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An Interface Anticrack in a Periodic Two–Layer Piezoelectric Space under Vertically Uniform Heat Flow

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This paper aims to investigate 3D static thermoelectroelastic problem of a uniform heat flow in a bi-material periodically layered space disturbed by a thermally and electrically-insulated rigid sheet-like inclusion (so-called anticrack) situated at one of the interfaces. An approximate analysis of the considered laminated composite is given in the framework of the homogenized model with microlocal parameters. Accurate results are obtained by constructing suitable potential solutions and reducing to the corresponding homogeneous thermoelectromechanical (or thermomechanical) anticrack problems. The governing boundary integral equation for a planar interface anticrack of arbitrary shape is derived in terms of a normal stress discontinuity. As an illustration, a complete solution for a rigid circular inclusion is obtained in terms of elementary functions and interpreted from the failure perspective. Unlike existing solutions for defects at the interface of materials, the solution obtained displays no oscillatory behavior.

Keywords: periodic two-layered composite, homogenized model, interface anticrack, thermo-electroelasticity, heat flow, thermal stress singularity.

1. Introduction

Due to the inherent coupling between mechanical and electrical properties, multilayered piezoelectric electronic structures (e.g., filters, radiators and converters) are in the focus of special attention in the manufacture of composite materials (Rao and Sunar [1]). In many cases, various defects such as cracks and inclusions may appear at the interface between the layers. These defects cause high thermal, stress and electric field concentrations, which may lead to failure, fracture and dielectric breakdown. Some industrial applications involve temperature changes producing the pyroelectric effect. In this context, there is tremendous interest in studying the failure behaviors of interface defects in thermopiezoelectric materials under thermal loadings.

A comprehensive overview in the field of interface cracks in piezoelectric materials was presented by Govorukha et al. [2]. In addition to cracks, rigid lamellate inclusions (also called anticracks, for brevity) are objects around which stress concentrations occur and further affect the behavior of material. In comparison with thermoelastic problems of cracks in homogeneous materials or bi-materials, the study of rigid inclusion problems are rather limited. Most investigations was conducted on two-dimensional problems. Due to mathematical complexity, the more practical and realistic three-dimensional problems involving anticracks subjected to thermal actions seem to remain inadequately treated. Certain progress in homogeneous media has been achieved lately by Kaczyński [3]; see also extensive references therein.

The present contribution is a sequel to some earlier investigations (Kaczyński and Matysiak [4-5], Kaczyński and Monastyrskyy [6]) dealing with interface crack or rigid inclusion problems in a bimaterial periodically layered space subjected to thermal loading. It is devoted to examine the piezoelectric effect in analyzing the obstruction of a uniform steady heat flow in a two-layer periodic stratified medium by an interface rigid sheet-like inclusion (anticrack) that is isothermal and electrically impermeable.

The paper is organized as follows. In Section 2, the description of the problem under study and the use of the homogenized model of the considered composite are demonstrated. Section 3 presents the resulting boundary-value problem and and its solution method within the framework of this homogenized model that is almost identical to that for the corresponding homogeneous thermoelectromechanical (or thermomechanical) anticrack problem considered by Kaczyński and Kaczyński [7]. As an application of the theory, a closed form solution in terms of elementary functions is given and discussed from the point of view of material failure for a circular rigid disc-inclusion in Section 4. Finally, some conclusions are made in Section 5.

2. Problem description and governing equations of the homogenized model

The composite being considered is a periodic stratified space consisting of a repeated thin fundamental layer of thickness δ which is composed of two homogeneous bonded sublayers with different mechanical and thermopiezoelectric properties, denoted by 1 and 2, with thicknesses δ_1 and δ_2 (so $\delta = \delta_1 + \delta_2$) as shown in Fig. 1. In the following, all quantities (material constants, stresses, etc.) pertinent to these sublayers will be denoted with the superscript (r) taking the values 1 and 2, respectively. Assume that the piezoelectric materials filling sublayers are chosen with 6 mm hexagonal symmetry (Nye [8]), characterized by the following system of constants: $c_{11}^{(r)}$, $c_{12}^{(r)}$, $c_{13}^{(r)}$, $c_{33}^{(r)}$, $c_{44}^{(r)}$– elastic stiffnesses, $e_{31}^{(r)}$, $e_{33}^{(r)}$, $e_{15}^{(r)}$ – piezoelectric constants, $\varepsilon_{11}^{(r)}$, $\varepsilon_{33}^{(r)}$ – dielectric permittivities, $k_1^{(r)}$, $k_3^{(r)}$– thermal conductivity coefficients, $\alpha_1^{(r)}$, $\alpha_3^{(r)}$– thermal coefficients of linear expansion. Besides,

$\beta_1^{(r)} = \left(c_{11}^{(r)} + c_{12}^{(r)}\right)\alpha_1^{(r)} + c_{13}^{(r)}\,\alpha_3^{(r)}$, $\beta_3^{(r)} = 2\,c_{13}^{(r)}\alpha_1^{(r)} + c_{33}^{(r)}\,\alpha_3^{(r)}$ are the thermomechanical moduli and $p_3^{(r)} = 2\,e_{31}^{(r)}\alpha_1^{(r)} + e_{33}^{(r)}\,\alpha_3^{(r)}$ are the pyroelectric constants.

