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To the Question of a Logarithmic Velocity Profile Correspondence with the Experimental Data

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The methodology of obtaining a logarithmic velocity profile describing the velocity distribution in the cross section of the boundary layer, which is based on the well-known equation of L. Prandtl, based on its semi-empirical turbulence theory, is considered.

It is shown that the logarithmic velocity profile obtained in this way does not satisfy any boundary condition arising from the classical definition of such concept as the boundary layer.

The perfect coincidence of this velocity profile with the experimental data of Nikuradze demonstrated in the world scientific literature is a consequence of making these profiles not in a fixed, but in a floating coordinate system. When rebuilding the velocity profiles obtained at different Reynolds numbers, all the profiles lose their versatility and do not coincide with the actual velocity profiles in cylindrical pipes.

Keywords: velocity profile, transverse gradient of friction stress, turbulence, boundary layer, coordinate system, boundary conditions, universal velocity profile.

1. Introduction

In classical fluid dynamics, a logarithmic velocity profile in a turbulent boundary layer is regarded as a perfect example of matching theoretical solutions with the experimental data. On the basis of this, such concept as the laminar sublayer was formulated, a physical picture of the occurrence of the regime of rough flow of surfaces was explained, a concept such as "dynamic" speed was introduced into consideration, which entered the structure of the so - called coordinates, which allowed to obtain a semi-experimental formula determining the value of turbulent friction stress on the streamlined surface and to introduce the idea of two-and even three-layer structure of the turbulent boundary layer into consideration.

Up to the present days all these concepts have been unshakable axioms that determine the mechanism of interaction of moving media with streamlined surfaces.

Explaining the structure of the turbulent boundary layer in [1] L. Sedov writes: in the turbulent boundary layer on the streamlined surface appears a very thin laminar sublayer, in which the liquid velocity is not large at all, pulsations are almost absent, but there is a very large transverse velocity gradient, causing large values of stresses of friction forces.

In [2] L. G. Loitsyansky writes: "the crucial fact is that the logarithmic equation (for the velocity profile in a turbulent boundary layer) preserves its form for all Reynolds numbers flow, or, as it is said, it is universal."

Further on, degree formulas for speed, which do not have the universal property, will be introduced. On the physical side, this property of logarithmic formulas is explained by the presence of a laminar sublayer, in which all the influence of viscosity is concentrated.

The following considerations of G. Schlichting [3] should also be noted: "In the immediate vicinity of the apparent turbulent friction wall, the stresses are small compared to the viscous stresses of the laminar flow. It follows that in a very thin layer in the immediate vicinity of the wall any turbulent flow behaves mainly as a laminar flow. In such a thin layer, called the laminar sublayer, the velocities are so small that the viscosity forces here are much greater than the inertia forces. This means that turbulence cannot exist here. This laminar sublayer is adjacent to the transition region, in which velocity pulsations are already so large that they entail the appearance of turbulent tangential stresses comparable to the viscosity forces. Finally, at still greater distance from the wall the turbulent stress tangent lines fully outweigh the laminar tension."

The statements and considerations above explain the unconditional recognition of the existence of the laminar sublayer in the turbulent boundary layer, initially introduced to give physical meaning to the logarithmic velocity profile. At the same time, the correspondence of the logarithmic profile of the velocity to the physical picture of the fluid flow within the boundary layer is not questioned and its universality is noted, regardless of the influence of Reynolds numbers.

Accordingly, the main purpose of these considerations is to physically justify the need for existence between the turbulent boundary layer and the streamlined surface of a certain laminar sublayer, within which the existence of a chaotic flow with pulsating components of the averaged velocity is impossible.

Let us consider further the extent to which the existing assessment of the logarithmic velocity profile and the consequence resulting of it correspond to the real picture of the flow in the turbulent boundary layer and what is the real picture of the interaction of moving media with streamlined surfaces.

2. Semi-empirical theory of turbulence by L. Prandtl and the logarithmic velocity profile. Major headings should be typeset in boldface with the first letter of important words capitalized.

