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# Optimal Control of an Inverted Pendulum Based on the New Method of Lyapunov Exponents Estimation

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This text covers optimization of an inverted pendulum control system according to the new control performance assessment criterion based on the optimal control theory. The novel control performance index is founded on the method of the Largest Lyapunov Exponent estimation. The detailed description of the new method is provided. Model of the control object is presented. A simple controller is proposed. Parameters of the controller are optimized with respect to the novel criterion by means of the Differential Evolution method. Results of numerical simulations are presented. It is shown that the new criterion can be successfully applied when the regulation time is crucial, whereas somewhat larger overshoot is acceptable.

*Keywords*: nverted pendulum, control, parameters optimization, Differential Evolution.

## 1. Introduction

Lyapunov Exponents (LE) are one of the commonly used tools for the analysis of non-linear dynamical systems. These exponents indicate the exponential convergence or divergence of trajectories that start close to each other. The existence of such numbers has been proved by Oseledec theorem [1], but the first numerical study of the system's behavior using Lyapunov exponents had been done by Henon and Heiles [2].

Recently, a simple and effective method of estimation of the Largest Lyapunov Exponent (LLE) from the perturbation vector and its derivative dot product has been presented. It is based on simple computations involving only basic mathematical operations such as summing, subtracting, multiplying, dividing. The LLE can be extracted from information known before each integration step. The method can be used in different aspects of the nonlinear systems control. The applications presented so far include: continuous systems [3], synchronization phenomena detection [4], time series in control systems [5-7].

In this paper, the new method of the LLE estimation is used as the foundation of the new control performance assessment criterion. To derive the novel control performance index, assumptions of the optimal control theory [8] are applied. Derivation of the new criterion is presented, its properties are explained. Features of the novel index are checked on an exemplary control system – the inverted pendulum. Equations of the control object are presented and the controller is proposed. Optimization of the controller parameters with respect to the new criterion is performed by means of the Differential Evolution method [9]. Finally, results of simulations are presented and conclusions are drawn.

## 2. The method

Consider a control system containing a control object (plant) and a controller (Fig. 1). Assume that the plant can be described as a non-autonomous, continuous-time dynamical system $\mathbf{f}(\mathbf{x}, \mathbf{u})$, where $\mathbf{x}$ is a state vector and $\mathbf{u}$ is an input (control) vector. Let the control error $\mathbf{e}$ be defined as the difference between the reference signal $\mathbf{x}_0$ and the current state $\mathbf{x}$ of the plant. The reference signal $\mathbf{x}_0$ determines the final value that the state vector of the plant is supposed to attain. Assume that the control signal $\mathbf{u}$ depends only on the value of the regulation error $\mathbf{e}$.

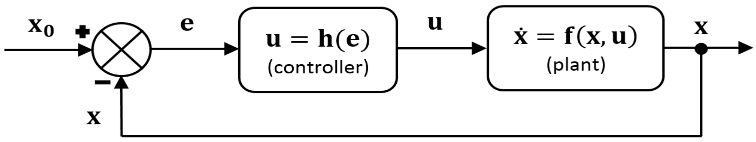


**Figure 1** Scheme of the control system under consideration

If the reference signal $\mathbf{x}_0$ is constant in time, i.e. the desired state of the plant is fixed, then evolution of the system, presented in the Fig. 1, can be described by the set of equations (1):

$$\frac{\mathrm{d}\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \qquad \mathbf{u} = \mathbf{h}(\mathbf{e}) = \mathbf{h}(\mathbf{x_0} - \mathbf{x}) \tag{1}$$

Simple rearrangements of (1) result in transformation to the form (2):

$$\frac{\mathrm{d}\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x_0} - \mathbf{x})) \tag{2}$$

Therefore, under conditions specified above, the control system presented in the Fig. 1, described by the set of equations (1), can be considered as an autonomous dynamical system according to (2). Consequently, all the methods applicable for autonomous dynamical systems, can be used for the specified class of control systems as well.

