Mechanics and Mechanical Engineering
Vol. 21, No. 2 (2017) 291–308
© Lodz University of Technology

The Effect of Initial Stress on Generalized Thermoviscoelastic Medium with Voids and Temperature–Dependent Properties under Green–Neghdi Theory

Mohamed I. A. Othman
Faculty of Science
Department of Mathematics
Zagazig University
P.O. Box 44519, Zagazig, Egypt
m_i_a_othman@yahoo.com

Montaser Fekry
Faculty of Science
Department of Mathematics
South Valley University
P.O. Box 83523, Qena, Egypt
montaser.fekry@yahoo.com

Received (25 Januaryt 2017)
Revised (16 February 2017)
Accepted (19 March 2017)

The present paper aims to study the effect of initial stress on the 2–D problem of a homogeneous, isotropic, generalized thermo-viscoelastic material with voids in the context of Green–Naghdi theory. The modulus of elasticity is taken as a linear function of reference temperature. The analytical expressions for the physical quantities are obtained in the physical domain by using the normal mode analysis. These expressions are calculated numerically for a specific material and explained graphically. Comparisons are made with the results predicted by (G–N II) and (G–N III) theory in the presence and absence of the initial stress and temperature-dependent properties.

Keywords: Green–Naghdi, thermo–viscoelasticity, initial stress, temperature dependence, voids.

1. Introduction

The linear viscoelasticity remains an important area of research. Gross [1], Staverman, Schwarzl [2], Alfrey and Gurnee [3] and Ferry [4] investigated the mechanical model representation of linear viscoelastic behavior results. Solution of the boundary value problems for linear viscoelastic materials, including temperature variations in both quasi–static and dynamic problems made great strides in the last decades, in the work of Biot [5, 6] and Huilgol and Phan-Thien [7]. Bland [8] linked the solution of linear-viscoelasticity problems with corresponding linear elastic solutions. A notable works in this field were the work of Gurtin and Sternberg [9], and Ilioushin [10] offered an approximation method for the linear thermal viscoelastic problems. One can refer to the book of Ilioushin and Pobedria [11] for a formulation of the mathematical theory of thermal viscoelasticity and the solutions of some boundary value problems, as well as, to the work of Pobedria [12] for the coupled problems in continuum mechanics. Results of important experiments determining the mechanical properties of viscoelastic materials were involved in the book of Koltunov [13]. Othman [14] studied the uniqueness and reciprocity theorems for generalized thermo–viscoelasticity.

The heat conduction equations for the classical linear uncoupled and coupled thermoelasticity theories are of the diffusion type predicting infinite speed of propagation of heat wave contrary to physical observations. To eliminate this paradox inherent in the classical theories, generalized theories of thermoelasticity were developed. The generalized thermoelasticity theories admit so-called second-sound effects, that is, they predict the finite velocity of propagation for heat field. The first attempt towards the introduction of generalized thermoelasticity was headed by Lord and Shulman [15], who formulated the theory by incorporating a flux-rate term into conventional Fourier's law of heat conduction. The Lord-Shulman theory introduces a new physical concept which called a relaxation time. Since the heat conduction equation of this theory is of the wave-type, it automatically ensures finite speed of propagation of heat wave. The second generalization was developed by Green and Lindsay [16]. This theory contains two constants that act as relaxation times and modifies all the equations of coupled theory, not the heat conduction equation only. Later on, Green and Naghdi [17, 18, 19] proposed another three models, which are subsequently referred to as (GN–I), (GN–II), and (GN–III) models. The linearized version of model-I corresponds to the classical thermoelastic model–II for which the internal rate of production of entropy is taken to be identically zero, implying no dissipation of thermal energy. This model assumes un-damped thermoelastic waves in a thermoelastic material and is best known as the theory of thermoelasticity without energy dissipation. Model–III includes the previous two models as special cases, and assumes dissipation of energy in general.

The theory of elastic materials with voids is one of the most important generalizations of the classical theory of elasticity. This theory is concerned with elastic materials consisting of a distribution of small pores (voids, which contain nothing of mechanical or energetic significance) in which the void volume is included among the kinematic variables. Practically, this theory is useful for investigating various types of geological and biological materials for which elastic theory is inadequate. Nunziato and Cowin [20] studied a non-linear theory of elastic materials with voids. They showed that the changes in the volume fraction cause an internal dissipation in the material and this internal dissipation leads to a relaxation property in the material. Cowin and Nunziato [21] developed a theory of linear elastic materials with voids for the mathematical study of the mechanical behavior of porous solids. This linearized theory of elastic materials with voids is a generalization of the classical theory of elasticity and reduces to it when the dependence of change in volume fraction and its gradient are suppressed. In this theory, the volume fraction corresponding to void volume is taken as an independent kinematic variable. Puri and Cowin [22] studied the behavior of plane waves in a linear elastic material with voids. Domain of influence theorem in the linear theory of elastic materials with voids was discussed by Dhaliwal and Wang [23]. Dhaliwal and Wang [24] also developed a heat-flux dependent theory of thermoelasticity with voids. Cowin [25] studied the viscoelastic behavior of linear elastic materials with voids. Othman [26] has developed a linear theory of generalized thermo-viscoelasticity under three theories. While a nonlinear and linear theory of thermo–viscoelastic materials with voids studied by Ieşan [27]. Othman [28] investigated the uniqueness and reciprocity theorem for generalized thermo–viscoelasticity with thermal relaxation times, Othman *et. al* [29, 30] have studied much interest applications dealing of thermoelasticity with voids.

Most of the investigations were done under the assumption of temperature–independent material properties, which limit the applicability of the solutions obtained to certain ranges of temperature. Modern structural elements are often subjected to temperature change of such magnitude that their material properties may be longer be regarded as

having constant values even in an approximate sense. At high temperature the material characteristics such as modulus of elasticity, thermal conductivity and the coefficient of linear thermal expansion are no longer constants. The thermal and mechanical properties of the materials vary with temperature, so the temperature-dependent on the material properties must be taken into consideration in the thermal stress analysis of these elements. Noda [31] studied the thermal stresses in materials with temperature-dependent properties.

The initial stresses are developed in the medium due to many reasons, resulting from difference of temperature, process of quenching, shot pinning and cold working, slow process of creep, differential external of forces, gravity variations, etc. The Earth is supposed to be under high initial stresses. It is therefore of great interest to study the effect of these stresses on the propagation of stress waves. During the last five decades, considerable attenuation has been directed towards this phenomenon. Biot [32] depicted that the acoustic propagation under initial stresses would be fundamentally different from that under stress free state.

The present work is to obtain the physical quantities in a homogenous, isotropic, thermo-visco-elastic material with voids in the case of absence and presence of initial stress and temperature dependent. The model is illustrated in the context of (GN–II), and (GN–III) theories. The normal mode analysis is used to obtain the exact expressions for physical quantities. The distributions of considered variables are represented graphically.