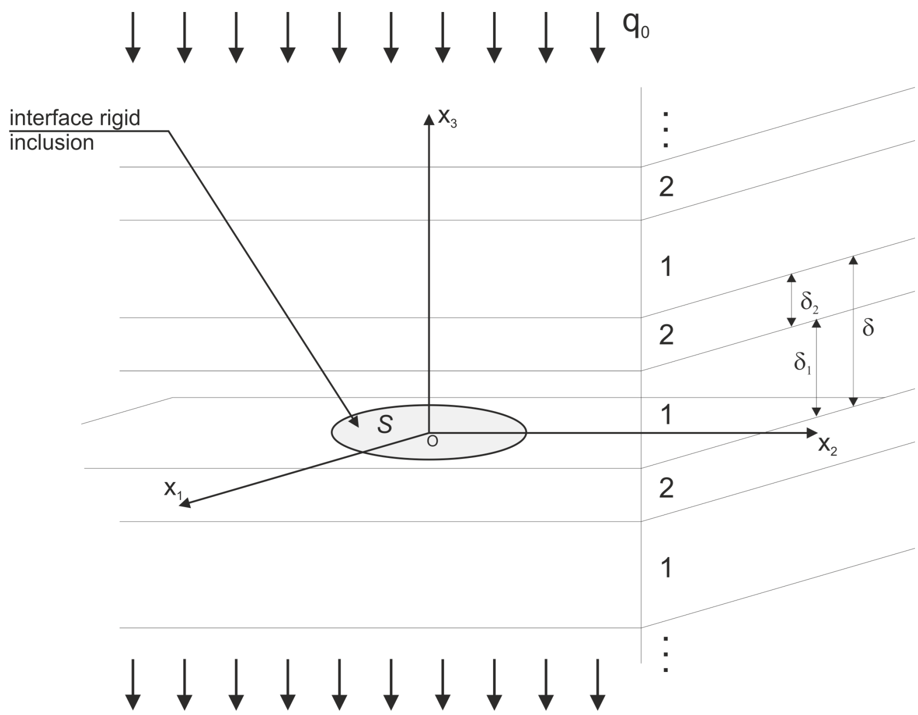

Figure 1 A bi-material periodically layered space with an interface anticrack

Referring to the Cartesian coordinate system $O\,x_1,\,x_2,\,x_3$ with the x_3-axis normal to the layering and being the axis of symmetry and polarization, denote at the typical point $\mathbf{x} = (x_1,\,x_2,\,x_3)$ the temperature (strictly, a deviation of the temperature from the reference state) by θ, the electric potential by Φ and the components of the displacement vector, stress tensor and electric induction vector by $u_i,\,\sigma_{i\,j},\,D_i$ $(i, j \in \{1,2,3\})$, respectively.

Suppose that a rigid sheet-like inclusion (anticrack) serving as a mechanical defect in this periodically layered composite occupies a domain S with smooth boundary at the interface $x_3 = 0$, and there is a constant heat flow $\mathbf{q}\,(\infty) = [0,\,0,-q_0]$, $q_0 > 0$ in the direction of the negativex_3-axis (Fig. 1). The anticrack S is assumed to be thermally insulated and electrically impermeable. The perfect mechanical, thermal and electrical contacts between the layers (excluding the inclusion region S) are required. Moreover, the stress and electric displacement field should decay to zero at infinity.

To analyze the thermal stress and electric displacement field disturbed by this defect, a direct analytical approach becomes intricate because of the complicated geometry and complex boundary conditions. Therefore, a specific homogenization procedure called microlocal modelling (Woźniak [9]) will be employed in order to seek an approximate solution within certain homogenized model of the considered composite. We utilize this approach for periodically layered piezoelectric composites presented in Evtushenko et al. [10] and attempt to derive the governing equations of this model by using the homogenized process based on the uncoupled thermo-electroelasticity. However, we omit the presentation of mathematical assumptions and detailed calculations; see the papers cited above for details.

In the subsequent considerations the following notation will be used: Latin subscripts always assume values 1, 2, 3 and the Greek ones 1,2. The Einstein summation convention holds and a comma followed by an index denotes the partial differentiation with respect to the corresponding coordinate variable.

The following representations and microlocal approximations for the temperature $\theta^{(r)}$, fluxes $q_i^{(r)}$, the electric potential $\Phi^{(r)}$, displacements $u_i^{(r)}$, stresses $\sigma_{ij}^{(r)}$ and electric displacements $D_i^{(r)}$ are postulated within the stationary thermo-electroelasticity with microlocal parameters (see Evtushenko et al. [10], Kaczyński [11], Chen [12]):

$$\theta^{(r)} = \vartheta + h\,\Gamma \cong \vartheta, \qquad \theta^{(r)}_{,\,\alpha} \cong \vartheta_{,\,\alpha}, \qquad \theta^{(r)}_{,\,3} \cong \vartheta_{,\,3} + h'\Gamma \tag{1}$$

$$q_\alpha^{(r)} \cong -\,k_r\,\vartheta_{,\,\alpha}, \qquad q_3^{(r)} \cong -\,k_r\,(\vartheta_{,\,3} + h'\Gamma_{,\,3}) \tag{2}$$

$$\Phi^{(r)} = \varphi + h\,\mathrm{H} \cong \varphi, \qquad \Phi^{(r)}_{,\,\alpha} \cong \varphi_{,\,\alpha}, \qquad \Phi^{(r)}_{,\,3} \cong \varphi_{,\,3} + h'\mathrm{H} \tag{3}$$

$$u_i^{(r)} = w_i + h\,d_i \cong w_i, \qquad u^{(r)}_{i,\,\alpha} \cong w_{i,\,\alpha}, \qquad u^{(r)}_{i,\,3} \cong w_{i,\,3} + h'd_3 \tag{4}$$

$$\sigma_{3\alpha}^{(r)} \cong c_{44}^{(r)}\,(w_{\alpha\,,3} + w_{3\,,\alpha} + h'd_3) + e_{15}^{(r)}\,\varphi_{,\alpha} \tag{5}$$

$$\sigma_{33}^{(r)} \cong c_{13}^{(r)}\,w_{\gamma\,,\gamma} + c_{33}^{(r)}\,(w_{3\,,\,3} + h'd_3) + e_{33}^{(r)}\,(\varphi_{,\,3} + h'H) - \beta_r\,\vartheta \tag{6}$$

$$\sigma_{12}^{(r)} \cong 0,5\left(c_{11}^{(r)} - c_{12}^{(r)}\right)(w_{1\,,2} + w_{2\,,1}) \tag{7}$$

$$\sigma_{11}^{(r)} \cong c_{1\gamma}^{(r)}\,w_{\gamma,\gamma} + c_{13}^{(r)}\,(w_{3\,,\,3} + h'd_3) + e_{31}^{(r)}\,(\varphi_{,\,3} + h'H) - \beta_r\,\vartheta \tag{8}$$

$$\sigma_{22}^{(r)} \cong c_{1\,3-\gamma}^{(r)}\,w_{\gamma,\gamma} + c_{13}^{(r)}\,(w_{3\,,\,3} + h'd_3) + e_{31}^{(r)}\,(\varphi_{,\,3} + h'H) - \beta_r\,\vartheta \tag{9}$$

$$D_\alpha^{(r)} = e_{15}^{(r)}\,(w_{\alpha\,,3} + w_{3\,,\alpha} + h'd_3) - \varepsilon_{11}^{(r)}\,\varphi_{,\alpha} \tag{10}$$

$$D_3^{(r)} = e_{31}^{(r)}\,w_{\gamma\,,\gamma} + e_{33}^{(r)}\,(w_{3\,,\,3} + h'd_3) - \varepsilon_{33}^{(r)}\,(\varphi_{,\,3} + h'H) + p_3\,\vartheta \tag{11}$$