One of the techniques used to determine properties of dynamics of an autonomous system is estimation of the Largest Lyapunov Exponent (LLE) by means of the dot product of a perturbation vector and its derivative [3]. On the basis of this method, a new control performance assessment criterion has been proposed [5]. This criterion is applied in the following manner. As the control system operates, a value $\lambda^*$ is calculated in subsequent moments of time. The formula for $\lambda^*$ is as follows (3):

$$\lambda^* = \frac{\frac{d\mathbf{e}}{dt} \cdot \mathbf{e}}{|\mathbf{e}|^2} \tag{3}$$

Throughout the evaluation, values of $\lambda^*$ are averaged, resulting in the value $\lambda_e$ of the performance index. The averaged performance index $\lambda_e$ has the following features, which result from properties of the original method of the LLE estimation [3]:

- The index $\lambda_e$ indicates an average exponential increase or decrease of the regulation error length.
- Positive value of $\lambda_e$ shows an average exponential increase of the regulation error length.
- Negative value of $\lambda_e$ is the sign of the average exponential decrease of the regulation error length.
- For negative values of $\lambda_e$, lower value of the performance index indicates faster convergence of the state vector $\mathbf{x}$ to the desired state $\mathbf{x}_0$.
- If $\mathbf{x}_0$ is a fixed point of the system (2), then $\mathbf{x}(t) = \mathbf{x}_0$ is a trajectory of this system; therefore, if $|\mathbf{e}| \to 0$, then $\mathbf{e}$ can be treated as an infinitesimal perturbation of the system (2) and the value $\lambda_e$ approaches the LLE.

One of important branches of the control theory is the optimal control [8]. The aim of this theory is to determine the control function $\mathbf{u}$ that minimizes the cost index $J$:

$$J = g[\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f, \mathbf{p}] + \int_{t_0}^{t_f} f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}) dt \tag{4}$$

where $t_0$ is the initial time, $t_f$ is the final time and $\mathbf{p}$ is a vector of constant parameters. By applying the substitution (5):

$$f = \lambda^* = \frac{\frac{d\mathbf{e}}{dt} \cdot \mathbf{e}}{|\mathbf{e}|^2} \qquad g = 0 \tag{5}$$

the following cost index $J$ is obtained (6):

$$J = \int_{t_0}^{t_f} \frac{\frac{d\mathbf{e}}{dt} \cdot \mathbf{e}}{|\mathbf{e}|^2} dt = (t_f - t_0) \frac{1}{(t_f - t_0)} \int_{t_0}^{t_f} \frac{\frac{d\mathbf{e}}{dt} \cdot \mathbf{e}}{|\mathbf{e}|^2} dt = (t_f - t_0)\lambda_e \tag{6}$$

where the last equality in (6) results from the fact that $\lambda_e$ is an average value of $\lambda^*$. The index $J$ defined by the equation (6) is a new index of performance assessment that can be used to evaluate quality of regulation in control systems. The further part of the paper focuses on application of the index (6) to optimize parameters of a control system.

## 3. Description of the control object

The control object analyzed in this paper is an inverted pendulum (Fig. 2). The inverted pendulum is a kind of pendulum in which the axis of rotation is fixed to a cart. The cart is able to move along the horizontal axis $x$ in a controlled way. The fundamental problem of the inverted pendulum is to find such a control of the cart that keeps the pendulum's bar in the vicinity of the upright vertical position $\alpha(t) = 0$ even if external disturbances appear.

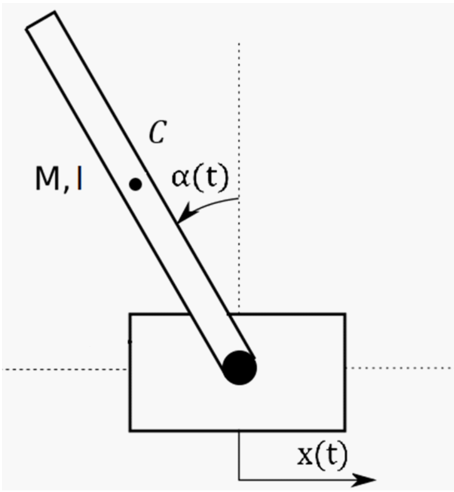


**Figure 2** Sketch of the considered control object – the inverted pendulum

It has been assumed that the pendulum's drive is velocity-controlled. It means that the control signal $u(t)$ supplied to the drive is equal to the desired velocity of the cart. If the drive is stiff enough, then the motion of the pendulum's bar does not influence position of the cart $x(t)$. Providing that the drive can be approximated by a linear differential equation of the first order, the dependence between acceleration of the cart and the control signal is as follows (7):

$$\ddot{x}(t) = a[u(t) - \dot{x}(t)] \tag{7}$$

where $u(t)$ is the control signal and $a$ is a drive constant, which can be determined in the identification process.