2. Formulation of the problem

We consider a homogeneous, isotropic, thermally conducting viscoelastic half-space $z \geq 0$ with voids and temperature-dependent mechanical properties. For the two -dimensional problem we assume the dynamic displacement vector as $u = (u, 0, w)$. All quantities considered will be functions of the time variable t and of the coordinates x and z. The whole body is at a constant temperature T_0. The basic governing equations for a linear generalized visco-thermoelastic media with voids under the effect of initial stress and temperature-dependent properties in the absence of body forces are written by Ieşan [27] and Green and Naghdi [19]:

$$(\mu^* + p)\nabla^2 u + (\lambda^* + \mu^*)\nabla(\nabla \cdot u) - \beta^* \nabla T + b^* \nabla \phi = \rho\, \ddot{u} \tag{1}$$

$$A^* \nabla^2 \phi - \xi_1 \phi - \xi_2 \dot{\phi} - B^* (\nabla \cdot u) + (\tau \nabla^2 + m) T = \rho \chi \ddot{\phi} \tag{2}$$

$$\rho C_e \ddot{T} + \beta^* T_0 \ddot{e} + (m T_0 - \varsigma\, \nabla^2)\dot{\phi} = K \nabla^2 T + K^* \nabla^2 \dot{T} \tag{3}$$

And the constitutive relations are given by:

$$\sigma_{ij} = (\mu^* + p) u_{i,j} + \mu^* u_{j,i} + [\lambda^* u_{k,k} \; - \beta^* T + b^* \phi]\delta_{ij} \tag{4}$$

The parameters $\lambda^*, \mu^*, \beta^*, A^*, B^*$and $b^*$are defined as:

$$\lambda^* = \lambda(1 + \alpha_0 \frac{\partial}{\partial t}) \qquad \mu^* = \mu(1 + \alpha_1 \frac{\partial}{\partial t}) \qquad \beta^* = \beta(1 + \beta_0 \frac{\partial}{\partial t})$$

$$A^* = A(1 + \alpha_3 \frac{\partial}{\partial t}) \qquad B^* = b(1 + \alpha_4 \frac{\partial}{\partial t}) \qquad b^* = b(1 + \alpha_2 \frac{\partial}{\partial t}) \tag{5}$$

where: $\beta_0 = \frac{1}{\beta}(3\lambda\alpha_0 + 2\mu\alpha_1)\alpha_t,\ \beta = (3\lambda + 2\mu)\alpha_t,\ \lambda,\ \mu$ are the Lame' constants, σ_{ij} is the components of the stress tensor, p is the initial stress, ϕ is the volume fraction field, $A,\ \xi_1,\ \xi_2,\ B, \tau,\ \varsigma,\ m,\ \chi$ are the material constants due to the presence of voids, $T$is the temperature deviation from the reference temperature T_0, K, ρ and C_e are the thermal conductivity, density and specific heat at constant strain, $\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$ are the visco–elastic parameters, α_t is the coefficient of linear thermal expansion, e is the dilatation and δ_{ij} is Kronecker's delta. The dot notation is used to denote time differentiation. The strain tensor is $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),\ i, j = 1, 3$.

Our aim is to investigate the effect of temperature dependence of modulus of elasticity, keeping the other elastic and thermal parameters as constant. Therefore we may assume that:

$$\{\lambda, \mu, \beta, b, A, \xi_1, \xi_2, \tau, m, \chi, \varsigma, K\} = \left\{\bar{\lambda}, \bar{\mu}, \bar{\beta}, \bar{b}, \bar{A}, \bar{\xi}_1, \bar{\xi}_2, \bar{\tau}, \bar{m}, \bar{\chi}, \bar{\varsigma}, \bar{K}\right\} f(T) \tag{6}$$

where $\bar{\lambda},\ \bar{\mu},\ \bar{\beta},\ \bar{b},\ \bar{A},\ \bar{\xi}_1,\ \bar{\xi}_2,\ \bar{\tau},\ \bar{m},\ \bar{\chi},\ \bar{\varsigma},\ \bar{K}$ and $\bar{K}$ are constants and $f(T)$ is a given non–dimensional function of temperature such that $f(T) = 1 - \alpha^* T_0$ (where α^* is an empirical material constant). In the case of a temperature independent modulus of elasticity we have $f(T) = 1$. For $x - z$ plane, Eq. (1) gives rise to the following two equations:

$$[\mu(1 + \alpha_1 \frac{\partial}{\partial t}) + p]\nabla^2 u + [\lambda(1 + \alpha_0 \frac{\partial}{\partial t}) + \mu(1 + \alpha_1 \frac{\partial}{\partial t})]\frac{\partial e}{\partial x} - \beta(1 + \beta_0 \frac{\partial}{\partial t})\frac{\partial T}{\partial x} + b(1 + \alpha_2 \frac{\partial}{\partial t})\frac{\partial \phi}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{7}$$

$$[\mu(1 + \alpha_1 \frac{\partial}{\partial t}) + p]\nabla^2 w + [\lambda(1 + \alpha_0 \frac{\partial}{\partial t}) + \mu(1 + \alpha_1 \frac{\partial}{\partial t})]\frac{\partial e}{\partial z} - \beta(1 + \beta_0 \frac{\partial}{\partial t})\frac{\partial T}{\partial z} + b(1 + \alpha_2 \frac{\partial}{\partial t})\frac{\partial \phi}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{8}$$

For simplifications we shall use the following non-dimensional variables:

$$
\begin{array}{lllll}
x_i' = \dfrac{\varpi}{c_1} x_i & u_i' = \dfrac{\rho c_1 \varpi}{\bar{\beta} T_0} u_i & T' = \dfrac{T}{T_0} & \phi' = \dfrac{\varpi^2 \bar{\chi}}{c_1^2}\phi & t' = \varpi t \\
\sigma_{ij}' = \dfrac{\sigma_{ij}}{\bar{\beta} T_0} & p' = \dfrac{p}{\rho c_1^2} & \multicolumn{3}{l}{\{\alpha_0', \alpha_1', \alpha_2', \alpha_3', \alpha_4'\} = \varpi\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}} \\
c_1^2 = \dfrac{\bar{\lambda} + 2\bar{\mu}}{\rho} & c_2^2 = \dfrac{\bar{\mu}}{\rho} & \varpi = \dfrac{C_e(\bar{\lambda} + 2\bar{\mu})}{K} & &
\end{array} \tag{9}
$$

where, ϖ is the characteristic frequency of the material and c_1, c_2 are the longitudinal and shear wave velocities in the medium, respectively.
Using Eq. (9), then Eqs. (7), (8), (2) and (3) become respectively (dropping the dashed for convenience):

