 \tag{4}$$

The boundary conditions of the problem are:

$$\tau_{\theta z}^{(1)}(r,0^+)=0 \qquad r<a \tag{5}$$

$$u_\theta^{(1)}(r,0)=0 \qquad r\ge a \tag{6}$$

$$u_\theta^{(2)}(r,h^+)=u_\theta^{(1)}(r,h^-)=\omega r \qquad r\le b \tag{7}$$

The continuity conditions of the problem in the planes $z=h$ can be written as:

$$\tau_{\theta z}^{(2)}(r,h^+)-\tau_{\theta z}^{(1)}(r,h^-)=0 \qquad r>b \tag{8}$$

$$u_\theta^{(2)}(r,h^+)-u_\theta^{(1)}(r,h^-)=0 \qquad r>b \tag{9}$$

With the regularity conditions at infinity given by Eq. (4), the expressions of displacements and stresses in the two regions take the following forms:

$$u_\theta^{(1)}(r,z)=\int_0^\infty \left[A_1(\lambda)e^{-\lambda z}+B_1(\lambda)e^{\lambda z}\right]J_1(\lambda r)d\lambda \tag{10}$$

$$\tau_{\theta z}^{(1)}(r,z)=G\int_0^\infty \lambda\left[-A_1(\lambda)e^{-\lambda z}+B_1(\lambda)e^{\lambda z}\right]J_1(\lambda r)d\lambda \tag{11}$$

$$u_\theta^{(2)}(r,z)=\int_0^\infty \left[A_2(\lambda)e^{-\lambda z}\right]J_1(\lambda r)d\lambda \tag{12}$$

$$\tau_{\theta z}^{(2)}(r,z)=-G\int_0^\infty \lambda\left[A_2(\lambda)e^{-\lambda z}\right]J_1(\lambda r)d\lambda \tag{13}$$

The unknown functions $A_1(\lambda)$, $B_1(\lambda)$ and $A_2(\lambda)$ can be determined from the boundary and continuity conditions. The boundary and continuity conditions Eq. (7) and Eq. (9) show that:

$$u_\theta^{(2)}(r,h^+)-u_\theta^{(1)}(r,h^-)=0 \tag{14}$$

From the above condition, we obtain

$$A_2(\lambda) = A_1(\lambda) + B_1(\lambda)e^{2\lambda h} \tag{15}$$

The mixed boundary conditions Eqs. (6–8) are converted to the following system of dual integral equations for obtained the two unknowns A_1 and B_1:

$$\int_0^\infty \lambda[B_1(\lambda) - A_1(\lambda)]J_1(\lambda r)d\lambda = 0 \qquad 0 \le r < a \tag{16}$$

$$\int_0^\infty [A_1(\lambda) + B_1(\lambda)]J_1(\lambda r)d\lambda = 0 \qquad r \ge a \tag{17}$$

$$\int_0^\infty [A_1(\lambda)e^{-\lambda h} + B_1(\lambda)(e^{\lambda h})]J_1(\lambda r)d\lambda = \omega r \qquad 0 \le r \le b \tag{18}$$

$$\int_0^\infty \lambda B_1(\lambda)e^{\lambda h}J_1(\lambda r)d\lambda = 0 \qquad r > b \tag{19}$$

4. Reduction of the problem to a system of Fredholm integral equations

In reducing the system of dual equations to a system of Fredholm integral equations, we begin by introducing the the auxiliary functions $\phi(t)$ and $\psi(t)$ such that:

$$A_1(\lambda) + B_1(\lambda) = \sqrt{\lambda}\int_0^a \sqrt{t}\phi(t)J_{\frac{3}{2}}(\lambda t)dt \tag{20}$$

$$B_1(\lambda)e^{\lambda h} = \sqrt{\lambda}\int_0^b \sqrt{t}\psi(t)J_{\frac{1}{2}}(\lambda t)dt \tag{21}$$

witch give us:

$$A_1(\lambda) = \sqrt{\lambda}\int_0^a \sqrt{t}\phi(t)J_{\frac{3}{2}}(\lambda t)dt - e^{-\lambda h}\sqrt{\lambda}\int_0^b \sqrt{t}\psi(t)J_{\frac{1}{2}}(\lambda t)dt \tag{22}$$

