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Elastic Waves in Spatially Periodic Anisotropic Media: Diffraction by Periodic Arrays of Voids or Anisotropic Inclusions

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A two-scale asymptotic analysis coupled with the spatially periodic fundamental solutions are used for analyzing diffraction of elastic bulk waves propagating in anisotropic media containing periodic inclusions or voids. Explicit equations are derived for the scattering cross sections and velocities of bulk waves propagating in spatially periodic media with arbitrary elastic anisotropy.

Keywords: scattering cross section, bulk wave, anisotropy, dispersed composite.

1. Introduction

A theoretical method based on the two-scale asymptotic expansions and spatially periodic fundamental solutions is applied to the analysis of energy variation and diffraction of elastic waves propagating in a heterogeneous anisotropic elastic medium containing periodically distributed anisotropic inclusions or voids. The considered heterogeneous medium is modeled by the deterministic approach utilizing a regular spatial lattice with inclusions located in the corresponding nodes. The considered medium with inclusions can have different kinds of lattices, each having inclusions of specific geometry and orientation placed at the corresponding nodes.

It is assumed that both medium and inclusions are elastically anisotropic with no restrictions imposed on the kind of elastic anisotropy. It is also assumed that the displacement and strain fields are infinitesimally small, so equations of linear elasticity can be applied.

The closest solutions in mechanics of heterogeneous media with inclusions can be obtained by application of the two-scale asymptotic analysis [1 – 3]. In this method it is assumed that two fields exist: (i) the global field that can be described by "slow" variables; and, (ii) the local field, having high frequency oscillations; this rapidly oscillating field can be described by "fast" variables.

In the two-scale asymptotic methods the effective elasticity tensor can be represented in a following form:

$$\mathbf{C}_0 = \sum_{p=1}^{N} f_p \mathbf{C}_p + \mathbf{K} \qquad \sum_{p=1}^{N} f_p = 1 \tag{1}$$

where $\mathbf{C}_0$ is the effective elasticity tensor, f_p is the volume fracture of the p-th component, $\mathbf{C}_p$ is the elasticity tensor of the p-th component, N is the total number of different components of the heterogeneous medium, and $\mathbf{K}$ is a correcting tensor, or "corrector". The main difficulty lies in finding the corrector.

Remark: Equation (1) covers almost all existing methods of homogenization by choosing different expressions for the corrector:

a) At $\mathbf{K} = 0$ expression (1) yields Voigt's homogenization.

b) Taking:

$$\mathbf{K} = \sum_{p=1}^{N} -f_p \mathbf{C}_p + \left(\sum_{p=1}^{N} f_p \mathbf{C}_p^{-1} \right)^{-1} \tag{2}$$

and assuming that at any p tensors $\mathbf{C}_p$ are invertible along with $\left(f_p \mathbf{C}_p^{-1}\right)$, Eqs. (1) and (2) yield Reuss homogenization. Assumption that tensors $\mathbf{C}_p$ are invertible at any p is not valid for media with pores; in this case the Reuss homogenization cannot give a non-trivial solution for the homogenized elasticity tensor.

Determination of the corrector in the two-scale asymptotic method demands the solution of the cell problem consisting in (i) setting up a boundary-value problem on the internal boundaries between inclusion(s) and the matrix material in the translational invariant cell; and, (ii) formulating a periodic boundary-value problem on the outer boundary of the cell.

Along with FEM and finite differences methods, the following other methods for constructing solutions to the cell problem are known. In [4 – 6], methods based on the Eshelby's transformation strain were applied to analyses of isotropic media with ellipsoidal inclusions. In [7, 8], media with isotropic components were studied by applying a method based on the periodic fundamental solutions for isotropic medium originally constructed in [9]. Because of multipolar expansions used for the solution of the inner boundary value problem this method is confined to inclusions of spherical form. Galerkin's technique for solution of the inner boundary value problem was used in [10].

Periodic fundamental solutions for media with arbitrary anisotropy were developed in [11] coupled with the boundary integral equation method (BIEM); that approach was applied to solution of the cell problem for composites with anisotropic inhomogeneities and porous anisotropic media in [12, 13], analysis of microstructural stresses in the matrix material was considered in [14]. Some dynamic problems were studied in [15] by the same method.

Scattering of elastic waves in the dispersed composites and porous media are generally studied at the long wave assumption [15 – 20] when the wave length considerably surpasses the lattice period. Similar scattering problems were studied in [21, 22] with additional assumption of the constant wave speed in the cell components, this is known as the Rayleigh approximation. Non-linear effects related to wave scattering by inclusions or pores were analyzed in [23 – 25], along with some recent publications on nonlinear effects caused by scattering of elastic waves by periodically distributed voids or inclusions [27 – 29].

The principle target of the current research lies in deriving solutions for scattering cross sections for the plane harmonic waves scattered by the periodically distributed inclusions (or voids) in anisotropic elastic matrix.