Among the numerous semi-empirical and phenomenological turbulence theories, L. Prandtl's semi-empirical turbulence theory [4] occupies a special place in historical terms, since on its basis an attempt was made for the first time to establish a functional connection of the turbulent friction voltage τ_T with the transverse gradient of the averaged longitudinal velocity in the turbulent wall (boundary) layer.

It is important to note that the established connection between these values was used to directly determine the shape of the velocity profile in cylindrical tubes during turbulent motion of working environments inside of the tubes [2], and to determine the velocity profiles in turbulent boundary layers.

Based on these solutions, the whole physical picture of the interaction of moving liquid and gaseous media with streamlined surfaces is built. Since this problem from a practical point of view, is one of the crucial problems of fluid dynamics let's consider from a methodological point of view, all the stages of its solution, starting with the process of obtaining the classical logarithmic velocity profile as the result of the integration of the known equations of L. Prandtl derived from its semi-empirical theories of turbulence and having the following form:

$$\tau_{''} = \rho l^2 \left(\frac{dU}{dy}\right)^2 \tag{1}$$

In the equation above, the characteristic linear dimension l is interpreted as a certain way of mixing turbulent "moles" of the liquid, on which their transverse mixing the original longitudinal velocity is preserved.

Formally, equation (1) presents a relation between three unknown quantities: the shearing stress τ, the averaged longitudinal velocit U and the mixing path l.

Accordingly, if we use the above equation to determine the velocity profile within the boundary layer, it must be supplemented by two more functional relations linking the unknown values of τ and l with the transverse coordinate y. In the classical version of the solution of this problem it was accepted that:

$$\begin{gathered} \tau = const \\ l = \chi y \end{gathered} \tag{2}$$

where χ is some constant.

If near the wall in a very narrow zone, the above ratios (2) can still be given some physical meaning, then outside this zone they lose any connection with the real picture of the flow within the boundary layer.

Nevertheless, it was under these conditions that the equation (1) was integrated and recorded with taking into account (2) in the following form:

$$\frac{dU}{\upsilon^*} = \frac{1}{\chi}\frac{dy}{y} \tag{3}$$

Here $\upsilon^* = \sqrt{\frac{\tau}{\rho}}$. Since $\sqrt{\frac{\tau}{\rho}}$has the dimension of velocity, then the value of $l\upsilon^*$ is called dynamic speed. Integrating (3), we obtain (4):

$$\frac{U}{\upsilon^*} = \frac{1}{\chi}\ln{(y)} + C \tag{4}$$

It is easy to see that the equation (4) describing the velocity profile in a turbulent boundary layer does not satisfy the central boundary condition that the flow velocity on the streamlined surface (y = 0) must be zero (in this case U $\to \infty$).

The result is quite predictable, since near the wall the accepted conditions (2) are incompatible since they turn the original equation (1) into inequality. (In the left part of the equation (1.1) there is a constant, in the right part - zero). That is, the curve described by the equation (4) is in clear contradiction to the real velocity profile in the near-wall region of the flow.

In this situation, there can be only two solutions. Either recognize the fact of the error of the considered solution of the problem or create a model of the flow in the wall area, with which the peculiarity of the logarithmic velocity profile in this region of the flow can be eliminated.

It sounds paradoxically but to give the physical meaning to the equation (4) in classical fluid dynamics was chosen the second way – the way of correction of the real flowing at the already derived equation.

As a result, a virtual structural model of a two-layer turbulent boundary layer appeared. According to this model, a thin layer of laminar flow is preserved near the streamlined surface, which was named laminar sublayer, with a thickness of y_π with the speed at the outer boundary of the U_π, which is mated with the outer turbulent part of the boundary layer.

To prove and determine such structural model of flow in a turbulent boundary layer, the following considerations are given in [3]: "in a laminar sublayer, the velocities are so small that the viscosity forces here are significantly greater than the inertia forces. This means that turbulence cannot exist here."

In [2] it is indicated that the turbulent friction voltage $\tau = -\rho\overrightarrow{U'\upsilon'}$on the wall is zero, as normal to its velocity fluctuations of υ'cannot exist on the wall and, consequently, on the wall $\tau_W = \mu\frac{dU}{dy}$.