In the above, ϑ, φ, w_i and Γ, H, d_i are unknown functions interpreted as the macro-temperature, the macro-electric potential, macro-displacements and the microlocal thermal, electric and kinematic parameters, respectively. Moreover, the postulated *a priori* function h, called the shape function, characterizes the special approximate model of the layered composite. As for the treated stratified body, this function (being sectional linear, δ- periodic) with its derivative is chosen as follows:

$$h\,(x_3) = \begin{cases} x_3 - 0,5\,\delta_1, & x_3 \in \langle 0, \delta_1\rangle \\ (\delta_1 - \eta\,x_3)\,/\,(1-\eta) - 0,5\,\delta_1, & x_3 \in \langle \delta_1, \delta\rangle \end{cases} \qquad \eta = \delta_1/\delta \tag{12}$$

$$h' = \begin{cases} 1 & \text{for } r = 1 \\ -\eta/(1-\eta) & \text{for } r = 2 \end{cases} \tag{13}$$

According to the results of Evtushenko et al. [10], the asymptotic approach to the macro-modelling of the considered layered composite leads to the governing relations of certain macro-homogeneous medium (the homogenized model), given in terms of the macroscopic temperature ϑ, the macroscopic electric potential φ and macroscopic displacements w_i (after eliminating all microlocal parameters Γ, H, d_i and in the absence of heat sources, body forces and electric charges) as follows:

- The governing equation of heat conduction for the macro-temperature ϑ:

$$\vartheta_{,11} + \vartheta_{,22} + K_0^{-2}\vartheta_{,33} = 0 \;\Leftrightarrow\; \Delta\vartheta + K_0^{-2}\vartheta_{,33} = 0 \tag{14}$$

- The governing equations for macro-displacements w_i and the macro-electric potential φ:

$$C_{1j}\, w_{j,1j} + C_{66}\,(w_{1,22} + w_{2,12}) + C_{44}\,(w_{1,33} + w_{3,13}) + E\,\varphi_{,31} = K_1\,\vartheta_{,1} \tag{15}$$

$$C_{2j}\, w_{j,2j} + C_{66}\,(w_{1,12} + w_{2,11}) + C_{44}\,(w_{2,33} + w_{3,23}) + E\,\varphi_{,32} = K_1\,\vartheta_{,2} \tag{16}$$

$$C_{3j}\, w_{j,3j} + C_{44}\,(w_{1,13} + w_{2,23} + w_{3,\gamma\gamma}) + E_{15}\,\varphi_{,\gamma\gamma} + E_{33}\,\varphi_{,33} = K_3\,\vartheta_{,3} \tag{17}$$

$$E\,(w_{1,13} + w_{2,23}) + E_{15} w_{3,\gamma\gamma} + E_{33} w_{3,33} - \breve{E}_{11}\varphi_{,\gamma\gamma} - \breve{E}_{33}\varphi_{,33} = -P_3\vartheta_{,3} \tag{18}$$

- The constitutive relations for fluxes $q_i^{(r)}$, stresses $\sigma_{ij}^{(r)}$ and electric displacements $D_i^{(r)}$:

$$q_\alpha^{(r)} = -k_r\,\vartheta_{,\alpha}, \qquad q_3 = -K\,\vartheta_{,3} \tag{19}$$

$$\sigma_{\alpha 3} = C_{44}\,(w_{\alpha,3} + w_{3,\alpha}) + E_{15}\,\varphi_{,\alpha} \tag{20}$$

$$\sigma_{33} = C_{13}\,(w_{1,1} + w_{2,2}) + C_{33}\,w_{3,3} + E_{33}\,\varphi_{,3} - K_3\,\vartheta \tag{21}$$

$$\sigma_{12}^{(r)} = c_{66}^{(r)}\,(w_{1,2} + w_{2,1}) \tag{22}$$

$$\sigma_{11}^{(r)} = d_{11}^{(r)}\,w_{1,1} + d_{12}^{(r)}\,w_{2,2} + d_{13}^{(r)}\,w_{3,3} + E_{31}^{(r)}\varphi_{,3} - K_1^{(r)}\vartheta \tag{23}$$

$$\sigma_{22}^{(r)} = d_{12}^{(r)}\,w_{1,1} + d_{11}^{(r)}\,w_{2,2} + d_{13}^{(r)}\,w_{3,3} + E_{31}^{(r)}\varphi_{,3} - K_1^{(r)}\vartheta \tag{24}$$

$$D_\alpha = E_{15}\,(w_{\alpha,3} + w_{3,\alpha}) - \breve{E}_{15}\,\varphi_{,\alpha} \tag{25}$$

$$D_3 = E_{31}\,(w_{1,1} + w_{2,2}) + E_{33}\,w_{3,3} - \breve{E}_{33}\,\varphi_{,3} + P_3\,\vartheta \tag{26}$$

The coefficients appearing in Eqs. (14)–(26) are given in Appendix A. They depend in a complicated way on material and geometrical properties of the composite constituents.