The equation of motion of the inverted pendulum can be easily derived using Lagrange approach [10]. Assume that the pendulum's bar is uniform, its mass center $C$ is in the middle of its length and it is loaded by a friction torque $\tau * ml^2/3$. Then, the following equation of motion (8) is obtained:

$$\ddot{\alpha}(t) = \frac{3g}{2l}\sin(\alpha(t)) + \frac{3\ddot{x}(t)}{2l}\cos(\alpha(t)) - \tau(\dot{\alpha}(t)) \tag{8}$$

where $l$ is the length of the bar. Equations (7) and (8) constitute a complete mathematical description of the inverted pendulum.

In this paper a linear control of the pendulum has been assumed. Therefore, the control function $u(t)$ can be described in the form (9):

$$u(t) = [k_1, k_2, k_3, k_4]^T \cdot \mathbf{x}(t) = \mathbf{k} \cdot \mathbf{x}(t) \tag{9}$$

where $\mathbf{k} = [k_1,\ k_2,\ k_3,\ k_4]^T$ is the vector of controller parameters and $k_1, \ldots, k_4$ are constants to be determined.

## 4. Controller optimization

This section is devoted to optimization of the controller (9), i.e. selection of such values of parameters $k_1, \ldots, k_4$ that minimize value of the index $J$ (6). For fixed initial conditions of the system (7–9), $J$ can be treated as a function of regulator constants: $J = J(k_1, \ldots, k_4)$. Values of this function can be obtained by numerical simulation of the system (7–9) and direct calculation of the index $J$ (6). Therefore, optimization of the controller can be regarded as minimization of the function $J(k_1, \ldots, k_4)$. This task has been solved by means of the Differential Evolution (DE) method [9]. DE is a heuristic method which does not require differentiability or even continuity of the optimized function. Therefore, it suits very well to the presented task, because properties of the function $J(k_1, \ldots, k_4)$ are not known. However, it is not guaranteed that the result of DE method is strictly the optimal one.

Let the state vector of the system (7–9) be defined as:

$$\mathbf{x} = [x_1, x_2, x_3, x_4]^T = [\alpha, \dot{\alpha}, x, \dot{x}]^T \tag{10}$$

Assume that for small angular velocities of the bar, the following cubic friction model is valid (11):

$$\tau(\dot{\alpha}) = c_1\dot{\alpha} + c_2(\dot{\alpha})^3 \tag{11}$$

Under such conditions, the whole inverted pendulum control system can be described in the state space as follows (12):

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{3g}{2l}\sin(x_1) + \frac{3a}{2l}[\mathbf{k}\cdot\mathbf{x} - x_4]\cos(x_1) - c_1x_2 - c_2(x_2)^3 \\ x_4 \\ a[\mathbf{k}\cdot\mathbf{x} - x_4] \end{bmatrix} \tag{12}$$

The following parameters values have been used for simulations: $g = 9.81$, $a = 19.72688$, $c_1 = 0.21075$, $c_2 = -0.11161$. These numbers have been obtained from identification of a real inverted pendulum. Please note that $c_2 < 0$. Therefore, the friction model (11) can be applied in limited range of velocities only. It has been assumed that the system starts with the following initial conditions: $x_1(0) = 0.3$, $x_2(0) = 1.0$, $x_3(0) = x_4(0) = 0.0$. The boundaries for optimized parameters have been selected as follows: [2.0, 100.0] for $k_1$, [0.0, 10.0] for $k_2$, [-5.0, 0.0] for $k_3$ and [-5.0, 0.0] for $k_4$. Such boundaries approximate the region in the parameters space for which the control system (12) is stable. Integration of the system (12) has been performed by means of Runge-Kutta method implemented in the SciPy package for Python programming language. The maximum integration step has been set to $10^{-3}$. The implementation of Differential Evolution method provided by the SciPy package has been applied to find the best parameters $\mathbf{k}$.

Throughout the optimization process, the system (12) has been simulated for 7821 different controller parameter vectors $\mathbf{k}$. In each trial the index $J$ (6) has been calculated. Apart from $J$, the simulation program estimated the value of the integral of squared control error (ISE) and the integral of absolute regulation error multiplied by time (ITAE) [5].