It follows from the above statement that the increase in resistance under turbulent flow regime within the boundary layer occurs only because of the growth of the transverse velocity gradient $\frac{dU}{dy}$and is still determined only by the amount of molecular friction. Leaving the given statements without comments and comparing the velocities on the outer boundary of the introduced laminar sublayer with the velocity in the turbulent part of the boundary layer determined by the formula (4), we obtain a well-known classical formula defining the logarithmic velocity profile

$$\frac{U}{\upsilon^*} = \frac{1}{\chi} \cdot \ln\frac{\upsilon^* y}{\gamma} + B \tag{5}$$

where χ and B are empiric constants equaled $\chi = 0.4$, $B = 5.5$.

Despite all the above assumptions, which clearly contradict the physical picture of the flow, Nikuradze's experiments (1932) aimed to determine velocity profiles in long cylindrical tubes at different values of Reynolds numbers brilliantly confirmed the validity of the theoretical formula (5). The results of these experiments, included in all monographs on fluid dynamics are shown in Fig. 1, from which all experimental data are arranged in a single line, regardless of Reynolds numbers.

In this regard, [2] contains the following statement: "it is of fundamental importance that the logarithmic formula retains its shape for all Reynolds numbers of the flow, or, as they say, is universal. The degree formulae for velocity do not have the property of universality. On the physical side, these properties of logarithmic formulas are explained by the presence of a laminar sublayer, in which all the influence of viscosity is concentrated."

Let us consider further to what extent the above statement corresponds to the real picture of the flow and whether the logarithmic profile of the velocity actually corresponds to the experimental data.

3. Coordinate focus and the degree of correspondence of the equation (5) to the actual velocity profile

For a clearer understanding of what is depicted in Fig. 1 we present the formula (5) as follows

$$\frac{U}{U_t}\frac{U_t}{\upsilon^*} = \frac{1}{\chi}\ln\frac{y}{\delta} + \frac{1}{\chi}\ln\frac{\delta\upsilon^*}{\gamma} + B \tag{6}$$

Here U_t is the flow rate at the outer limit of the boundary layer, and δ is the physical thickness of this layer. Since the experimental data given in Fig. 1 have been obtained for velocity profiles in a cylindrical tube on a section of stabilized flow, in this case the entire cross section is occupied by a closed boundary layer and, accordingly, $\delta = r - 0$ ($r - 0$ – the radius of the tube). Then, the formula (6) should be written as:

$$\frac{U}{U_t}\frac{U_t}{\upsilon^*} = \frac{1}{\chi}\ln\frac{r}{r_0} + \frac{1}{\chi}\ln\frac{r_0\upsilon^*}{\gamma} + B \tag{7}$$

where r – current radial coordinate (Re_{r0}).

In addition, it is easy to show that $\frac{U_{t\max}}{\upsilon^*} = f(\mathrm{Re}_{r0})$ and $\ln\frac{r_0\cdot\upsilon^*}{\gamma} = \varphi(\mathrm{Re}_{r0})$, and for the analysis it is necessary to consider the following formula describing, as already noted, the velocity profile in a tube under turbulent flow regime:

$$\frac{U}{U_{\max}} f\,(Re) = \frac{1}{\chi}\ln\frac{r}{r_0} + \frac{1}{\chi}\varphi\,(Re) + B \tag{8}$$

Consider using the formula entry (5) in the form (8), to what extent it corresponds to the conditions in the center of the tube, where $\mathbf{U}_t = \mathbf{U}_{max}$.

According to these conditions in the center of the tube $\frac{U}{U_{\max}} = 1$ where $\frac{r}{r_0} = 1$, and $\frac{d}{dr}\left(\frac{U}{U_{\max}}\right) = 0$.