It is noteworthy to point out the close relation of these equations to fundamental equations of piezothermoelasticity in stationary case for an elastic homogeneous body with transverse isotropy specified by five elastic stiffnesses C_{11}, C_{12}, C_{13}, C_{33}, C_{44}, three piezoelectric constants E_{31}, E_{33}, E_{15}, two dielectric permittivities $\breve{E}_{11}$, $\breve{E}_{33}$, one pyroelectric constant P_3 and two thermal conductivity coefficients $\tilde{k}_1$, K. The difference manifests itself in the fact that fluxes $q_\alpha^{(r)}$and stresses $\sigma_{\alpha\beta}^{(r)}$ suffer a discontinuity on the interfaces. Notice that the condition of ideal thermal and electromechanical contact between the layers is satisfied. Finally, assuming that the two sublayers have the same thermo-electromechanical properties, we pass directly to the well-known equations of classical theory of uncoupled stationary thermo-electroelasticity for a homogeneous transversely isotropic solid (Chen [12]).

3. Formulation of the anticrack problem and its potential solution

Within the scope of the presented homogenized model we are concerned with the following boundary-value problem: find fields ϑ and $\varphi,\ w_i,\ \sigma_{ij},\ D_i$ suitable smooth on $R^3\backslash S$ such that Eqs. (14)–(26) hold subject to:

- Thermal conditions:

$$q_3 = -K\,\vartheta_{,3} = 0 \qquad \forall\ \left(x_1,\, x_2,\, x_3 = 0^{\pm}\right) \in S \tag{27}$$

(heat-insulated anticrack)

$$q_3 = -K\,\vartheta_{,\,3}\ \to\ -q_0 \text{ as } \sqrt{x_1^2 + x_2^2 + x_3^2}\ \to\ \infty \tag{28}$$

(vertical flow of uniform heat)

- Mechanical conditions on S:

$$w_1 = w_2 = 0, \quad w_3 = \varepsilon,\ \forall\ \left(x_1,\, x_2,\, x_3 = 0^{\pm}\right) \in S \tag{29}$$

(rigid inclusion with a vertical unknown small translation ε)

- The condition of electrically-insulated (impermeable) rigid inclusion S:

$$D_3 = 0\,,\ \forall\ \left(x_1,\, x_2,\, x_3 = 0^{\pm}\right) \in S \tag{30}$$

- Mechanical and electrical conditions at infinity (stress and electric-free state):

$$\sigma_{ij} = 0, \quad D_i = 0 \tag{31}$$

It is noteworthy here that a similar boundary-value problem is formulated and solved in Kaczyński and Kaczyński [7]. In what follows, proceeding as in this paper, only main results will be presented in order to solve the above anticrack problem within the homogenized model.

To satisfy the global boundary conditions (28)–(31), superposition is applied to separate the problem into two parts: the first one (attached by 0) relating to a basic state of the homogenized space with no inclusion and the second, corrective part (with tilde) describing the perturbations caused by the existence of the rigid inclusion. Thus, the solution is written as:

$$\vartheta = \overset{0}{\vartheta} + \tilde{\vartheta}, \quad w_i = \overset{0}{w}_i + \tilde{w}_i\,, \quad \Phi = \overset{0}{\Phi} + \tilde{\Phi}, \quad \sigma_{ij} = \overset{0}{\sigma}_{ij} + \tilde{\sigma}_{ij}\,,\ \ D_i = \overset{0}{D}_i + \tilde{D}_i \tag{32}$$

The results for the first 0-problem involving the solution to the basic equations (14)–(18) with conditions (28) and (31) are computed as:

$$\begin{aligned} \overset{0}{\vartheta}(x_1, x_2, x_3) &= \tfrac{q_0}{K}x_3, \ \overset{0}{w}_\alpha = \tfrac{q_0}{k_3}n_\gamma x_\gamma \delta_{\alpha\gamma} x_3, \ \overset{0}{w}_3 = \tfrac{q_0}{2K}\left[n_2 x_3^2 - n_1\left(x_1^2+x_2^2\right)\right] \\ \overset{0}{\Phi} &= -\tfrac{q_0}{2K}n_3 x_3^2, \ \overset{0}{\sigma}_{ij}=0, \quad \overset{0}{D}_i = 0 \end{aligned} \tag{33}$$

where the constants n_i are found from the linear system of equations

$$\begin{bmatrix} C_{11}+C_{12} & C_{13} & -E_{31} \\ 2C_{13} & C_{33} & -E_{33} \\ 2E_{31} & E_{33} & \breve{E}_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} K_1 \\ K_3 \\ -P_3 \end{bmatrix} \tag{34}$$

Attention will be concentrated next on the non-trivial solution of the perturbed problem.

The disturbing thermal field $\tilde{\vartheta}$, which is odd in x_3 and vanishes at infinity, is determined by solving Eq. (14) in the half-space $x_3 \geq 0$ with the following boundary conditions:

$$\begin{aligned} &\Delta\tilde{\vartheta} + K_0^{-2}\tilde{\vartheta}_{,33} = 0, \ \forall\,(x_1, x_2, x_3), \ x_3 \geq 0 \\ &\tilde{\vartheta}_{,3}|_{S^+} = -\tfrac{q_0}{K}, \quad \tilde{\vartheta}|_{R^2-S^+} = 0, \quad \tilde{\vartheta} \to 0 \ \text{as} \ \sqrt{x_1^2+x_2^2+x_3^2} \to \infty \end{aligned} \tag{35}$$

Making use of the potential theory (Kellogg [13]), a solution is written via the thermal potential $\tilde{\omega}(x_1, x_2, z_0)$ such that

$$\tilde{\vartheta}(x_1, x_2, x_3) = -\frac{\partial^2 \tilde{\omega}(x_1, x_2, z_0)}{\partial z_0^2}, \ z_0 = K_0 x_3 \geq 0, \ \left(\Delta + \frac{\partial^2}{\partial z_0^2}\right)\tilde{\omega} = 0 \tag{36}$$

Assuming that

$$\tilde{\omega}(x_1, x_2, z_0) = \iint\limits_S \ln\left(\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + z_0^2} + z_0\right)\gamma(\xi_1, \xi_2)\,d\xi_1 d\xi_2 \tag{37}$$

we obtain from (35) the following integro-differential equation of Newton's potential type for the unknown density $\gamma(x_1, x_2)$:

$$\Delta \iint\limits_S \frac{\gamma(\xi_1, \xi_2)\ d\xi_1\, d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}} = -\frac{q_0}{K\,K_0} \tag{38}$$