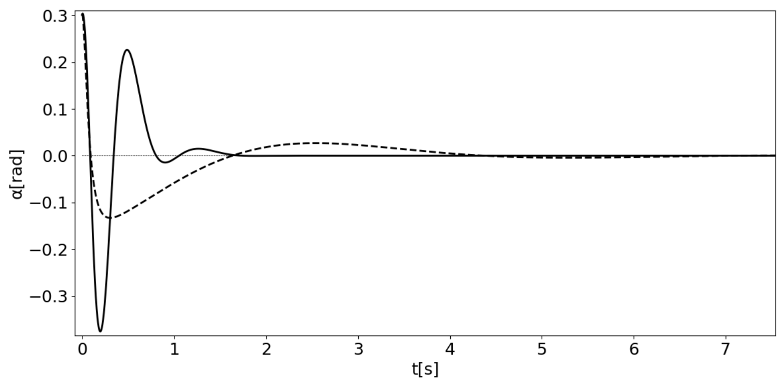


**Figure 3** Stabilization of the pendulum with a controller optimized with respect to $J$ (solid line) and with a controller that produces the lowest ISE (dashed line)

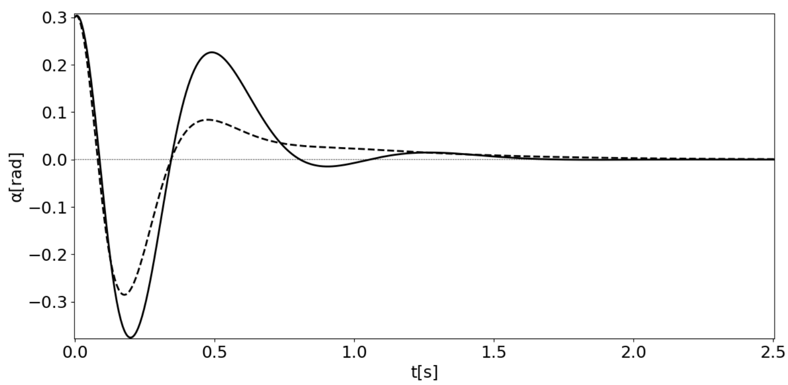


**Figure 4** Stabilization of the pendulum with a controller optimized with respect to $J$ (solid line) and with a controller that yields the lowest ITAE (dashed line)

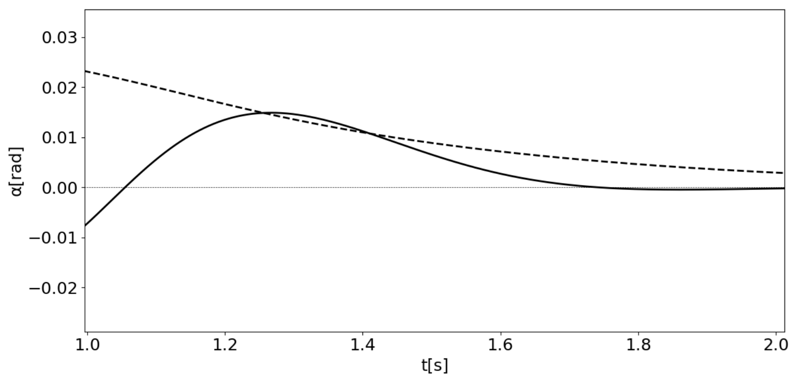


**Figure 5** Zoom of the Fig. 4

Features of the controller optimized with respect to the index $J$ (6) are presented by comparison with controllers that exhibit the lowest value of the ISE or the ITAE. Fig. 3, Fig. 4 and Fig. 5 present stabilization of the inverted pendulum for different control strategies, but with the same initial conditions. Solid lines are related to controllers optimized with respect to $J$ (6). The dashed line in the Fig. 3 depicts stabilization with a controller that yields the lowest ISE. On the other hand, the dashed line in the Fig. 4 and in the Fig. 5 show how the pendulum stabilizes when a controller with the lowest ITAE is selected.

## 5. Conclusions

This paper presents application of a new criterion of control performance assessment based on the assumptions of the optimal control theory. The index based on the new method of the largest Lyapunov exponent estimation has been introduced. An exemplary control system – the inverted pendulum – has been described. Equations of the control system have been presented. A simple linear controller has been proposed. The controller has been optimized with respect to the new control performance index. The Differential Evolution method has been applied in the optimization process.

Results show that the regulator optimized with respect to the new criterion assures much shorter regulation time than in the case of regulator with the lowest ISE value. On the other hand, much higher overshoot is obtained for the new index $J$. It turns out that optimization according to the new criterion yields similar results as selection of parameters that minimize the ITAE. It can be noticed that application of the regulator with the optimal $J$ produces larger overshoot and slightly shorter regulation time than in case of the regulator with the smallest ITAE. Therefore, the new criterion can be successfully applied when the regulation time is crucial, whereas somewhat larger overshoot is acceptable.

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