If you use the formula (8), then when $\frac{r}{r_0} = 1$ in the first case we get:

$$\frac{U}{U_{\max}} = \frac{1}{f\,(Re)}\left(\frac{1}{\chi}\varphi\,(Re) + B\right),$$

And in the second case $\frac{d}{dr}\left(\frac{U}{U_{\max}}\right) = 0 \to +\infty$

From the analysis above it follows that the logarithmic velocity profile does not correspond to any of the two natural boundary conditions in the center of the tube, and the boundary condition on the wall had to be corrected by introducing a certain laminar sublayer into consideration. How, in this case, was it possible to obtain such good coincidence of the logarithmic velocity profile with the experimental data?

To answer this question, let us turn again to the formula (8), which, at a fixed value of Reynolds number, establishes an unmistakable relationship between the dimensionless distance from the wall $\overline{y} = y/r_0$ and dimensionless speed $\overline{U} = U/U_{\max}$.

Accordingly, if we postpone the figure $\frac{U}{\upsilon^*} = \overline{U} f(\mathrm{Re})$ along the y-axis, then its maximum value $(\frac{U}{\upsilon^*})$max will be equal to the function value f(Re. This limit value of scale along the ordinate axis is reached at $y/r0 = 1$, which corresponds to the maximum value of the coordinate on the absciss axis, equaled $\left(\ln \frac{y\upsilon^*}{\gamma}\right)$ max $=$ $\left(\ln \frac{ro\upsilon^*}{\gamma}\right)$. That is, at a fixed value of the number Re the velocity profile corresponding to this value of the number Re, on the plane $\frac{U}{\upsilon^*} - \ln \frac{ro\upsilon^*}{\gamma}$ will take a quadrant $f(Re) - \ln \frac{ro\upsilon}{\gamma}$. If the number Re increases, the size of the specified square will also increase. As a result of using not fixed, but "floating" coordinates with the increase in the number Re there is a "stretching" of the velocity profiles along the shown in Fig. 1 straight line. For Fig. 2 this process is shown more clearly with the designation of those squares, inside of which are the experimental points obtained by measuring the number Re from Re $= 5 \cdot 10^3$ to Re $= 10^6$.

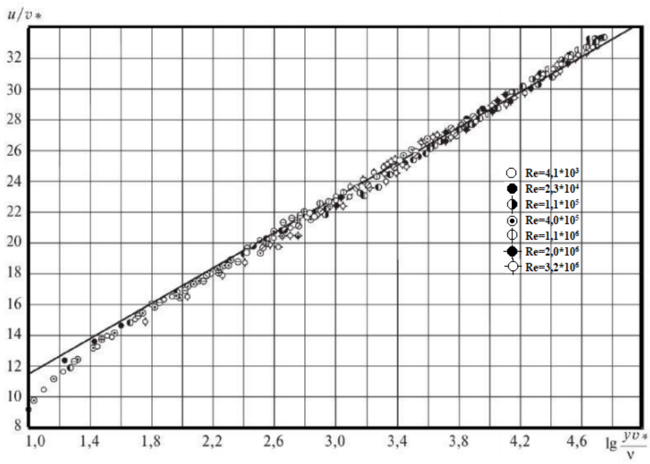

Figure 1 Logarithmic velocity profile in semi-logarithmic coordinates at different values of Reynolds numbers

If at $\mathrm{Re}_{\min} = 5 \cdot 10^3$ the logarithmic profile is located in the first quadrant along o – a line, then at $\mathrm{Re}_{\max} = 10^6$ it occupies the entire last quadrant. Accordingly, if, for example, in the second quadrant we take an experimental point "b", lying almost in the center of the tube, the coinciding point "b" of the last profile corresponding to the number Re, equals to 106 will be near the wall of the tube.

It is clear that the information value of such comparison of velocity profiles is close to zero and it is not necessary to talk about the "universality" of the logarithmic velocity profile.