$$
\begin{aligned}
&[\delta^2(1 + \alpha_1 \frac{\partial}{\partial t}) + a_0 p]\nabla^2 u + [(1 - 2\delta^2)(1 + \alpha_0 \frac{\partial}{\partial t}) + \delta^2(1 + \alpha_1 \frac{\partial}{\partial t})]\frac{\partial e}{\partial x} \\
&- (1 + \beta_0 \frac{\partial}{\partial t})\frac{\partial T}{\partial x} + a_1(1 + \alpha_2 \frac{\partial}{\partial t})\frac{\partial \phi}{\partial x} = q\frac{\partial^2 u}{\partial t^2}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&[\delta^2(1 + \alpha_1 \frac{\partial}{\partial t}) + a_0 p]\nabla^2 w + [(1 - 2\delta^2)(1 + \alpha_0 \frac{\partial}{\partial t}) + \delta^2(1 + \alpha_1 \frac{\partial}{\partial t})]\frac{\partial e}{\partial z} \\
&- (1 + \beta_0 \frac{\partial}{\partial t})\frac{\partial T}{\partial z} + a_1(1 + \alpha_2 \frac{\partial}{\partial t})\frac{\partial \phi}{\partial z} = q\frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{11}
$$

$$
(1 + \alpha_3 \frac{\partial}{\partial t})\nabla^2 \phi - a_2(\phi + \xi\dot{\phi}) - a_3(1 + \alpha_4 \frac{\partial}{\partial t})e + (a_4 \nabla^2 + a_5)T = \frac{\ddot{\phi}}{\delta_1^2} \tag{12}
$$

$$
\ddot{T} + \varepsilon_1(1 + \beta_0 \frac{\partial}{\partial t})\ddot{e} + a_6\dot{\phi} - a_7\nabla^2\dot{\phi} = \varepsilon_2\nabla^2 T + \varepsilon_3\nabla^2\dot{T} \tag{13}
$$

where: $a_0 = \frac{q}{\rho c_1^2}$, $a_1 = \frac{\bar{b} c_1^2}{\varpi^2 \bar{\chi} \bar{\beta} T_0}$, $a_2 = \frac{\bar{\xi}_1 c_1^2}{\bar{A}\varpi^2}$, $a_3 = \frac{\bar{b}\bar{\chi}\bar{\beta} T_0}{\bar{A}\rho c_1^2}$, $a_4 = \frac{\bar{\tau}\varpi^2\bar{\chi} T_0}{\bar{A} c_1^2}$,

$$
a_5 = \frac{\bar{m}\bar{\chi} T_0}{\bar{A}}, \quad a_6 = \frac{\bar{m} c_1^2}{q\rho C_e \varpi^3 \bar{\chi}}, \quad a_7 = \frac{\bar{\varsigma}}{q\rho C_e \varpi \bar{\chi} T_0}, \quad \varepsilon_1 = \frac{\beta^2 T_0}{q\rho^2 c_1^2 C_e}, \quad \varepsilon_2 = \frac{\bar{K}}{q\rho C_e c_1^2},
$$

$$
\varepsilon_3 = \frac{K^*\varpi}{\rho C_e c_1^2}, \quad \delta^2 = \frac{c_2^2}{c_1^2}, \quad c_3^2 = \frac{\bar{A}}{\rho\bar{\chi}}, \quad \delta_1^2 = \frac{c_3^2}{c_1^2}, \quad \xi = \frac{\bar{\xi}_2\varpi}{\bar{\xi}_1}, \quad q = \frac{1}{f(T)}, \quad i, j = 1, 3.
$$

The non–dimensional constitutive relations are given by:

$$
\begin{aligned}
\sigma_{ij} = &\frac{1}{q}[(\delta^2(1 + \alpha_1 \frac{\partial}{\partial t}) + a_0 p)u_{i,j} + \delta^2(1 + \alpha_1 \frac{\partial}{\partial t})u_{j,i}] \\
&+ \frac{1}{q}[(1 - 2\delta^2)(1 + \alpha_0 \frac{\partial}{\partial t})u_{k,k} - (1 + \beta_0 \frac{\partial}{\partial t})T + a_1(1 + \alpha_2 \frac{\partial}{\partial t})\phi]\delta_{ij}
\end{aligned} \tag{14}
$$

The expressions relating displacement components $u(x,z,t)$, $w(x,z,t)$ to the potentials are:

$$
u = \Phi_{,x} + \Psi_{,z} \quad w = \Phi_{,z} - \Psi_{,x} \quad e = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \nabla^2\Phi \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \nabla^2\Psi \tag{15}
$$

Substituting from Eq. (15) into Eqs. (10)-(13), we obtain:

$$[\delta^2(1+\alpha_1\frac{\partial}{\partial t})+a_0p]\nabla^2\Psi = q\ddot{\Psi} \tag{16}$$

$$(1+a_0p+\delta_0\frac{\partial}{\partial t})\nabla^2\varPhi - (1+\beta_0\frac{\partial}{\partial t})T + a_1(1+\alpha_2\frac{\partial}{\partial t})\phi = q\ddot{\varPhi} \tag{17}$$

$$(1+\alpha_3\frac{\partial}{\partial t})\nabla^2\phi - a_2(\phi+\xi\dot{\phi}) - a_3(1+\alpha_4\frac{\partial}{\partial t})\nabla^2\varPhi + (a_4\nabla^2+a_5)T = \frac{\ddot{\phi}}{\delta_1^2} \tag{18}$$

$$\ddot{T}+\varepsilon_1(1+\beta_0\frac{\partial}{\partial t})\nabla^2\ddot{\varPhi} + a_6\dot{\phi} - a_7\nabla^2\dot{\phi} = \varepsilon_2\nabla^2 T + \varepsilon_3\nabla^2\dot{T} \tag{19}$$

where: $\delta_0 = \alpha_0 + 2\delta^2(\alpha_1 - \alpha_0)$.

3. Normal mode method

The solution of the considered physical variable can be decomposed in terms of normal modes as the following form:

$$[\varPhi, \varPsi, T, \phi, \sigma_{ij}](x,z,t) = [\bar{\varPhi}, \bar{\varPsi}, \bar{T}, \bar{\phi}, \bar{\sigma}_{ij}](z)\exp[\omega t + iax] \tag{20}$$

where, ω is the frequency, a is the wave number in the $x-$direction and $i=\sqrt{-1}$. Eqs. (16)-(19) with the aid of Eq. (20) become respectively:

$$(\mathrm{D}^2 - k_1^2)\,\bar{\varPsi} = 0, \tag{21}$$

$$(b_1\mathrm{D}^2 - b_2)\,\bar{\varPhi} - b_3\bar{T} + b_4\bar{\phi} = 0, \tag{22}$$

$$(b_5\mathrm{D}^2 - b_6)\,\bar{\varPhi} - (a_4D^2 - b_7)\bar{T} - (b_8D^2 - b_9)\bar{\phi} = 0, \tag{23}$$

$$(b_{10}\mathrm{D}^2 - b_{11})\,\bar{\varPhi} - (b_{12}D^2 - b_{13})\bar{T} - (b_{14}D^2 - b_{15})\bar{\phi} = 0 \tag{24}$$

where:

$$\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}z} \qquad k_1^2 = a^2 + \frac{q\omega^2}{\delta^2\,(1+\alpha_1\omega)+a_0p} \qquad b_1 = 1+a_0p+\omega\delta_0 \qquad b_2 = b_1a^2+q\omega^2$$

$$b_8 = 1+\alpha_3\omega \qquad b_9 = b_8a^2+a_2(1+\xi\omega)+\frac{\omega^2}{\delta_1^2} \qquad b_{10} = \varepsilon_1\omega^2(1+\beta_0\omega) \qquad b_{11} = b_{10}a^2$$

$$b_{12} = \varepsilon_2+\varepsilon_3\omega \qquad b_{13} = b_{12}a^2+\omega^2 \qquad b_{14} = a_7\omega \qquad b_{15} = b_{14}a^2+a_6\omega$$