Note that this equation is similar to that which arises in mode I crack problem for the case of constant loads (Fabrikant [14]). Moreover, the desired macro-temperature has a jump on the surface S:

$$\tilde{\vartheta}\left(x_1, x_2, x_3 = 0^+\right) - \tilde{\vartheta}\left(x_1, x_2, x_3 = 0^-\right) = 4\pi\gamma(x_1, x_2), \quad \forall\,(x_1, x_2) \in S \tag{39}$$

We proceed now to the associated problem of electroelasticity that is governed by Eqs. (15)–(18) and (20)–(26) with the unknowns marked by the tilde. Because of the anti-symmetry of the temperature and stress system, and bearing in mind Eqs. (32), (33), (29), (30) and the resulting conditions for the displacements (i.e., ($\tilde{w}_1$, $\tilde{w}_2$, $\tilde{D}_3$ are odd in x_3,$\tilde{w}_3$ is even in x_3), the anticrack perturbed problem may be formulated as a mixed problem over a half-space $x_3 \geq 0$ with the following boundary conditions:

$$\tilde{w}_\alpha \left(x_1, x_2, x_3 = 0^+\right) = 0\,, \quad \forall\,(x_1, x_2) \in R^2 \quad (\alpha = 1,\ 2) \tag{40}$$

$$\tilde{w}_3 \left(x_1, x_2, x_3 = 0^+\right) = \frac{q_0\, n_1}{2K} \left(x_1^2 + x_2^2\right) + \varepsilon\,, \quad \forall\ (x_1, x_2) \in S \tag{41}$$

$$\tilde{\sigma}_{3\,3} \left(x_1, x_2, x_3 = 0^+\right) = 0\,, \quad \forall\ (x_1, x_2) \in R^2 - S, \quad \forall\,(x_1, x_2) \in R^2 \tag{42}$$

$$\tilde{D}_3 \left(x_1, x_2, x_3 = 0^+\right) = 0\,, \quad \forall\,(x_1, x_2) \in R^2 \tag{43}$$

$$\tilde{w}_i = \mathrm{O}\left(\frac{1}{|\mathbf{x}|}\right) \quad \text{as} \quad |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty \tag{44}$$

For solving this boundary-value problem we use the potential function approach with some modifications well suited to the above boundary conditions, derived in Kaczyński and Kaczyński [7]. It is based on constructing a special representation of governing equations (15)–(18) expressed by a single harmonic function $\tilde{f}\,(x_1, x_2, x_3)$ as follows:

$$\begin{aligned} \tilde{w}_\alpha &= b_i \left[\tilde{f}\,(x_1, x_2, z_i)\right]_{,\alpha} + A_i \left[\tilde{\omega}\,(x_1, x_2, z_i)\right]_{,\,\alpha} \\ &+ c_1 \left[\tilde{\omega}\,(x_1, x_2, z_0)\right]_{,\alpha} \end{aligned} \tag{45}$$

$$\begin{aligned} \tilde{w}_3 &= m_i s_i b_i \left[\tilde{f}\,(x_1, x_2, z_i)\right]_{,z_i} + m_i s_i A_i \left[\tilde{\omega}\,(x_1, x_2, z_i)\right]_{,z_i} \\ &- c_2 K_0 \left[\tilde{\omega}\,(x_1, x_2, z_0)\right]_{,z_0} \end{aligned} \tag{46}$$

$$\begin{aligned} \tilde{\Phi} &= l_i s_i b_i \left[\tilde{f}\,(x_1, x_2, z_i)\right]_{,\,z_i} + l_i s_i A_i \left[\tilde{\omega}\,(x_1, x_2, z_i)\right]_{,\,z_i} \\ &- c_3 K_0 \left[\tilde{\omega}\,(x_1, x_2, z_0)\right]_{,z_0} \end{aligned} \tag{47}$$

$$\begin{aligned} \tilde{\sigma}_{3\,\alpha} &= a_i s_i b_i \left[\tilde{f}\,(x_1, x_2, z_i)\right]_{,\,z_i\,\alpha} + a_i s_i A_i \left[\tilde{\omega}\,(x_1, x_2, z_i)\right]_{,\,z_i\,\alpha} \\ &+ \delta_1 \left[\tilde{\omega}\,(x_1, x_2, z_0)\right]_{,z_0\,\alpha} \end{aligned} \tag{48}$$

$$\begin{aligned} \tilde{\sigma}_{3\,3} &= a_i b_i \left[\tilde{f}\,(x_1, x_2, z_i)\right]_{,\,z_i\,z_i} + a_i A_i \left[\tilde{\omega}\,(x_1, x_2, z_i)\right]_{,\,z_i\,z_i} \\ &- \delta_3 \left[\tilde{\omega}\,(x_1, x_2, z_0)\right]_{,\,z_0\,z_0} \end{aligned} \tag{49}$$

$$\begin{aligned} \tilde{D}_\alpha &= d_i s_i b_i \left[\tilde{f}\,(x_1, x_2, z_i)\right]_{,\,z_i\,\alpha} + d_i s_i A_i \left[\tilde{\omega}\,(x_1, x_2, z_i)\right]_{,\,z_i\,\alpha} \\ &+ \tau_1 \left[\tilde{\omega}\,(x_1, x_2, z_0)\right]_{,z_0\,\alpha} \end{aligned} \tag{50}$$

$$\begin{aligned} \tilde{D}_3 &= d_i b_i \left[\tilde{f}\,(x_1, x_2, z_i)\right]_{,\,z_i\,z_i} + d_i A_i \left[\tilde{\omega}\,(x_1, x_2, z_i)\right]_{,\,z_i\,z_i} \\ &- \tau_3 \left[\tilde{\omega}\,(x_1, x_2, z_0)\right]_{,\,z_0\,z_0} \end{aligned} \tag{51}$$

here, $z_0 = K_0\, x_3\,,\ z_i = s_i\, x_3$ and the material constants s_i, m_i, l_i, c_i, a_i, b_i, d_i, δ_1, δ_3, τ_1, τ_3, A_i are defined in Appendix B.