$$B_1(\lambda) = e^{-\lambda h}\sqrt{\lambda}\int_0^b \sqrt{t}\psi(t)J_{\frac{1}{2}}(\lambda t)dt \tag{23}$$

where $\phi(t)$ and $\psi(t)$ are continuous unknown functions of t, defined over two intervals $0 \le t < a$ and $0 \le t \le b$ respectively. This choice implies that the homogeneous equations Eq. (17) and Eq. (19) are identically satisfied while the equations Eq. (16) and Eq. (18) lead to the Fredholm's integral equations.
Substituting $A_1(\lambda)$ and $B_1(\lambda)$ in the equations Eq. (16)and Eq. (18), we get:

$$\begin{aligned}&\int_0^a \sqrt{t}\phi(t)dt\int_0^\infty \lambda^{\frac{3}{2}}f_{11}(\lambda)J_{\frac{3}{2}}(\lambda t)J_1(\lambda r)d\lambda \\ &+\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty \lambda^{\frac{3}{2}}f_{12}(\lambda)J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda = 0 \qquad r < a\end{aligned} \tag{24}$$

$$\begin{aligned}&\int_0^a \sqrt{t}\phi(t)dt\int_0^\infty \sqrt{\lambda}f_{21}(\lambda)J_{\frac{3}{2}}(\lambda t)J_1(\lambda r)d\lambda \\ &+\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty \sqrt{\lambda}f_{22}(\lambda)J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda = \omega r \qquad r < b\end{aligned} \tag{25}$$

where:

$$f_{11}(\lambda)=1 \qquad f_{12}(\lambda)=-2e^{-\lambda h} \qquad f_{21}(\lambda)=e^{-\lambda h} \qquad f_{22}(\lambda)=1-e^{-2\lambda h}$$

Eq. (24) can be converted to the Abel integral equation by means of the relation:

$$\lambda J_1(\lambda r)=\frac{1}{r^2}\frac{d}{dr}[r^2 J_2(\lambda r)]$$

and the integral formula:

$$\int_0^\infty \sqrt{\lambda} J_{\frac{3}{2}}(\lambda t) J_2(\lambda r) d\lambda = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{t^{\frac{3}{2}}}{r^2\sqrt{r^2-t^2}} & t<r \\ 0 & t>r \end{cases}$$

we obtain Abel equation corresponding to equation Eq. (24):

$$\sqrt{\frac{2}{\pi}}\int_0^r \frac{t^2\phi(t)}{\sqrt{r^2-t^2}}dt + r^2\int_0^a \sqrt{t}\phi(t)dt\int_0^\infty \lambda^{\frac{1}{2}}(f_{11}(\lambda)-1)J_{\frac{3}{2}}(\lambda t)J_2(\lambda r)d\lambda$$
$$+r^2\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty \lambda^{\frac{1}{2}} f_{12}(\lambda)J_{\frac{1}{2}}(\lambda t)J_2(\lambda r)d\lambda = 0 \qquad r<a \tag{26}$$

Next, we invert the last equation by applying the Abel's transform formula:

$$\int_0^r \frac{f(t)}{\sqrt{r^2-t^2}}dt = g(r) \quad \text{then} \quad f(t)=\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{rg(r)}{\sqrt{t^2-r^2}}dr$$

to obtain:

$$t^2\phi(t)=\sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t \frac{r^3}{\sqrt{t^2-r^2}}[-\int_0^a \sqrt{\delta}\phi(\delta)d\delta\int_0^\infty \lambda^{\frac{1}{2}}(f_{11}(\lambda)-1)J_{\frac{3}{2}}(\lambda\delta)J_2(\lambda r)d\lambda$$
$$-\int_0^b \sqrt{\delta}\psi(\delta)d\delta\int_0^\infty \lambda^{\frac{1}{2}} f_{12}(\lambda)J_{\frac{1}{2}}(\lambda\delta)J_2(\lambda r)d\lambda]dr \quad r<a \tag{27}$$