Eliminating $\bar{T}$ and $\bar{\phi}$ between Eqs. (22)-(24) we get the following ordinary differential equation satisfied with $\bar{\varPhi}$:

$$(\mathrm{D}^6 - d_1\mathrm{D}^4 + d_2\mathrm{D}^2 - d_3)\,\bar{\varPhi} = 0 \tag{25}$$

where:

$$d_1 = \frac{f_1}{f_0} \qquad d_2 = \frac{f_2}{f_0} \qquad d_3 = \frac{f_3}{f_0} \qquad f_0 = b_1(a_4b_{14} - b_8b_{12})$$

$$f_1 = a_4(b_1b_{15}+b_2b_{14}-b_4b_{10}) - b_1(b_9b_{12}+b_8b_{13}-b_7b_{14}) + b_5(b_3b_{14}+b_4b_{12}) \\ -b_8(b_2b_{12}+b_3b_{10})$$

$$f_2 = b_1(b_7b_{15}-b_9b_{13})+b_2(a_4b_{15}+b_7b_{14}-b_8b_{13}-b_9b_{12})+b_3(b_5b_{15}+b_6b_{14}-b_8b_{11}-b_9b_{10}) + b_4(b_5b_{13}+b_6b_{12}-a_4b_{11}-b_7b_{10})$$

$$f_3 = b_2(b_7b_{15}-b_9b_{13})+b_3(b_6b_{15}-b_9b_{11})+b_4(b_6b_{13}-b_7b_{11})$$

In a similar manner we arrive at:

$$(\mathrm{D}^6 - d_1\mathrm{D}^4 + d_2\mathrm{D}^2 - \ d_3)\{\bar{T}, \bar{\phi}\} = 0 \tag{26}$$

Eq. (25) can be factored as:

$$(\mathrm{D}^2 - k_2^2)(\mathrm{D}^2 - k_3^2)(\mathrm{D}^2 - k_4^2)\,\bar{\Phi} = 0 \tag{27}$$

where, k_j^2 $(j = 2, 3, 4)$ are the roots of the characteristic equation of Eq. (25). The solution of (21) bound as $z \to \infty$, can be written as:

$$\bar{\Psi}(z) = R_1 e^{-k_1 z} \tag{28}$$

The solution of Eq. (27) bound as $z \to \infty$, is given by:

$$\bar{\Phi}(z) = \sum_{j=2}^{4} R_j e^{-k_j z} \tag{29}$$

Similarly, the solution of Eq. (26), can be written as:

$$\bar{T}(z), \bar{\phi}(z)\} = \sum_{j=2}^{4} \{S_{1j}, S_{2j}\} R_j e^{-k_j z} \tag{30}$$

where:

$$S_{1j} = \frac{b_1b_{14}k_j^4 + (b_4b_{10} - b_1b_{15} - b_2b_{14})k_j^2 + b_2b_{15} - b_4b_{11}}{(b_3b_{14} + b_4b_{12})k_j^2 - b_3b_{15} - b_4b_{13}}$$

$$S_{2j} = \frac{-\ b_1b_{12}k_j^4 + (b_3b_{10} + b_1b_{13} + b_2b_{12})k_j^2 - b_2b_{13} - b_3b_{11}}{(b_3b_{14} + b_4b_{12})k_j^2 - b_3b_{15} - b_4b_{13}} \quad j = 2, 3, 4$$

Substituting Eqs. (28), (29) and (30) into Eq. (20) we get:

$$\Psi(z) = R_1 e^{(\omega\, t + iax - k_1 z)}$$

$$\Phi(z), T(z), \phi(z)\} = \sum_{j=2}^{4} \{1, S_{1j}, S_{2j}\} R_j e^{(\omega\, t + iax - k_j z)} \tag{31}$$

Inserting Eq. (31) in Eq. (15), the displacement components u and w, bound as $z \to \infty$ are obtained as:

$$u = (\sum_{j=2}^{4} iaR_j e^{-k_j z} - k_1 R_1 e^{-k_1 z}) e^{(\omega t + iax)} \tag{32}$$

$$w = -\ (\sum_{j=2}^{4} k_j R_j e^{-k_j z} +\ iaR_1 e^{-k_1 z}) e^{(\omega t + iax)} \tag{33}$$

The stress components and the chemical potential are of the form:

$$\sigma_{xx} = \{\sum_{j=2}^{4}[(1-2\delta^2)(1+\alpha_0\omega)k_j^2 - (1+a_0p+\delta_0\omega)a^2 - b_3S_{1j} + b_4S_{2j}]R_je^{-k_jz}$$

$$+\, iak_1[(1-2\delta^2)(1+\alpha_0\omega) - (1+a_0p+\delta_0\omega)]R_1e^{-k_1z}\}f(T)\, e^{(\omega t+iax)} \tag{34}$$

$$\sigma_{zz} = \{\sum_{j=2}^{4}[(1+a_0p+\delta_0\omega)k_j^2 - (1-2\delta^2)(1+\alpha_0\omega)a^2 - b_3S_{1j} + b_4S_{2j}]R_je^{-k_jz}$$

$$+\, iak_1[(1+a_0p+\delta_0\omega) - (1-2\delta^2)(1+\alpha_0\omega)]R_1e^{-k_1z}\}f(T)\, e^{(\omega t+iax)} \tag{35}$$

$$\sigma_{xz} = \big\{[\delta^2(1+\alpha_1\omega)(a^2+k_1^2) + a_0pk_1^2]R_1e^{-k_1z}$$

$$-\, ia[2\delta^2(1+\alpha_1\omega)+a_0p]\sum_{j=2}^{4}k_jR_je^{-k_jz}\big\}\, f(T)\, e^{(\omega t+iax)} \tag{36}$$

4. The boundary conditions

In order to determine the parameters $R_j (j = 1, 2, 3, 4)$ we need to consider the boundary condition at $z = 0$ as follows:
The mechanical boundary conditions:

$$\sigma_{zz} = -p_1N(x,t), \quad \sigma_{xz} = 0, \quad \frac{\partial\phi}{\partial z} = 0 \tag{37}$$