Considering Nikuradze's experiments on the influence of the Re number on the shape of velocity profiles, it is necessary to note their very important feature, which

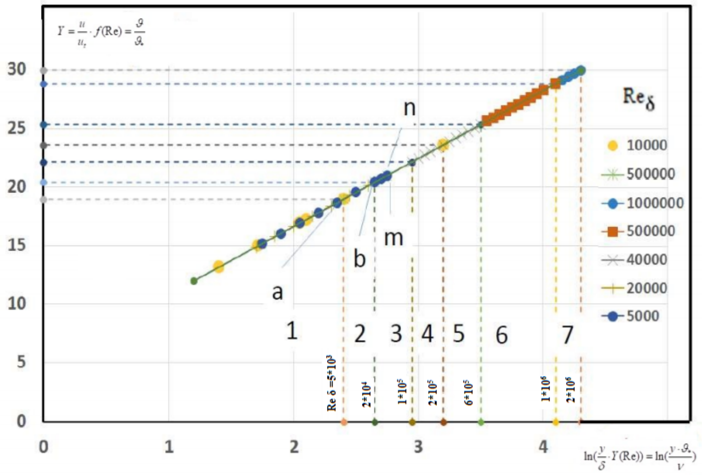

Figure 2 The effect of Re numbers on the dimensions of the quadrants in which the corresponding logarithmic velocity profile is located

is the fact that they were carried out on the tube section, where there was a stabilized turbulent flow at a fixed coordinate $y_{\max} = r_0$, corresponding to the center of the tube with maximum speed $U_{\max}$. Accordingly, the change of number Re in this case was carried out only as a result of increase in speeds (growth of the expense of liquid).

Accordingly, if at a fixed average speed U_{av} number $\mathrm{Re} = (U_{av}\ r_0)/\gamma$ was lower than its critical value, then the laminar flow regime remains along the entire length of the tube.

The situation is fundamentally different in the flow of the plane viscous flow. In this case at a constant velocity in the outer part of the boundary layer $\overline{U}$, a continuous increase in the number $\mathrm{Re} = \frac{U\delta}{\gamma}$ occurs along the plate due to the increase in the thickness of the boundary layer δ (in the laminar flow regime $\delta \sim \mathrm{X}^{0,5}$). As a result, with a sufficient length of the plate, the laminar flow at a certain distance from the input edge of the X_{ed} will necessarily go into turbulent flow mode.

We will show further, to what extent in a fixed coordinate system $\frac{U}{Ut} - \frac{y}{\delta}$ the logarithmic velocity profile corresponds to the real velocity profile on a plate at a gradient-free flow. With that purpose we present the function (6) as follows

$$\frac{U}{U_t} = \frac{\upsilon^*}{U_t}\frac{1}{\chi}\left(\ln\frac{\delta\upsilon^*}{\gamma}\frac{y}{\delta} + \chi B\right) \tag{9}$$

Built on the basis of this formula three logarithmic profile corresponding to the Re numbers, equaling $\mathrm{Re}_1 = 10^5$, $\mathrm{Re}_2 = 10^6$, $\mathrm{Re}_3 = 10^7$ are shown in Fig. 3. Here you can also see the degree profile $\frac{U}{Ut} = \left(\frac{y}{\delta}\right)^n$where the exponent n = 1/7. The given comparison shows that the logarithmic velocity profile in a fixed coordinate system strongly depends on the number Re and is very far from the real (degree) velocity profile.

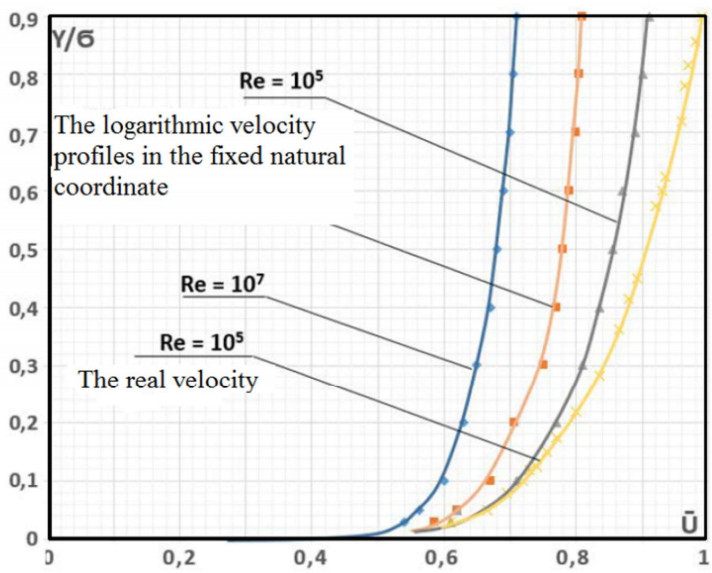

Figure 3 Logarithmic velocity profile in a natural fixed coordinate system $\overline{U} - \overline{Y}$with different Re numbers

The reason for this is obvious and, as already stated, it is the fact that the logarithmic velocity profile does not satisfy any of the two basic boundary conditions on the outer limit of the boundary layer.