For the left hand side of the above equation, the integral is further simplified by using the following relationship:

$$\sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t \frac{r^3}{\sqrt{t^2-r^2}}J_2(\lambda r)dr = \sqrt{\lambda}t^{\frac{5}{2}}J_{\frac{3}{2}}(\lambda t)$$

we obtain the first Fredholm integral equation of second kind:

$$\phi(t)+\sqrt{t}\int_0^a \sqrt{\delta}\phi(\delta)K(t,\delta)d\delta + \sqrt{t}\int_0^b \sqrt{\delta}\psi(\delta)L(t,\delta)d\delta = 0, \qquad r<a \tag{28}$$

where:

$$K(t,\delta)=\int_0^\infty \lambda(f_{11}(\lambda)-1)J_{\frac{3}{2}}(\lambda t)J_{\frac{3}{2}}(\lambda\delta)d\lambda$$
$$L(t,\delta)=\int_0^\infty \lambda f_{12}(\lambda)J_{\frac{3}{2}}(\lambda t)J_{\frac{1}{2}}(\lambda\delta)d\lambda$$

Following the same procedure as before to reduce Eq. (24) to the second Fredholm integral equation. Using the formula:

$$\int_0^\infty \sqrt{\lambda} J_{\frac{1}{2}} \lambda t J_1(\lambda r) d\lambda = \begin{matrix} \sqrt{\frac{2t}{\pi}} \frac{1}{r\sqrt{(r^2-t^2)}} & t < r \\ 0 & t > r \end{matrix} \tag{29}$$

we obtain the following Abel type equation:

$$\frac{1}{r}\sqrt{\frac{2}{\pi}} \int_0^r \frac{t\psi(t)}{\sqrt{r^2-t^2}} dt + \int_0^b \sqrt{t}\psi(t)dt \int_0^\infty \sqrt{\lambda}(f_{22}(\lambda)-1) J_{\frac{1}{2}}(\lambda t) J_1(\lambda r) d\lambda$$
$$+ \int_0^a \sqrt{t}\phi(t)dt \int_0^\infty \sqrt{\lambda} f_{21}(\lambda) J_{\frac{3}{2}}(\lambda t) J_1(\lambda r) d\lambda = \omega r \qquad r < b \tag{30}$$

Now, we invert the above equation by applying the Abel's transform formula to get:

$$t\psi(t) = \sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^t \frac{r^2}{\sqrt{t^2-r^2}} \omega r$$
$$- \int_0^b \sqrt{\delta}\psi(\delta) d\delta \int_0^\infty \sqrt{\lambda}(f_{22}(\lambda)-1) J_{\frac{1}{2}}(\lambda\delta) J_1(\lambda r) d\lambda \tag{31}$$
$$- \int_0^a \sqrt{\delta}\phi(\delta) d\delta \int_0^\infty \sqrt{\lambda} f_{21}(\lambda) J_{\frac{3}{2}}(\lambda\delta) J_1(\lambda r) d\lambda] dr \quad r < b$$

Using the following relationship:

$$\frac{d}{dt} \int_0^t \frac{r^3}{\sqrt{(t^2-r^2)}} dr = 2t^2$$
$$\sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^t \frac{r^2 J_1(\lambda r)}{\sqrt{(t^2-r^2)}} dr = t\sqrt{\lambda t} J_{\frac{1}{2}}(\lambda t) \tag{32}$$

we finally get the second Fredholm integral equation of second kind:

$$\psi(t) + \sqrt{t} \int_0^a \sqrt{\delta}\phi(\delta) M(t,\delta) d\delta + \sqrt{t} \int_0^b \sqrt{\delta}\psi(\delta) N(t,\delta) d\delta = \frac{4\omega}{\sqrt{2\pi}} t, \quad 0 < t < b \tag{33}$$

with the kernel:

$$M(t,\delta) = \int_0^\infty \lambda f_{21}(\lambda) J_{\frac{1}{2}}(\lambda t) J_{\frac{3}{2}}(\lambda\delta) d\lambda \tag{34}$$

$$N(t,\delta) = \int_0^\infty \lambda (f_{22}(\lambda)-1) J_{\frac{1}{2}}(\lambda t) J_{\frac{1}{2}}(\lambda\delta) d\lambda \tag{35}$$

The system (28) and (31) can be written in the dimensionless form as follows. By putting:

$$\begin{cases} \delta = as, & 0 < \delta < a; \quad t = au \quad 0 < t < a \\ \delta = bs, & 0 < \delta < b; \quad t = bu \quad 0 < t < b \end{cases}$$