The thermal boundary condition: the surface of the half-space is subjected to a thermal shock:

$$T = p_2N(x,t) \tag{38}$$

where, p_1 is the magnitude of the mechanical force, p_2 is the constant of temperature applied to the boundary, and $N(x,t)$ is known function.
Substituting from the expressions of the variables considered into the boundary conditions (37), (38) respectively, we can obtain the following equations:

$$h_{11}R_1 + \sum_{j=2}^{4}h_{1j}R_j = -p_1 \tag{39}$$

$$h_{21}R_1 + \sum_{j=2}^{4}h_{2j}R_j = 0 \tag{40}$$

$$\sum_{j=2}^{4}h_{3j}R_j = 0 \tag{41}$$

$$\sum_{j=2}^{4}S_{1j}R_j = p_2 \tag{42}$$

where: $h_{11} = iak_1[(1 + a_0p + \delta_0\omega) - (1 - 2\delta^2)(1 + \alpha_0\omega)]f(T)$

$$h_{21} = [\delta^2(1 + \alpha_1\omega)(a^2 + k_1^2) + a_0pk_1^2]f(T)$$

$$h_{1j} = [(1 + a_0p + \delta_0\omega)k_j^2 - (1 - 2\delta^2)(1 + \alpha_0\omega)a^2 - b_3S_{1j} + b_4S_{2j}]f(T)$$

$$h_{2j} = -ia\,[2\delta^2(1 + \alpha_1\omega) + a_0p]k_jf(T) \qquad h_{3j} = -k_jS_{2j} \qquad j = 2, 3, 4$$

Solving Eqs. (39)-(42) for $R_j(j = 1,\ 2\ ,3\ ,4)$ by using the inverse of matrix method as follows:

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ 0 & h_{32} & h_{33} & h_{34} \\ 0 & S_{12} & S_{13} & S_{14} \end{pmatrix}^{-1} \begin{pmatrix} -p_1 \\ 0 \\ 0 \\ p_2 \end{pmatrix} \tag{43}$$

5. Numerical results and discussions

We will present some numerical results to illustrate the problem. The material chosen for the purpose of numerical computation is copper, the physical data for which are given by Ref. [33] in SI units:

$$\lambda = 7.76 \times 10^{10}\text{kg m}^{-1}\,\text{s}^{-2} \quad \mu = 3.86 \times 10^{10}\text{kg m}^{-1}\text{s}^{-2} \quad K = 386\,\text{W m}^{-1}\text{K}^{-1}$$

$$T_0 = 293\text{K}, \rho = 8954\ \text{kg m}^{-3} \quad \alpha_t = 1.78 \times 10^{-5}\text{K}^{-1} \quad C_e = 383.1\ \text{J kg}^{-1}\text{K}^{-1}$$

The voids parameters are:

$$A = 1.688\times 10^{-5}\text{kg m s}^{-2} \quad b = 1.139\times 10^{10}\text{kg m}^{-1}\text{s}^{-2} \quad m = 2\times 10^5\text{kg m}^{-1}\text{s}^{-2}\text{K}^{-1}$$

$$\chi = 1.75 \times 10^{-15}\text{m}^2 \quad \xi_1 = 1.475 \times 10^{10}\text{kg m}^{-1}\text{s}^{-2} \quad \xi_2 = 3.8402 \times 10^{-4}\text{kg m}^{-1}\,\text{s}^{-3}$$

$$\tau = 0.2 \times 10^{-5}\text{kg m}^{-1}\text{s}^{-2}\text{K}^{-1} \quad \varsigma = 0.1 \times 10^{-5}\text{kg m s}^{-2}$$

The comparisons were carried out for:

$$p_1 = 0.8 \quad p_2 = 0.03 \quad t = 0.4 \quad x = 0.8 \quad \omega = 1.6{+}1.4i \quad a = 1.2, 0 \le z \le 3 \quad q = 1.4$$

$$p = 0.25 \quad \alpha_0 = 3.25\times 10^{-2} \quad \alpha_1 = 3.91\times 10^{-2} \quad \alpha_2 = 6.51\times 10^{-2} \quad \alpha_3 = 1.02\times 10^4$$

$$\alpha_4 = 1.95 \times 10^{-2}$$

The above numerical technique was used for the distribution of the real parts of the displacement components $u$and w, the temperature T, the stress components $\sigma_{xx},\ \sigma_{zz},\ \sigma_{xz},\ \sigma_{zx}$ and the change in the volume fraction fieldϕ with distance z for (G–N II) and (G–N III) with and without initial stress effect in the presence of temperature-dependent properties which are shown graphically in the 2–D Figures 1–7. At $p = 0$ the solid lines represent the solution in the context of the (G–N II) and the dashed line represents the solution for the (G–N III). In the case of $p = 0.25$, the solid lines with squares represent the solution in the context of the (G–N II) and the dashed line with circles represents the solution for the (G–N III). Figures 8–14 clarify 2–D figures on the distribution of the physical quantities with distance z for (G–N II) and (G–N III) with and without temperature-dependent properties in the presence of initial stress.

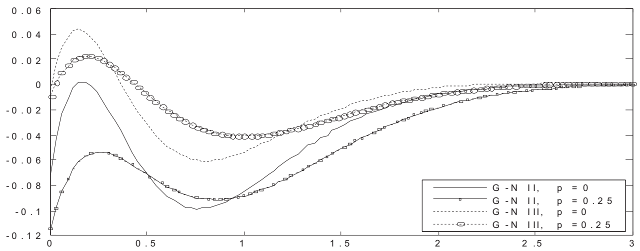

Figure 1 Variation of the displacement u ith horizontal distance $z$in the presence and absence of initial stress

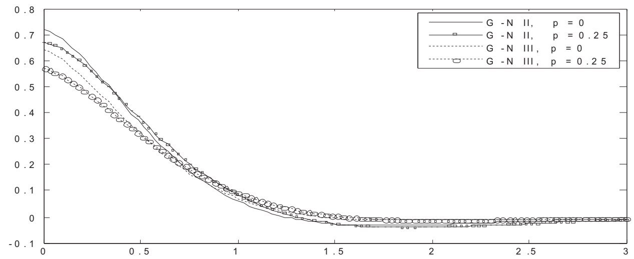

Figure 2 Variation of the displacement w with horizontal distance $z$in the presence and absence of initial stress

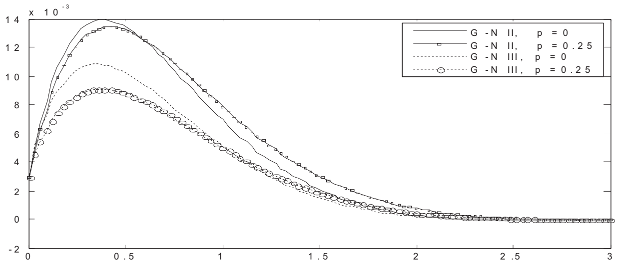

Figure 3 Variation of the temperature T with horizontal distance z in the presence and absence of initial stress