2. Principle equations

A homogeneous elastic anisotropic medium is considered. The equations of equilibrium can be written in the form:

$$\mathbf{A}(\partial_x)\mathbf{u} = -\mathrm{div_x}\mathbf{C} \cdot \nabla_x \mathbf{C} = 0 \tag{3}$$

where $\mathbf{u}$ is a displacement field. It is assumed that the tensor of elasticity satisfies the condition of positive definiteness, which is generally adopted for problems of mechanics.

Applying the Fourier transform:

$$f^{\wedge}(\boldsymbol{\xi}) = \int f(\mathbf{x})\exp(2\pi i \mathbf{x} \cdot \boldsymbol{\xi})dx, \quad \xi \in R^3 \tag{4}$$

to Eqs. (3) gives the following symbol of the operator A:

$$\mathbf{A}^{\wedge}(\boldsymbol{\xi}) = (2\pi)^2 \boldsymbol{\xi} \cdot \mathbf{C} \cdot \boldsymbol{\xi} \tag{5}$$

From the definition of the fundamental solution E, the following formula for the corresponding symbol can be obtained:

$$\mathbf{E}^{\wedge}(\xi) = \mathbf{A}^{\wedge}(\xi)^{-1} \tag{6}$$

Expression (6) shows that symbol $E^{\wedge}$ is also strongly elliptic, positively homogeneous of degree -2 with respect to $|\xi|$, and analytical everywhere in $R^3 \backslash 0$.

Remark: Fourier inversion of expression (6) and procedures for constructing the fundamental solution, are discussed in [26].

3. Spatially periodic fundamental solution

Consider a homogeneous anisotropic medium, loaded by periodically distributed force singularities, located in nodes $\mathbf{m}$ of a spatial lattice Λ.

Let $\mathbf{a}_i, (i = 1, 2, 3)$ be linearly independent vectors of the main periods of the lattice, so that each of the nodes can be represented in the form:

$$\mathbf{m} = \sum_i m_i \mathbf{a}_i \tag{7}$$

where L^1 are the integer-valued coordinates of the node $\mathbf{m}$ in the basis $(\mathbf{a}_i)$. The adjoint basis $\overline{L^1}(Q, R^3 \otimes R^3)$ is introduced in such a manner that $\mathbf{a}_i^* \cdot \mathbf{m} = m_i$. The lattice corresponding to the adjoint basis is denoted by Λ^*.

Now, periodic delta-function corresponding to the singularities disposed at the nodes of the lattice $\overline{L^1}$ has the form:

$$\delta_p(\mathbf{x}) = V_Q^{-1} \sum_{\mathbf{m}^* \in \Lambda *} \exp(-2\pi i \mathbf{x} \cdot \mathbf{x}^*) \tag{8}$$

where V_Q is the volume of the fundamental region (cell) Q. Expression (8) defines the periodic delta-function uniquely.

Substitution of the periodic fundamental solution $\mathbf{E}_p$ in Eq. (3) yields:

$$\mathbf{A}(\partial_{\mathbf{x}})\mathbf{E}_p(\mathbf{X}) = \delta_p(\mathbf{x})\mathbf{I} \tag{9}$$

where $\mathbf{I}$ is the identity matrix. Looking for $\mathbf{E}_p$ also in the form of harmonic series and taking into account representation (8), it is possible to get:

$$\mathbf{E}_p(\mathbf{x}) = V_Q^{-1} \sum_{\mathbf{m}^* \in \Lambda_0^*} \mathbf{E} \wedge (\mathbf{m}^*) \exp(-2\pi i \mathbf{x} \cdot \mathbf{m}^*) \tag{10}$$

where $\Lambda_0 *$ is the adjoint lattice without the zero node. It should be noted that expression (10) defines a periodic fundamental solution up to an additive (tensorial) constant.

Lemma 1: The series on the right side of Eq. (10) is convergent in the $\overline{L^1}$-topology, defining the fundamental solution of the class $\overline{L^1}(Q, R^3 \otimes R^3)$, where $\overline{L^1}$ is a class of integrable in Q functions with the zero mean value.

Proof of the lemma can be found in [11].

4. Scattering cross sections

For simplicity it will be assumed that the considered medium has only one kind of uniformly distributed inhomogeneities placed at nodes of the spatial lattice Λ. The region occupied by an individual inhomogeneity in a cell Q is denoted by Ω.

The two-scale asymptotic analyses being applied to such a medium produces the following expression for the corrector [12]:

$$\mathbf{K} = -V_Q^{-1} \int_{\partial\Omega} \mathbf{C} \cdot (\nu_{\mathbf{Y}} \otimes \mathbf{H}(\mathbf{Y}))dY \tag{11}$$

where $\mathbf{Y}$ are the "fast" variables, $\mathbf{H}$ is the third-order tensorial field, being a solution of the following boundary value problem:

$$\begin{array}{ll} \mathbf{A}(\partial_Y)\mathbf{H}(\mathbf{Y}) = 0 & \mathbf{Y} \in Q \backslash \Omega \\ \mathbf{T}(\boldsymbol{\nu}_Y, \partial_Y)\mathbf{H}(\mathbf{Y})|_{\partial\Omega} = -\boldsymbol{\nu}_Y \cdot \mathbf{C} & \end{array} \tag{12}$$

In Eqs. (11) and (12) ν_Y represents a field of external unit normals to the boundary $\partial\Omega$, and the elasticity tensor Ω is defined by:

$$\mathbf{C} = \mathbf{C}_2 - \mathbf{C}_1 \tag{13}$$

where $\mathbf{C}_2$ is referred to the matrix material, and $\mathbf{C}_1$ to inclusions. Strong ellipticity of the tensor $\mathbf{C}$ is also assumed.