Next, we multiply the above two equations of the system , respectively by $\frac{\sqrt{2\pi}}{4a\omega}\phi(au)$ and $\frac{\sqrt{2\pi}}{4b\omega}\psi(bu)$ and using the following substitutions:

$$\left\{ \begin{array}{l} \Phi(u) = \frac{\sqrt{2\pi}}{4a\omega}\phi(au) \quad \Psi(u) = \frac{\sqrt{2\pi}}{4b\omega}\psi(bu) \\ c = \frac{a}{b} \quad \lambda = \frac{x}{b} \quad H = \frac{h}{b} \end{array} \right. \tag{36}$$

hence:

$$\Phi(u) + c^2\sqrt{u}\int_0^1 \sqrt{s}\Phi(s)K(u,s)ds + \frac{1}{\sqrt{c}}\sqrt{u}\int_0^1 \sqrt{s}\Psi(s)L(u,s)ds = 0,\ u < 1 \tag{37}$$

$$\Psi(u) + c^{\frac{5}{2}}\sqrt{u}\int_0^1 \sqrt{s}\Phi(s)M(u,s)ds + \sqrt{u}\int_0^1 \sqrt{s}\Psi(s)N(u,s)ds = u,\ u < 1 \tag{38}$$

where:

$$\begin{aligned} K(u,s) &= 0 \\ L(u,s) &= \int_0^\infty x f_{12}(x) J_{\frac{3}{2}}(xcu) J_{\frac{1}{2}}(xs)dx \\ M(u,s) &= \int_0^\infty x f_{21}(x) J_{\frac{1}{2}}(xu) J_{\frac{3}{2}}(xcs)dx \\ N(u,s) &= \int_0^\infty x(f_{22}(x) - 1) J_{\frac{1}{2}}(xu) J_{\frac{1}{2}}(xs)dx \end{aligned}$$

5. Numerical results and discussion

The quadrature rule is used in evaluating the Fredholm integral equations given by Eq. (33) and Eq. (34). Let us divide the interval $[0,1]$ into N subintervals of length $\frac{1}{N}$, so that $u = u_m = \frac{2m-1}{2N}, s = u_n = \frac{2n-1}{2N}$ $m, n = 1, 2, ..., N$. For simplicity, we adopt the following notations:

$$\begin{aligned} &\Phi(u_m) = \Phi_m, \Psi(u_m) = \Psi_m \\ &L(u_m,u_n) = L_{mn},\ M(u_m,u_n) = M_{mn},\ N(u_m,u_n) = N_{mn} \end{aligned}$$

we get the following system of algebraic equations for determination of the coefficients Φ and Ψ:

$$\Phi_m + \frac{1}{N\sqrt{c}}\sqrt{u_m}\sum_{n=1}^{N}\sqrt{u_n}\Psi_n L_{mn} = 0 \quad m = 1, 2, ..., N \tag{39}$$

$$\Psi_m + \frac{c^{\frac{5}{2}}}{N}\sqrt{u_m}\sum_{n=1}^{N}\sqrt{u_n}\Phi_n M_{mn} + \frac{1}{N}\sqrt{u_m}\sum_{n=1}^{N}\sqrt{u_n}\Psi_n N_{mn} = u_m \tag{40}$$

$$m = 1, 2, ..., N$$

where:

$$K_{mn} = 0$$
$$L_{mn} = \int_0^\infty x f_{12}(x) J_{\frac{3}{2}}(xcu_m) J_{\frac{1}{2}}(xu_n) dx$$
$$M_{mn} = \int_0^\infty x f_{21}(x) J_{\frac{1}{2}}(xu_m) J_{\frac{3}{2}}(xcu_n) dx$$
$$N_{mn} = \int_0^\infty x(f_{22}(x) - 1) J_{\frac{1}{2}}(xu_m) J_{\frac{1}{2}}(xu_n) dx$$