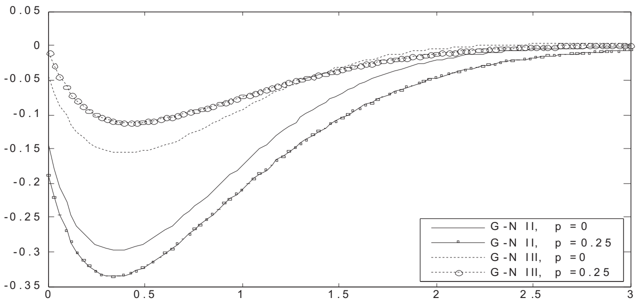

Figure 4 Variation of the stress σ_{xx} with horizontal distance $z$in the presence and absence of initial stress

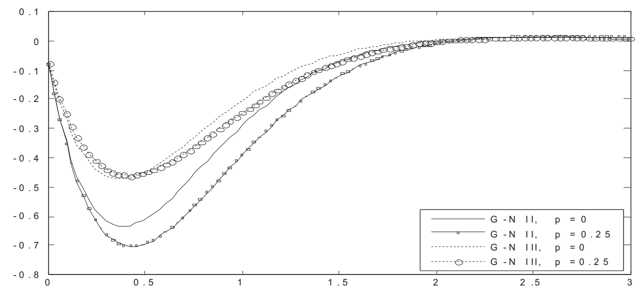

Figure 5 Variation of the stress σ_{zz} with horizontal distance $z$in the presence and absence of initial stress

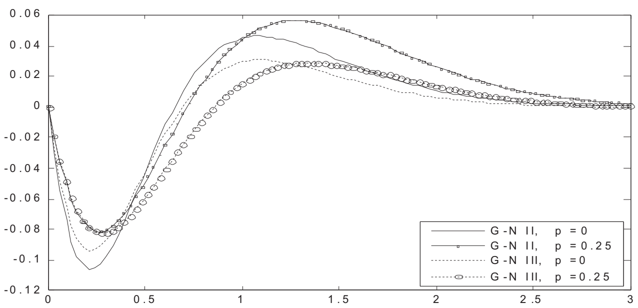

Figure 6 Variation of the stress σ_{xz} with horizontal distance $z$in the presence and absence of initial stress

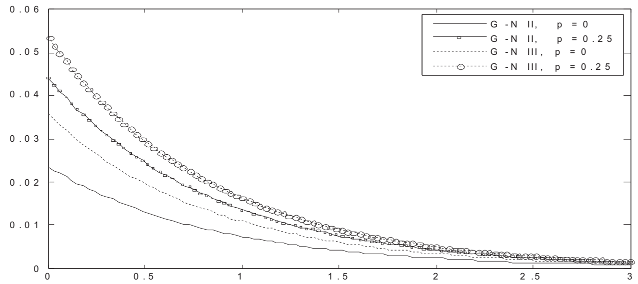

Figure 7 Variation of the volume fraction field ϕ with horizontal distance z in the presence and absence of initial stress

At $q = 1$ the solid line represents the solution in the context of the (G–N II) and the dashed line represents the solution for the (G–N III). In the case of $q = 1.4$, the solid line with squares represents the solution in the context of the (G–N II) and the dashed line with circles represents the solution for the (G–N III). Here all the variables are taken in non–dimensional form. Fig. 1 depicts that the distribution of the horizontal displacement component u, always begins from negative values for $p = 0,\ p = 0.25$. In the context of (G–N II) the distribution of u at $p = 0.25$ is higher than that at $p = 0$ in the range $0.53 < z < 0.95$, while conversely in the other ranges, and in the context of (G-N III) at $p = 0.25$, it is higher than that at $p = 0$ in the range $0.3 < z < 1.25$ and conversely in the other ranges. Fig. 2 shows the distribution of the displacement component w in the case of $p = 0$, $p = 0.25$. In the context of (G–N II (the distribution of w at $p = 0.25$ is larger than that at $p = 0$ in the range $0.4 < z < 1.6$, then, conversely in the other ranges. But in the context of (G-N III) at $p = 0.25$ it is larger than that at $p = 0$ in the range $0.55 < z < 2.5$, and conversely in the other ranges. Fig. 3 explains that the distribution of temperature Tbegins from a positive value in the case of $p = 0,\ p = 0.25$. In the context of (G–N II) the distribution of T at $p = 0$ is higher than that at $p = 0.25$ in the range $0 < z < 0.6$, but conversely in the range $z > 0.6$, while in the context of (G–N III) at $p = 0$ it is higher than that at $p = 0.25$ in the range $0 < z < 1.1$, and conversely in the range $z > 1.1$. Fig.e 4 expresses the distribution of the stress component σ_{xx}in the case of $p = 0,\ p = 0.25$. In the context of (G–N II) the distribution of σ_{xx} at $p = 0$ is larger than that at $p = 0.25$ for $z > 0$, while in the context of (G–N III) at $p = 0.25$ it is larger than that at $p = 0$ in the range $0 < z < 1.35$, but conversely in the range $z > 1.35$. Figure 5 expresses the distribution of the stress component σ_{zz} in the case of $p = 0,\ p = 0.25$. In the context of (G–N II) the distribution of σ_{zz} at $p = 0$ is higher than that at $p = 0.25$ in the range $0 < z < 2.4$, and conversely in the range $z > 2.4$. However, in the context of (G-N III) at $p = 0.25$ it is higher than it at $p = 0$ in the range $0 < z < 0.48$, then conversely in the range $z > 0.48$. Fig. 6 expresses the distribution of the stress component σ_{xz} in the case of $p = 0,\ p = 0.25$. In the context of (G–N II) the distribution of σ_{xz} at $p = 0$ is greater than that at $p = 0.25$ in the range $0.45 < z < 1$, and conversely in the other ranges, while in the context of (G-N III) at $p = 0$ is greater than that at $p = 0.25$ in the range

$0.35 < z < 1.3$, and conversely in the other ranges. Fig. 7 depicts the distribution of the change in the volume fraction field ϕ for $p = 0$, $p = 0.25$. In the context of (G-N II) and (G-N III) the distribution of ϕ at $p = 0.25$ is larger than that at $p = 0$ for $z > 0$. It explains that all the curves converge to zero, and the initial stress is significant on the distributions of all physical functions. Fig. 8 depicts that the distribution of the horizontal displacement component u, always begins from negative values for $q = 1,\ 1.4$. In the context of (G-N II) the distribution of u at $q = 1.4$ is higher than that at $q = 1$ for $z > 0$, and in the context of (G-N III) at $q = 1$ it is higher than that at $q = 1.4$ in the range $0.4 < z < 1.2$,, then conversely in the other ranges. Fig. 9 shows the distribution of the displacement component w in the case of $q = 1,\ 1.4$. In the context of (G–N II) the distribution of $w$at $q = 1$ is greater than that at $q = 1.4$ in the range $0.4 < z < 1.9$, and conversely in the other ranges, but in the context of (G-N III) at $q = 1$ it is greater than that at $q = 1.4$ in the range $0.6 < z < 2.4$, then conversely in the other ranges.