Here, after considering the degree of "universality" with respect to the Re number of the logarithmic velocity profile in tubes, it is advisable to review L. G. Loitsyansky's statement [2] that degree velocity profiles, in contrast to the logarithmic ones, do not possess property of universality, regarding the Reynolds numbers.

Indeed, if we construct degree velocity profiles obtained experimentally in a fixed coordinate system $U/U_t \to y/\delta$ [2] (Fig. 4), we can observe a strong influence of Re numbers on their form. Accordingly, in the formula $\frac{U}{Ut} = \left(\frac{y}{\delta}\right)^n$the influence of this complex must be taken into account by adjusting the index of degree n. If at Re $= 10^5$ n $= 1/7$, then at Re $= 10^7$ the index of degree decreases and is n $= 1/10$.

The resulting difference in the form of velocity profiles at three values of Re numbers is clearly visible in Fig. 4. But if we postpone along the ordinate axis the function of this argument $(y/\delta)^n$ and not the argument y/δ itself, then as in the logarithmic velocity profile, all described points will be located strictly along the line drawn from the origin of coordinates of the quadrant under consideration. The transfer of the experimental points from exponential velocity profiles to a specified straight line is shown in Fig. 4 with the corresponding arrows.

For example, if you select relative velocity on the X-axis, then in the natural coordinate system this velocity value will be achieved for profile 1 in the coordinate section (point a), for profile 2 in the section (point b), and for profile 3 in the section (point d). When using the argument as not the argument itself, but the function of this argument , points a, b, d on the bisector of the rectangle's coordinate angle will be the same.

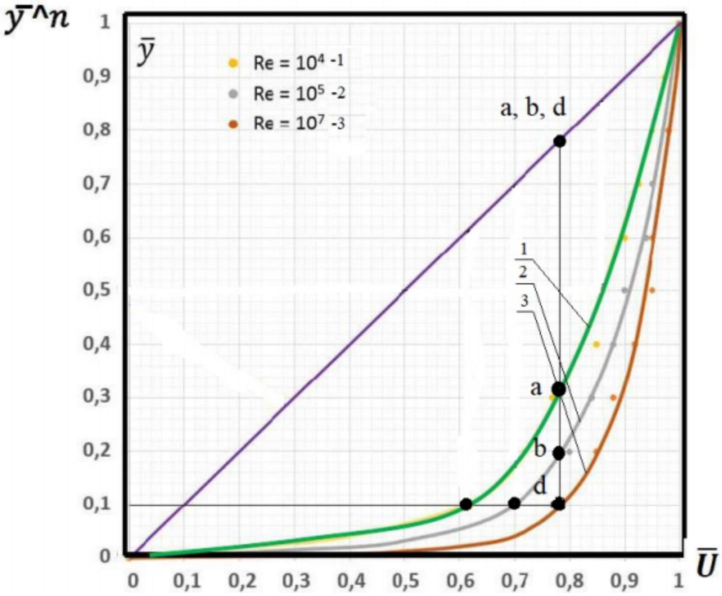

Figure 4 The effect of the coordinate system on the graphical representation of the degree velocity profile in the boundary layer

As a result, it is clearly seen that the experimental points coinciding on the considered line (as well as in Fig. 2) belong to completely different sections of real speed profiles.

Based on the analysis, the following axiom related to the coordinate focus under consideration can be formulated.

In the case when the graphical comparison of any functional dependence describing any process where experimental data do not use the argument itself, but a function from this argument containing several parametric parameters, all experimental points will inevitably be located along a straight line, regardless of the values of those parameters that are included in the function under consideration. However, in this case, it is impossible to determine the degree of influence of these parameters on the considered process due to the use of so called "universal" coordinates.