Next, we evaluate numerically the infinite integral L, M, and N by simpson rule. After solving the above system, the unknown coeffcients can be obtained. Then we get the numerical approximation of the unknown functions A_1, B_1 and A_2 given by Eq. (22), Eq. (23) and Eq. (15):

$$A_1(x) = \frac{4b^2\omega}{N\sqrt{2\pi}}\sqrt{x}\sum_{m=1}^{N}\sqrt{u_m}[c^{\frac{5}{2}}\Phi_m J_{\frac{3}{2}}(xcu_m) - e^{-xH}\Psi_m J_{\frac{1}{2}}(xu_m)] \quad (41)$$

$$B_1(x) = \frac{4b^2\omega}{N\sqrt{2\pi}} e^{-xH}\sqrt{x}\sum_{m=1}^{N}\sqrt{u_m}\Psi_m J_{\frac{1}{2}}(xu_m) \quad (42)$$

and $$A_2(x) = A_1(x) + B_1(x)e^{2\lambda h} \quad (43)$$

5.1. ***Stress intensity factors***

The stress intensity factors at the edge of the crack and at the rim of disc are defined respectively as:

$$K^a_{\mathrm{III}} = \lim_{r\to a^+}\sqrt{2\pi(r-a)}\tau^{(1)}_{\theta z}(r,z)|_{z=0} \quad (44)$$
$$K^b_{\mathrm{III}} = \lim_{r\to b^-}\sqrt{2\pi(b-r)}\tau^{(1)}_{\theta z}(r,z)|_{z=h} \quad (45)$$

On the plane $z = 0$ for $r \geq a$, the expression of stress is given by:

$$\tau^{(1)}_{\theta z}(r,0) = G\int_0^\infty \lambda^{\frac{3}{2}}[-\int_0^a \sqrt{t}\phi(t)J_{\frac{3}{2}}(\lambda t)dt + 2e^{-\lambda h}\int_0^b \sqrt{t}\psi(t)J_{\frac{1}{2}}(\lambda t)dt]J_1(\lambda r)d\lambda \quad (46)$$

On the plane $z = h$, the expression of stress is given by:

$$\tau^{(1)}_{\theta z}(r,h) = G\int_0^\infty \lambda^{\frac{3}{2}}[-e^{-\lambda h}\int_0^a \sqrt{t}\phi(t)J_{\frac{3}{2}}(\lambda t)dt + (1+e^{-2\lambda h})\int_0^b \sqrt{t}\psi(t)J_{\frac{1}{2}}(\lambda t)dt]J_1(\lambda r)d\lambda \quad (47)$$

Using the relation:

$$J_1(\lambda R) = -\frac{1}{\lambda}\frac{d}{dR}J_0(\lambda R)$$

we obtain:

$$\tau_{\theta z}^{(1)}(r,0) = G\int_0^a \sqrt{t}\phi(t)dt\int_0^\infty \lambda^{\frac{1}{2}}J_{\frac{3}{2}}(\lambda t)J_0(\lambda r)d\lambda$$
$$+2G\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty \lambda^{\frac{3}{2}}e^{-\lambda h}J_{\frac{1}{2}}(\lambda t)dtJ_1(\lambda r)d\lambda \tag{48}$$

$$\tau_{\theta z}^{(1)}(r,h) = -G\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty \lambda^{\frac{1}{2}}J_{\frac{1}{2}}(\lambda t)J_0(\lambda r)d\lambda - G\int_0^a \sqrt{t}\phi(t)dt$$
$$\int_0^\infty e^{-\lambda h}\lambda^{\frac{3}{2}}J_1(\lambda r)d\lambda + G\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty \lambda^{\frac{3}{2}}e^{-2\lambda h}J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda \tag{49}$$

We use the following asymptotic behavior of the Bessel function of the first kind, for large values of λ:

$$J_\nu(\lambda) \simeq \sqrt{\frac{2}{\lambda\pi}}\cos(\lambda - \frac{\pi}{2}\nu - \frac{\pi}{4})$$

then we get:

$$J_{3/2}(\lambda t) \simeq \sqrt{\frac{2}{\lambda t\pi}}\cos(\lambda t - \pi) = -\sqrt{\frac{2}{\lambda t\pi}}\cos(\lambda t)$$
$$J_{1/2}(\lambda t) \simeq \sqrt{\frac{2}{\lambda t\pi}}\cos(\lambda t - \frac{\pi}{2}) = \sqrt{\frac{2}{\lambda t\pi}}\sin(\lambda t)$$