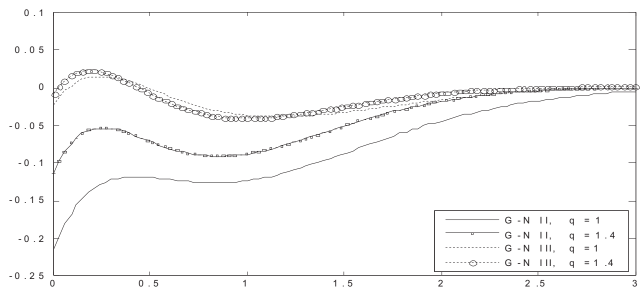

Figure 8 Variation of the displacement $u$with horizontal distance z in the presence and absence of temperature–dependent properties

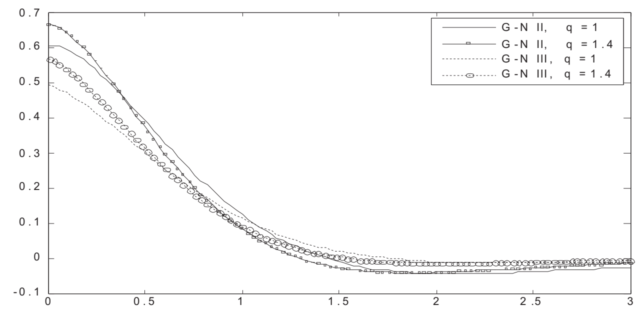

Figure 9 Variation of the displacement $w$with horizontal distance z in the presence and absence of temperature–dependent properties

Fig. 10 explains the distribution of temperature T begins from a positive value in the case of $q = 1,\ 1.4$. In the context of (G-N II) the distribution of Tat $q = 1.4$ is higher than that at $q = 1$ in the range $0 < z < 0.75$, and conversely in the range $z > 0.75$, while in the context of (G-N III) at $q = 1.4$ it is higher than that at $q = 1$ in the range $0 < z < 1.25$, then conversely in the range $z > 1.25$. Fig. 11 expresses the distribution of the stress component σ_{xx} in the case of $q = 1,\ 1.4$. In the context of (G-N II) the distribution of σ_{xx} at $q = 1.4$ is larger than that at $q = 1$ for $z > 0$, but in the context of (G-N III) at $q = 1$ it is larger than that at $q = 1.4$ in the range $0.2 < z < 0.7$, and conversely in the range $z > 0.7$.

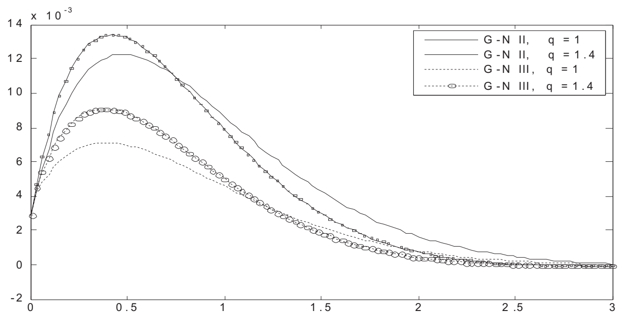

Figure 10 Variation of the temperature $T$with horizontal distance $z$in the presence and absence of temperature–dependent properties

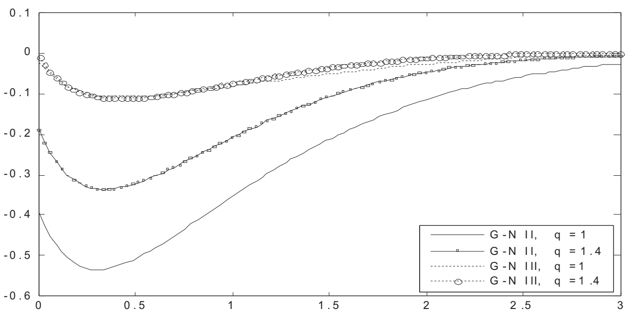

Figure 11 Variation of the stress $\sigma_{xx}$with horizontal distance $z$in the presence and absence of temperature–dependent properties

Fig. 12 expresses the distribution of the stress component σ_{zz} in the case of $q = 1,\ 1.4$. In the context of (G–N II) the distribution of σ_{zz} at $q = 1.4$ is larger than that at $q = 1$ for $z > 0$, while in the context of (G–N III) at $q = 1.4$ it is larger than that at $q = 1$ in the range $0 < z < 0.4$, and conversely in the range $z > 0.4$. Figure 13 expresses the distribution of the stress component σ_{xz} in the case of $q = 1,\ 1.4$. In the context of (G–N II) the distribution of σ_{xz} at $q = 1$ is greater than that at $q = 1.4$ for $z > 0$, but in the context of (G–N III) at $q = 1.4$ it is greater than that at $q = 1$ in the range $0 < z < 1.4$ and conversely in the range $z > 1.4$. Fig. 14 depicts the distribution of the change in the volume fraction field ϕ for $q = 1,\ 1.4$. In the context of (G-N II) the distribution of ϕ at $q = 1.4$ is higher than that at $q = 1$ for $z > 0$, while in the context of (G–N III) at $q = 1$ it is higher than that at $q = 1.4$ for $z > 0$. It explains that all the curves converge to zero, and the temperature–dependent properties is significant of the distributions of all physical functions.

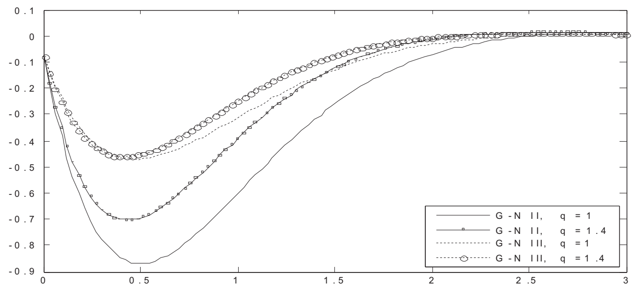

Figure 12 Variation of the stress $\sigma_{zz}$with horizontal distance $z$in the presence and absence of temperature-dependent properties

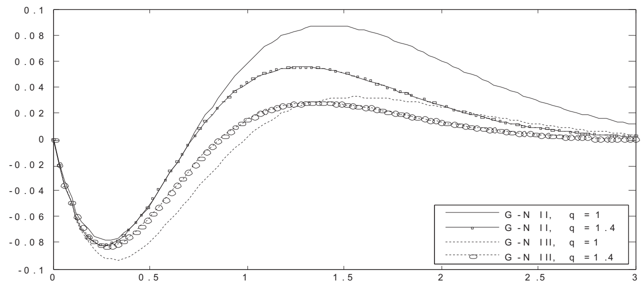

Figure 13 Variation of stress $\sigma_{xz}$with horizontal distance $z$in the presence and absence of temperature–dependent properties