Lemma 2: Under assumptions stated above, boundary-value problem admits the unique solution.

Proof of the lemma can be found in [11, 12].

Remark: Supposition that the tensor $\mathbf{C}$ in the left-hand side of Eq. (13) is not strong elliptic, violates proof of Lemma 2.

Now, the solution of the boundary value problem (12) for the traction field can be constructed by applying boundary integral equation method, giving the following representation for the desired solution [12]:

$$(12\mathbf{I} + \mathbf{S})\,\mathbf{H}(\mathbf{Y}') = \mathbf{H}_c \qquad \mathbf{Y}' \in \partial\Omega \tag{14}$$

where $\mathbf{H}_c$ is a constant tensor, and $\mathbf{S}$ is a singular integral operator resulting from a restriction of the double-layer potential on the surface $\partial\Omega$. Some of the relevant properties of operator $\mathbf{S}$ are discussed in [13].

Substitution of Eq. (10) for periodic fundamental solutions in the expression for the operator S allows to obtain a lower (on energy) bound for the corrector; i.e.

$$\mathbf{K}_l = -8\pi^2 V_Q^{-2} \times \sum_{m^* \in \Lambda_0^*} \left(\hat{\chi}_\Omega(\mathbf{m}^*)\right)^2 \mathbf{C} \cdot \mathbf{m}^* \otimes \mathbf{E} \wedge (\mathbf{m}^*) \otimes \mathbf{m}^* \cdot \mathbf{C} \tag{15}$$

where $\chi \wedge_\Omega$ is the Fourier image of the characteristic function of the region Ω. An expression for the upper bound can be obtained similarly [12, 13].

Theorem: Series appearing on the right side of Eq. (15) is absolutely convergent, provided Ω is a proper open region in Q.

Proof of the theorem can be found in [12, 13].

Remark: Proof of convergence of the series analogous to (15) for very thin inclusions or cracks, is to be studied separately, as in this case a special asymptotic analysis is needed.

As was shown in [13, 14], the energy level W_{osc} of the microstructural highly oscillating stresses for the case of porous medium is defined by:

$$W_{osc} = 12\varepsilon_0 \cdot \mathbf{K} \cdot \varepsilon_0 \tag{16}$$

where ε_0 represents the uniform deformation field, and $\mathbf{K}$ is the corrector obtained by Eq. (15).

Similarly, having applied terminology used in quantum mechanics, the scattering cross-section S for the porous medium can be obtained by the following expression [15]:

$$S = (1-f)^{-1} \left| \frac{\varepsilon_0 \cdot \mathbf{K} \cdot \varepsilon_0}{\varepsilon_0 \cdot \mathbf{C} \cdot \varepsilon_0} \right| \tag{17}$$

where f is the porous ratio and $\mathbf{C}$ is the elasticity tensor for the matrix material, in expression (17) the homogeneous deformation field ε_0 corresponds to the amplitude deformation on the wave front:

$$\boldsymbol{\varepsilon}_0 = \frac{1}{2}\left(\mathbf{n} \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{a}\right) \tag{18}$$

In (18) a is the polarization vector of the bulk wave and n is the unit vector normal to the plane wave front. Polarization vector a in the right-hand side of (18) should satisfy the propagation condition

$$(\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n}) \cdot \mathbf{a} = \rho c^2 \mathbf{a} \tag{19}$$

where c is the speed of the corresponding bulk wave, and ρ is the density.

Remark: In [12-15] some examples for the corrector obtained by Eq. (15), and corresponding to inclusions or voids of some canonical shapes, are presented.

As can be seen from Eq. (17), the scattering cross-section heavily depends upon the corrector **K** (and the applied homogenization technique). For example, Voigt's homogenization (see Remark in Sec. 1) necessary leads to absence of any scattering irrespective of nature of a dispersed composite or porous media, while Reuss homogenization leads to infinite scattering cross-section for any porous medium. This underlines the fact of necessity to choose the closest technique for evaluating the corrector.

5. Conclusions

A two-scale asymptotic analyses method coupled with the spatially periodic fundamental solutions is developed for analyzing scattering elastic bulk waves propagating in anisotropic media with periodic inclusions or voids. Explicit equations are derived for the scattering cross sections and velocities for bulk waves propagating in spatially periodic media with arbitrary elastic anisotropy.

Finally, the developed two-scale asymptotic analysis coupled with the spatially periodic fundamental solutions allowed us to construct the converged series for the corrector tensor and to get the exact expressions for the scattering cross sections.

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