Next, we use the following integral formula for the first infinite integral in the right part of the Eq. (44) and Eq. (45) respectively:

$$\int_0^\infty \cos(\lambda t)J_0(\lambda r)d\lambda = \begin{cases} \frac{1}{\sqrt{r^2-t^2}} & r > t \\ 0 & r < t \end{cases}$$

$$\int_0^\infty sin(\lambda t)J_0(\lambda r)d\lambda = \begin{cases} 0 & r > t \\ \frac{1}{\sqrt{t^2-r^2}} & r < t \end{cases}$$

we obtain:

$$\tau_{\theta z}^{(1)}(r,0) = -\sqrt{\frac{2}{\pi}}G\frac{d}{dr}\int_0^a \frac{\phi(t)}{\sqrt{r^2-t^2}}dt + R_1(r) \tag{50}$$

$$\tau_{\theta z}^{(2)}(r,h) = -\sqrt{\frac{1}{2\pi}}G\frac{d}{dr}\int_0^b \frac{\psi(t)}{\sqrt{t^2-r^2}}dt + R_2(r) \tag{51}$$

where:

$$R_1(r) = 2G\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty \lambda^{\frac{3}{2}}e^{-\lambda h}J_{\frac{1}{2}}(\lambda t)dtJ_1(\lambda r)d\lambda$$
$$R_2(r) = -G\int_0^a \sqrt{t}\phi(t)dt\int_0^\infty e^{-\lambda h}\lambda^{\frac{3}{2}}J_1(\lambda r)d\lambda$$
$$+G\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty \lambda^{\frac{3}{2}}e^{-2\lambda h}J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda$$

Now integrating by parts, we get:

$$\tau_{\theta z}^{(1)}(r,h_1)=\frac{G\sqrt{2}}{\sqrt{\pi}}\left[\frac{a\phi(a)}{r\sqrt{r^2-a^2}}-\int_0^a\frac{t\phi'(t)}{r\sqrt{r^2-t^2}}dt\right]+R_1(r) \quad (52)$$

$$\tau_{\theta z}^{(1)}(r,h_2)=\frac{G}{\sqrt{2\pi}}\left[\frac{b\psi(b)}{r\sqrt{b^2-r^2}}-\int_r^b\frac{1}{r}\frac{t\psi'(t)}{\sqrt{t^2-r^2}}dt\right]+R_2(r) \quad (53)$$

The stress intensity factors at $r=a$ and at $r=b$, may be calculated as:

$$K_{\text{III}}^a=\lim_{r\to a^+}\sqrt{2\pi(r-a)}\frac{G\sqrt{2}}{\sqrt{\pi}}\left(\frac{a\phi(a)}{r\sqrt{r^2-a^2}}\right) \quad (54)$$

$$K_{\text{III}}^b=\lim_{r\to b^-}\sqrt{2\pi(b-r)}\frac{G}{\sqrt{2\pi}}\left(\frac{b\psi(b)}{r\sqrt{b^2-r^2}}\right) \quad (55)$$

By using the following transformations:

$$\phi(a)=\frac{4a\omega}{\sqrt{2\pi}}\Phi_N \qquad \psi(b)=\frac{4b\omega}{\sqrt{2\pi}}\Psi_N \qquad \rho=\frac{r}{b}$$

we obtain:

$$K_{\text{III}}^a=\frac{4G\omega\sqrt{a}}{\sqrt{\pi}}\Phi_N \quad (56)$$

$$K_{\text{III}}^b=\frac{4G\omega\sqrt{b}}{\sqrt{\pi}}\Psi_N \quad (57)$$

Fig. 2 shows the results of the effect of the normalized crack size a/b on the stress intensity factor K_{III}^a defined by Eq. (56) for different disc locations $H=1;0.75;0.5$ and 0.25. It is observed that the values of the stress intensity factor versus a/b increase, attain its maximum values and then decrease to zero. The effect of the distance between the crack and the disc H on the stress intensity factor is also shown in Fig. 2. The increase of the height H induces the decrease of stress intensity factor for all values of a/b.
Fig. 3 illustrates the variation of the normalized stress intensity factor K_{III}^b at the edge of the rigid inclusion defined by Eq. (57) versus a/b for $H=1;0.75;0.5$ and 0.25. Relatively, small variation for smaller values of a/b and considerable variation for larger values of a/b are observed. Also, it can be seen that the interaction between the crack and the rigid discs is greater when the discs are closer to the crack. In addition to the interaction, the stress intensity factor K_{III}^b decreases as the crack radius increases.