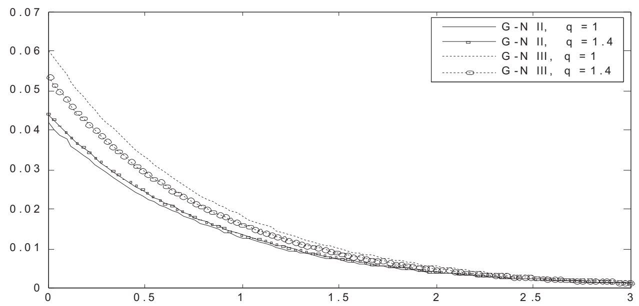

Figure 14 Variation of the volume fraction field ϕ with horizontal distance z in the presence and absence of temperature–dependent properties

6. Conclusions

According to the above analysis, we can conclude that the reference temperature–dependent modulus play an important role on all the physical quantities. The presence and absence of the initial stress in the current model has a significant effect. The normal mode analysis has been used which is applicable to a wide range of problems in thermoviscoelasticity. This method gives exact solutions without any assumed restrictions on the actual physical quantities that appear in the governing equations of the physical problem considered. The value of all physical quantities converges to zero with the increase of distance and all of them are continuous. It noticed that the thermoviscoelastic materials with voids have an important role in the distribution of the field quantities, since the amplitudes of these quantities is varying (increasing or decreasing) with the changes of the initial stress and the reference temperature–dependent modulus. Finally, it deduced that the deformation of a body depends on the nature of the applied forces and the initial stress effect as well as the type of boundary conditions.

References

[1] **Gross, B.**: Mathematical structure of the theories of viscoelasticity, *Hermann*, Paris, **1953**.

[2] **Staverman, A., Schwarzl, F. and Stuart, H.**: Die physik der Hochpolymeren, Band IV, *Springer, Berlin*, **1956**.

[3] **Alfrey, T. and Gurnee, E. F.**: Rheology Theory and Applications, *Ed. Eirich F. R.*, 1, *Academic Press*, New York, **1956**.

[4] **Ferry, J. D.**: Viscoelastic properties of polymers, *John Wiley & Sons*, **1980**.

[5] **Biot, M.**: Variational principles in irreversible thermodynamics with application to viscoelasticity, *Physical Review*, 97, 6, 1463, **1955**.

[6] **Biot, M.**: Theory of stress–strain relations in anisotropic viscoelsticity and relaxation phynomenoa, *J. Appl. Phys.*, 25, 11, 1385–1391, **1954**.

[7] **Huilgol, R. R. and Phan–Thien, N.**: Fluid Mechanics of Viscoelasticity: General Principles, Constitutive Modelling, Analytical and Numerical Techniques, 6, *Elsevier*, **1997**.

[8] **Bland, D. R.**: The Theory of Linear Viscoelasticity, 10, *Pergamon Press, New York*, **1960**.

[9] **Gurtin, M. E. and Sternberg, E.**: On the linear theory of viscoelasticity, *Archive for Rational Mechanics and Analysis*, 11, 1, 291–356, **1962**.

[10] **Ilioushin, A.**: The approximation method of calculating the constructures by linear thermal viscoelastic theory, *Mekhanika Polimerov*, 2, 168178, **1968**.

[11] **Ilioushin, A. and Pobedria, B.**: Mathematical Theory of Thermal Viscoelasticity, *Nauka*, Moscow, **1970**.

[12] **Pobedria, B.**: Coupled Problems in Continuum Mechanics, *J. Durability and Plasticity*, 1, **1984**.

[13] **Koltunov, M.**: Creeping and Relaxation, *Vysshaya Shkola*, Moscow, **1976**.

[14] **Othman, M. I. A.**: The uniqueness and reciprocity theorems for generalized thermovisco- elasticity with thermal relaxation times. *Mech. and Mech. Eng.*, 7, 2, 77–87, **2004**.

[15] **Lord, H. W. and Shulman, Y.**: A generalized dynamical theory of thermoelasticity, *J. the Mech. and Phys. of Solids*, 15, 5, 299–309, **1967**.

[16] **Green, A. and Lindsay, K.**: Thermoelasticity, *J. Elast.*, 2, 1, 1–7, **1972**.

[17] **Green, A. and Naghdi, P.**: A re-examination of the basic postulates of thermomechanics, *Proceedings of the Royal Society of London. Series A: Math. and Phys. Sci.*, 432, 171–194, **1991**.

[18] **Green, A. and Naghdi, P.**: On undamped heat waves in an elastic solid, *J. Thermal Stresses*, 15, 2, 253–264, **1992**.

[19] **Green, A. and Naghdi, P.**: Thermoelasticity without energy dissipation, *J. Elast.*, 31, 3, 189–208, **1993**.

[20] **Nunziato, J. W. and Cowin S. C.**: A nonlinear theory of elastic materials with voids, *Arch. for Rational Mech. and Anal.*, 72, 2, 175–201, **1979**.

[21] **Nunziato, J. W. and Cowin S. C.**: Linear elastic materials with voids, *J. Elast.*, 13, 3, 125–147, **1983**.

[22] **Puri, P. S. and Cowin, C.**: Plane waves in linear elastic materials with voids, *J. Elast.*, 15, 2, 167–183, **1985**.

[23] **Dhaliwal, R. S. and Wang, J.**: Domain of influence theorem in the theory of elastic materials with voids, *Int. J. Eng. Sci.*, 32, 11, 1823–1828, **1994**.

[24] **Dhaliwal, R. S. and Wang, J.**: A heat–flux dependent theory of thermoelasticity with voids, *Acta Mechanica*, 110, 1–4, 33–39, **1995**.

[25] **Cowin, S. C.**: The viscoelastic behavior of linear elastic materials with voids, *J. Elast.*, 15, 2, 185191, **1985**.

[26] **Othman, M. I. A.**: Generalized thermoviscoelasticity under three theories, *Mech. and Mechanical Engineering*, 13, 1, 25–44, **2009**.

[27] **Ieşan, D.**: On a theory of thermoviscoelastic materials with voids, *J. Elast.*, 104, 1–2, 369–384, **2011**.

[28] **Othman, M. I. A.**: The uniqueness and reciprocity theorem for generalized thermoviscoelasticity with thermal relaxation times, *Mech. and Mechanical Engineering*, 7, 2, 77–87, **2004**.

[29] **Othman, M. I. A. and Abd-Elaziz, E. M.**: The effect of thermal loading due to laser pulse in generalized thermoelastic medium with voids in dual phase lag model, *J. Thermal Stresses*, 38, 9, 1068–1082, **2015**.

[30] **Othman, M. I. A., Zidan, M. E. and Hilal, M. I.**: The effect of initial stress on thermoelastic rotating medium with voids due to laser pulse heating with energy dissipation, *J. Thermal Stresses*, 38, 8, 835–853, **2015**.

[31] **Noda, N.**: Thermal stresses in materials with temperature-dependent properties, *J. Thermal Stresses*, 1, 391–483, **1986**.

[32] **Biot, M.**: Mechanics of incremental deformations, *Wiley*, New York, **1965**.

[33] **Othman, M. I. A., Hasona, W. and Eraki, E.**: The effect of initial stress on generalized thermoelastic medium with three–phase–lag model under temperature dependent properties, *Can. J. Phys.*, 92, 448–457, **2014**.