It is then shown that the anticrack perturbed problem described by Eqs. (40)–(44) is reduced to the classical mixed potential problem (cf. Sneddon [15]) of finding the harmonic function $\tilde{f}$ in the half-space $x_3 \geq 0$ with the boundary conditions:

$$m_i s_i b_i \left[\frac{\partial \tilde{f}\left(x_1, x_2, x_3\right)}{\partial x_3}\right]_{x_3=0^+} = \tilde{r}_3\left(x_1, x_2\right), \quad \forall\left(x_1, x_2\right) \in S \tag{52}$$

$$\left[\frac{\partial^2 \tilde{f}\left(x_1, x_2, x_3\right)}{\partial x_3^2}\right]_{x_3=0^+} = 0, \quad \forall\left(x_1, x_2\right) \in R^2 - S \tag{53}$$

with:

$$\begin{aligned} &\tilde{r}_3\left(x_1, x_2\right) = \tilde{\beta}\left[\frac{\partial \tilde{\omega}\left(x_1, x_2, z_0\right)}{\partial z_0}\right]_{z_0=0} + \frac{q_0 n_1}{2K}\left(x_1^2 + x_2^2\right) + \varepsilon \\ &\tilde{\beta} = c_2 K_0 - m_i s_i A_i \end{aligned} \tag{54}$$

A well-known solution to this classical boundary problem in potential theory (Kellogg [13]) may be written in the following form:

$$\begin{aligned} &\tilde{f}\left(x_1, x_2, x_3\right) \\ &= \frac{-1}{2\pi\, a_i b_i} \iint\limits_S \tilde{\sigma}_{33}\left(\xi_1, \xi_2, 0^+\right) \ln\left(\sqrt{\left(x_1 - \xi_1\right)^2 + \left(x_2 - \xi_2\right)^2 + x_3^2} + x_3\right) d\xi_1 d\xi_2 \end{aligned} \tag{55}$$

Then, enforcing the displacement boundary condition (52), we arrive at the governing two-dimensional singular integral equation of Newtonian potential type to determine the unknown normal stress $\tilde{\sigma}_{33}^{+}\left(x_1, x_2\right) \equiv \tilde{\sigma}_{33}\left(x_1, x_2, 0^+\right)$, $\left(x_1, x_2\right) \in S$ on the upper side of S:

$$\frac{m_j s_j b_j}{2\pi\, a_i b_i} \iint\limits_S \frac{\tilde{\sigma}_{33}^{+}\left(\xi_1, \xi_2\right)\, d\xi_1\, d\xi_2}{\sqrt{\left(x_1 - \xi_1\right)^2 + \left(x_2 - \xi_2\right)^2}} = -\tilde{r}_3\left(x_1, x_2\right), \quad \forall\left(x_1, x_2\right) \in S \tag{56}$$

The above equation is analogous to the well-known governing equation of the frictionless contact problem of an isotropic half-space under the action of a rigid punch (see, for instance, Fabrikant [14]).

Having found the stress $\tilde{\sigma}_{33}^{+}|_S$ from the solution to this equation, the parameter εwill be obtained from the equilibrium condition having a form:

$$\iint\limits_S \tilde{\sigma}_{33}^{+}\left(x_1, x_2\right)\, dx_1\, dx_2 = 0 \tag{57}$$

Moreover, the main potential $\tilde{f}$is found from Eq. (55) and the whole perturbed electroelastic fields can be obtained from relations (45)–(51).

It is worth mentioning that for a rigid inclusion with an arbitrary shape S, the governing equations (38) and (56) can be generally solved by numerical methods.

4. Example: interface circular anticrack in a uniform heat flow

For illustration, making use of the results of Kaczyński and Kaczyński [7], a solution expressed in elementary functions will be presented for a rigid circularly shaped inclusion, i.e.:

$S = \left\{ (x_1 = r\cos\theta, x_2 = r\sin\theta,\ x_3 = 0) : 0 \le r = \sqrt{x_1^2 + x_2^2} \le a \wedge 0 \le \theta \le 2\pi \right\}$

Accordingly, the axially-symmetric solution to the thermal perturbed problem is given by:

$$\gamma(x_1, x_2) = \tilde{\gamma}(r) = \frac{q_0}{\pi^2 \sqrt{\tilde{k}_1 K}} \sqrt{a^2 - r^2}, \quad 0 \le r \le a \tag{58}$$

$$\begin{aligned} \tilde{\omega}(r, z_0) = \tfrac{q_0}{2\pi \sqrt{\tilde{k}_1 K}} \Big[z_0 \left(2a^2 + \tfrac{2}{3} z_0^2 - r^2\right) \sin^{-1} \tfrac{a}{l_{20}} + \\ + \tfrac{1}{3} \sqrt{a^2 - l_{10}^2} \left(5r^2 - \tfrac{10}{3} a^2 - 2l_{20}^2 - \tfrac{11}{3} l_{10}^2\right) + \tfrac{4}{3} a^3 \ln \left[l_{20} + \sqrt{l_{20}^2 - r^2} \right] \Big] \end{aligned} \tag{59}$$

$$\tilde{\vartheta}(r, z_0) = -\frac{\partial^2 \tilde{\omega}}{\partial z_0^2} = -\frac{2\, q_0}{\pi \sqrt{\tilde{k}_1\, K}} \left(z_0 \sin^{-1} \frac{a}{l_{20}} - \sqrt{a^2 - l_{10}^2} \right) \tag{60}$$

where Fabrikant's [14] notation has been used:

$$\begin{aligned} L_1 \equiv L_1(a,\, r,\, x_3) = \frac{1}{2} \left[\sqrt{(r+a)^2 + x_3^2} - \sqrt{(r-a)^2 + x_3^2} \right], l_{10} = L_1(a, r, z_0) \\ L_2 \equiv L_2(a,\, r,\, x_3) = \frac{1}{2} \left[\sqrt{(r+a)^2 + x_3^2} + \sqrt{(r-a)^2 + x_3^2} \right], l_{20} = L_2(a, r, z_0) \end{aligned} \tag{61}$$

along with the following properties:

$$[L_1]_{x_3=0} = [l_{10}]_{z_0=0} = \min(a, r), \qquad [L_2]_{x_3=0} = [l_{20}]_{z_0=0} = \max(a, r) \tag{62}$$