5.1.1. *Displacement and stress fields*

By substituting the Eqs. (37)–(39) in the expressions of the displacements and the stresses Eqs. (10)–(132), we get the numerical results of displacements and stresses for the two regions.

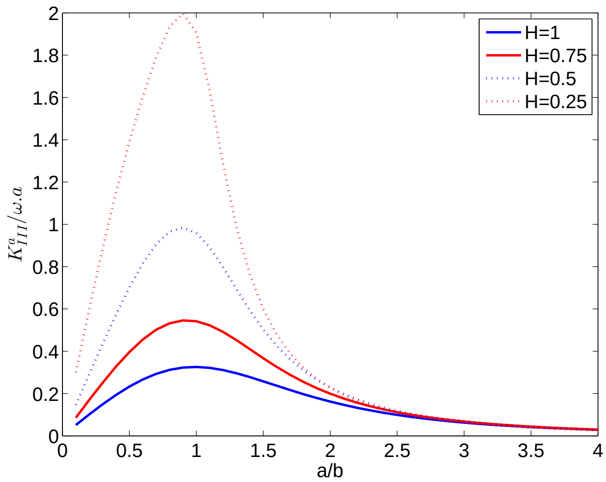

Figure 2 Variation of the normalized stress intensity factor at the edge of the crack K^a_{III} with a/b

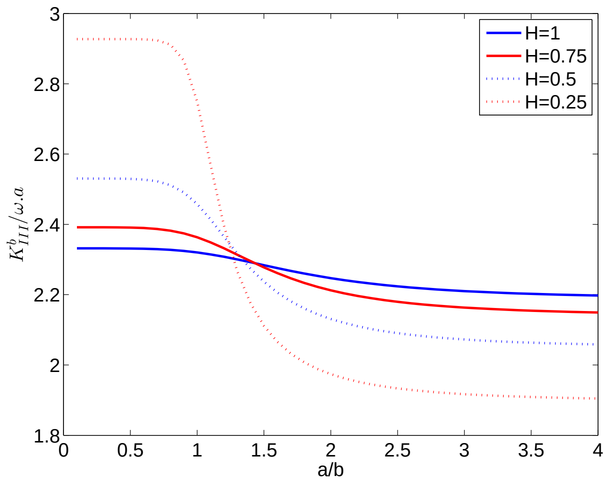

Figure 3 Variation of the normalized stress intensity factor at the edge of the rigid disc K^b_{III} with a/b

The results for the variation of the normalized displacement $u^{(i)}_\theta(\rho, \xi)/\omega a$ and stress $\tau^{(i)}(\rho, \xi)/\omega a$ with $\rho = r/b$ are shown graphically in Fig. 4 to Fig. 5 for the different values of the dimensionless axial distances $\xi = z/a$. For each region, five different axial distances are selected as $I(\xi = 0; H/4; H/2; 3H/4; H)$ and $I(\xi = H; 5H/4; 3H/2; 7H/4; 2H)$, with the particular values of the height $H = 1$ and the dimensionless disc size $c = 1$.

The variation of the normalized displacements are shown in Fig. 4 and Fig. 5. We notice that the displacements in the two regions increase at first, reach maximum values at $\rho = 1$ then decrease out of the disc band with increasing ρ.