For a clearer idea of the essence of the coordinate focus under consideration, we take into account that in a rectangular coordinate system, any function of one argument converts the linear absciss axis into some curve on the X-Y plane.

If we go to the plane where two arbitrary functions of the same argument are postponed along the coordinate axes (for example Y), φ(y) and ψ(y) between which on the basis of experimental data it is necessary to establish a certain relationship, the local connection between them is easiest to get with the help of a linear relationship of that kind:

$$\varphi\left(y\right)=Ki\psi\left(y\right)$$

where Ki is an experimental coefficient.

Then, determined experimentally for each coordinate y in the flow plane of the value of φ(y) and ψ(y) functions, it is easy to find the local correlation coefficient Ki according to these values. As a result, when transferring the experimental values of the coefficient Ki to the plane $\varphi - \psi$, we obtain a curve that determines the relationship between completely arbitrary functions. In particular case, the Ki coefficient can be the same for all values of the "y" coordinate.

Then, as a result of processing the experimental data on the plane $\varphi - \psi$, there appears a straight line, "brilliantly" coinciding with the experimental data.

In relation to the boundary layer $\varphi(y) = U/v^*$and $\psi = \ln \frac{yv^*}{\gamma}$the relationship between these functions is determined by the known equation:

$$U/v^* = 2,5 \ln \frac{yv^*}{\gamma} + 5,5$$

In conclusion, let us return to the history of the degree velocity profile mentioned above. Practically in all monographs and textbooks on fluid dynamics, where the logarithmic velocity profile is studied, it is noted that the use of coordinates $U/v^* - \frac{yv^*}{\gamma}$ in the logarithmic profile is to some extent universal. To verify this "universality", the velocity profile in the turbulent boundary layer was approximated by the degree function using the specified "universal" complexes of the following kind:

$$U/v^* = \xi \left(\frac{\delta v^*}{\gamma}\right)^n \tag{10}$$

Unlike the logarithmic velocity profile, the dependence (10) already satisfies two of the three boundary conditions arising from the boundary layer definition. So, at $y = 0$ $U/v^* = 0$and at $y = \delta U = U_t$, then:

$$U_t/v^* = \xi \left(\frac{\delta v^*}{\gamma}\right)^n \tag{11}$$

When dividing (10) to (11), we obtain:

$$U/U_t = \left(\frac{y}{\delta}\right)^n \tag{12}$$

where according to the experimental data, the indicator "n" turned out to be a function of Reynolds number.

As a result, it was mentioned in [2] that the power velocity profile is not, unlike logarithmic, universal. However, if we accept that in this case $\varphi(y) = U/v^*$and $\psi = \left(\frac{y \cdot v^*}{\gamma}\right)^n$ then on a plane $U/v^* - \left(\frac{y \cdot v^*}{\gamma}\right)^n$ a straight line is formed that exits the coordinate origin at an angle α equaled to arctg(D) and all sample points, regardless of the Re numbers will be placed along this straight line (Fig.4).

4. Conclusion

- It is shown that the logarithmic velocity profile in its original form does not satisfy any of these three boundary conditions arising from the definition of a boundary layer. With the introduction of the concept of laminar sublayer disappears the feature of the wall, but two conditions on the outer limit of the velocity profile remain to be unfulfilled. Item two

- The coincidence of experimental data on the velocity profile in cylindrical tubes with different values of Reynolds numbers with the theoretical logarithmic velocity profile demonstrated in the literature is a consequence of incorrect comparison of these values in the system of "floating" coordinates, where the function itself is used as an argument.

- The comparison of the logarithmic velocity profile with the experimental velocity profile in a fixed coordinate system showed that the calculated velocity profile has nothing in common with the real picture of the flow.

- On the basis of the analysis, an axiom is formulated, according to which when using argument not as an argument but the function of this argument (that happens in the case of a graphical representation of the logarithmic velocity profile) depending on some parameters is always provided with 100% location of the experimental data along a straight line corresponding to the rating formula.

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