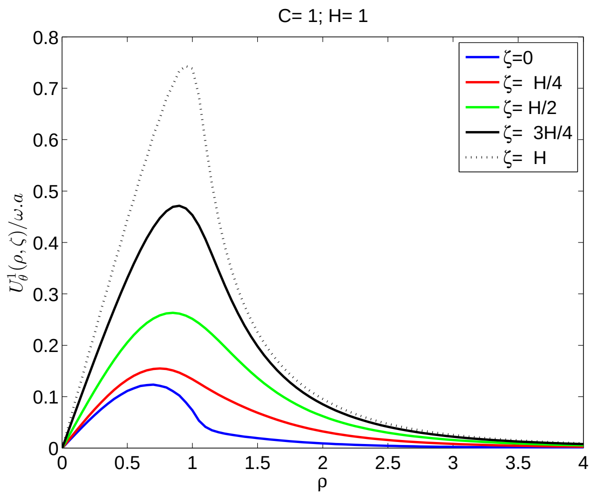

Figure 4 Tangential displacement u_θ^1 versus ρ for various ξ, $0 \leq z \leq h$

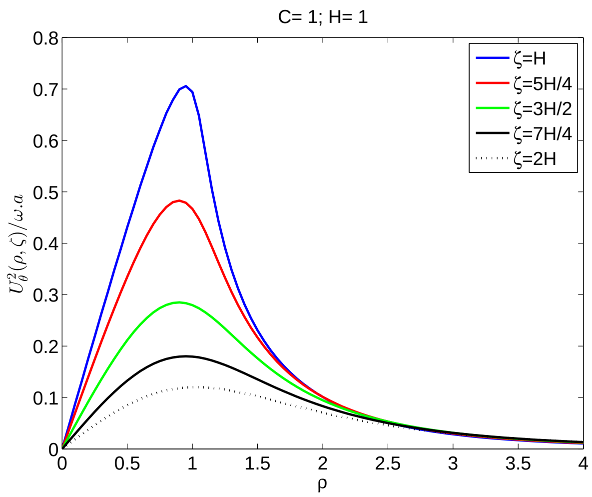

Figure 5 Tangential displacement u_θ^2 versus ρ for various ξ, $z \geq h$

The distribution of the shear stresses in the elastic medium is also discussed and shown in Fig. 6 and Fig. 7. The stresses are initially rise, attain its maximum values and with the increase in the value of ρ the stresses go on decreasing.

6. Conclusion

In this study, the axisymmetric torsion problem of two rigid discs symmetrically located embedded in the interior of a homogeneous elastic material is analytically addressed. The medium is weakened by a penny-shaped crack located parallel to the discs at the symmetry plane. Using the Hankel integral transformation method,

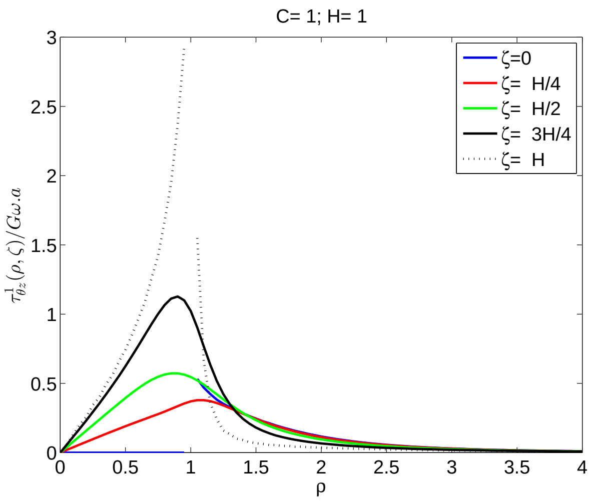

Figure 6 Shear stress $\tau^1_{\theta z}$ versus ρ for various ξ, $0 \leq z \leq h$

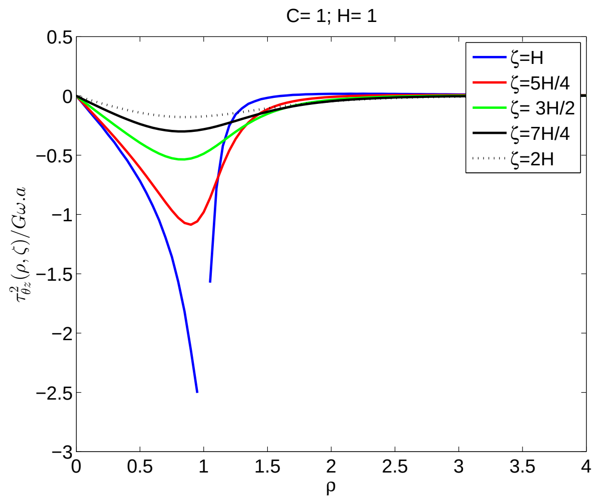

Figure 7 Tangential displacement $\tau^2_{\theta z}$ versus ρ for various ξ, $z \geq h$

the doubly mixed boundary value problem is reduced to a system of dual integral equations, which are transformed, to a Fredholm integral equations system of the second kind. The presented graphs show the variation of the displacements, the stresses and the stress intensity factor at the edge of the crack and at the rim of the disc for some dimensionless parameters. The numerical results show that the discontinuities around the crack and the inclusion cause a large increase in the stresses which decay with distance from the disc-loaded. Furthermore, it can be seen the dependence of the stress intensity factor on the crack size and the distance between the crack and the disc.

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