In turn, an analytical solution to the governing equation (56) is:

$$\tilde{\sigma}_{33}^{+}(r) = \frac{\tilde{\beta}_3^{(e)} q_0}{\pi} \frac{2a^2 - 3r^2}{\sqrt{a^2 - r^2}}, \qquad 0 \le r < a \tag{63}$$

where:

$$\tilde{\beta}_3^{(e)} = \frac{2a_i b_i}{3m_j s_j b_j} \beta_0; \quad \beta_0 = \frac{2n_1}{K} - \frac{\tilde{\beta}}{\sqrt{\tilde{k}_1 K}} \tag{64}$$

Moreover, the vertical rigid displacement is found from (57) as:

$$\varepsilon = -\frac{a^2 q_0}{3} \left(\frac{\tilde{\beta}}{\sqrt{\tilde{k}_1 K}} + \frac{n_1}{K} \right) \tag{65}$$

The main harmonic potential for the electroelastic perturbed problem is obtained by calculating integral (55) with the use of (63). As a result, we find that for $x_3 \ge 0$:

$$\begin{aligned} \tilde{f}(x_1, x_2,\, x_3) = -\tfrac{\tilde{\beta}_3^{(e)} q_0}{2\pi^2 a_i b_i} \Big[x_3 \sin^{-1} \tfrac{a}{L_2} \left(a^2 - \tfrac{3}{2} r^2 + x_3^2\right) + \\ + \sqrt{a^2 - L_1^2} \left(5r^2 + \tfrac{1}{3} a^2 - L_2^2 - \tfrac{11}{6} L_1^2\right) \Big] \end{aligned} \tag{66}$$

The full-space piezothermoelastic field can be obtained from formulas (45)–(51). The derivation is omitted here to save the space of the paper. To investigate the singular behavior of the thermal-electric-stress field near the disc edge, however, some relevant interfacial quantities in the plane of the anticrack are presented below:

$$\vartheta\left(x_1, x_2, 0^{\pm}\right) = \begin{cases} \pm \frac{2\,q_0}{\pi\,K\,K_0}\,\sqrt{a^2 - r^2}\ , & 0 \le r \le a \\ 0\,, & r > a \end{cases} \tag{67}$$

$$q_3\left(r, 0^{\pm}\right) = -K\,\vartheta_{,3}\left(r,\,0^{\pm}\right) = \begin{cases} 0\,, & 0 \le r < a \\ \frac{2\,q_0}{\pi}\left(\sin^{-1}\frac{a}{r} - \frac{a}{\sqrt{r^2-a^2}}\right) - q_0\,, & r > a \end{cases} \tag{68}$$

$$\sigma_{33}\left(r, 0^{\pm}\right) = \begin{cases} \pm\,\frac{\tilde{\beta}_3^{(e)} q_0}{\pi}\,\frac{2a^2-3r^2}{\sqrt{a^2-r^2}}\,, & 0 \le r < a \\ 0\,, & r > a \end{cases} \tag{69}$$

$$\begin{aligned} \sigma_{3\,r}\left(r, 0^{\pm}\right) &= \sigma_{31}\left(r, 0^{\pm}\right)\cos\theta + \sigma_{32}\left(r, 0^{\pm}\right)\sin\theta = \\ &= \begin{cases} \tilde{\beta}^{(e)} q_0\, r\,, & 0 \le r < a \\ \frac{2q_0}{\pi}\left(\tilde{\beta}^{(e)} r \sin^{-1}\frac{a}{r} - \frac{\tilde{\beta}_r^{(e)} a^3}{r\,\sqrt{r^2-a^2}} - \frac{\tilde{\beta}^{(e)} a\,\sqrt{r^2-a^2}}{r}\right), & r > a \end{cases} \end{aligned} \tag{70}$$

$$\begin{aligned} D_r\left(r, 0^{\pm}\right) &= D_1\left(r, 0^{\pm}\right)\cos\theta + D_2\left(r, 0^{\pm}\right)\sin\theta = \\ &= \begin{cases} \tilde{\beta}^{(ed)} q_0\, r\,, & 0 \le r < a \\ \frac{2q_0}{\pi}\left(\tilde{\beta}^{(ed)} r \sin^{-1}\frac{a}{r} - \frac{\tilde{\beta}_r^{(ed)} a^3}{r\,\sqrt{r^2-a^2}} - \frac{\tilde{\beta}^{(ed)} a\,\sqrt{r^2-a^2}}{r}\right), & r > a \end{cases} \end{aligned} \tag{71}$$

with the following constants:

$$\begin{aligned} \tilde{\beta}^{(e)} &= \frac{3\tilde{\beta}_3^{(e)} a_j s_j b_j}{4\,a_i b_i} - \frac{(a_k s_k A_k + \delta_1)}{2\sqrt{\tilde{k}_1 K}}\ , \quad & \tilde{\beta}_r^{(e)} &= \frac{a_i s_i b_i}{3\,m_j s_j b_j}\beta_0 \\ \tilde{\beta}^{(ed)} &= \frac{3\tilde{\beta}_3^{(e)} d_j s_j b_j}{4\,a_i b_i} - \frac{(d_k s_k A_k + \tau_1)}{2\sqrt{\tilde{k}_1 K}}\ , \quad & \tilde{\beta}_r^{(ed)} &= \frac{d_i s_i b_i}{3\,m_j s_j b_j}\beta_0 \end{aligned} \tag{72}$$

Now, it can be emphasized that the singularity of the thermal stresses and electric displacements close to the edge of the anticrack has the order $r^{-1/2}$, contrary to the oscillatory type observed in the elastic fields relating to bimaterial interfaces (Li and Fan [16]). Analyzing the above expressions, we reveal that:

- The anticrack S obstructs locally the heat flow, producing the jump of temperature and the drastic change of its gradient on the surface near its front.
- The sign of normal stress $\sigma_{33}\,|_S$ changes at $r = \sqrt{2/3}\,a$. Moreover, this stress suffers the jump across S and exhibits the inverse square root singularity at $r = a^-$. One would expect a separation (detachment) of the surrounding matrix material from the anticrack surface described by the stress singularity coefficient:

$$S_{\mathrm{I}}^{\pm} = \lim_{r \to a^-} \sqrt{2\pi\,(a-r)}\,\sigma_{33}\left(r,\,0^{\pm}\right) \ = \ \mp\frac{\tilde{\beta}_3^{(e)} q_0\, a\sqrt{a}}{\sqrt{\pi}} \tag{73}$$

- Another mechanism controlling the material cracking around the anticrack front is Mode II (edge-sliding) described by the thermal and electric stress intensity factors:

$$K_{\mathrm{II}}^{(e)} = \lim_{r\to a^+} \sqrt{2\pi\,(r-a)}\,\sigma_{3\,r}\,(r,\,0) \;=\; -\,\frac{2\,\tilde{\beta}_r^{(e)} q_0\, a\sqrt{a}}{\sqrt{\pi}} \tag{74}$$

$$K_{\mathrm{II}}^{(ed)} = \lim_{r\to a^+} \sqrt{2\pi\,(r-a)}\,D_r\,(r,\,0) \;=\; -\,\frac{2\,\tilde{\beta}_r^{(ed)} q_0\, a\sqrt{a}}{\sqrt{\pi}} \tag{75}$$

The above-mentioned parameters may be used in the appropriate criterion for initiating fractures at the edge of the inclusion.

Finally, the obtained solution for a stratified medium containing of a large number of alternating plane-parallel layers of two different piezoelectric materials can be converted to that considered in Kaczyński and Kaczyński [7] for homogeneous case.

5. Conclusions

The three-dimensional thermal stress problem for an interface insulated rigid inclusion obstructing a uniform heat flux in a two-layer microperiodic piezoelectric space has been investigated within the homogenized model with microlocal parameters. Using the potential function method, the problem involving the inclusion of arbitrary shape has been reduced to classical boundary problems of potential theory. Specifically, with the knowledge of the steady-state temperature distribution, the governing equation similar in form to that reported in the literature on contact problems in elasticity was derived. In particular, for a circularly-shaped inclusion the complete solution was obtained in terms of elementary functions. Explicit expressions for the thermo-piezoelastic fields at the plane of inclusion surface were given and interpreted from the point of view of linear fracture theory.

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Appendix A:

The effective constants in the governing equations (14)–(26) of the homogenized model are derived by using the following averaging operators on the material function M taking constant values $M^{(r)}$ in the r-sublayer:

$$\begin{aligned} &\tilde{M} = \eta M^{(1)} + (1-\eta)\, M^{(2)}, \quad [M] = \eta \left(M^{(1)} - M^{(2)} \right) \\ &\hat{M} = \eta M^{(1)} + \frac{\eta^2}{1-\eta} M^{(2)}, \quad \eta = \delta_1/\delta \end{aligned} \tag{A.1}$$

They are given as follows:

$$\begin{aligned} &K_0 = \left(\tilde{k}_1/K\right)^{1/2} \qquad \tilde{k}_1 = \eta\, k_1^{(1)} + (1-\eta)\, k_1^{(2)} \\ &K = \tilde{k}_3 - \frac{[k_3]^2}{\hat{k}_3} = \frac{k_3^{(1)} k_3^{(2)}}{(1-\eta)\, k_3^{(1)} + \eta\, k_3^{(2)}} \end{aligned} \tag{A.2}$$

$$\begin{aligned} &C_{11} = C_{22} = \tilde{c}_{11} - \Delta_{11}, \quad C_{12} = C_{21} = \tilde{c}_{12} - \Delta_{11}, \\ &C_{13} = C_{31} = C_{23} = C_{32} = \tilde{c}_{13} - \Delta_{11} \\ &\Delta_{11} = \frac{[c_{13}]\left(\hat{\varepsilon}_{33}[c_{13}] + \hat{e}_{31}[e_{31}]\right) + [e_{31}]\left(\hat{e}_{31}[c_{13}] - \hat{c}_{33}[e_{31}]\right)}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \end{aligned} \tag{A.3}$$

$$\begin{aligned} &C_{33} = \tilde{c}_{33} - \Delta_{33} \\ &\Delta_{33} = \frac{[c_{33}]\left(\hat{\varepsilon}_{33}[c_{33}] + \hat{e}_{31}[e_{33}]\right) + [e_{31}]\left(\hat{e}_{31}[c_{33}] - \hat{c}_{33}[e_{33}]\right)}{\hat{c}_{33}\,\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \end{aligned} \tag{A.4}$$

$$C_{44} = \tilde{c}_{44} - \frac{[c_{44}]^2}{\hat{c}_{44}} = \frac{c_{44}^{(1)} c_{44}^{(2)}}{(1-\eta)\, c_{44}^{(1)} + \eta\, c_{44}^{(2)}}, \quad C_{66} = (\tilde{c}_{11} - \tilde{c}_{12})/2 \tag{A.5}$$

$$E = \tilde{e}_{31} - \delta_{31} + \tilde{e}_{15} - \delta_{15}, \;\; E_{15} = \tilde{e}_{15} - \delta_{15}, \;\; E_{33} = \tilde{e}_{33} - \delta_{33}$$
$$\delta_{31} = \frac{[c_{13}]\,(\hat{\varepsilon}_{33}\,[e_{31}] + \hat{e}_{31}\,[\varepsilon_{33}]) + [e_{31}]\,(\hat{e}_{31}\,[e_{31}] - \hat{c}_{33}\,[\varepsilon_{33}])}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.6}$$
$$\delta_{15} = \frac{[c_{44}]\,[e_{15}]}{\hat{c}_{44}}, \;\; \delta_{33} = \frac{[c_{33}]\,(\hat{\varepsilon}_{33}\,[e_{31}] + \hat{e}_{31}\,[\varepsilon_{33}]) + [e_{33}]\,(\hat{e}_{31}\,[e_{31}] - \hat{c}_{33}\,[\varepsilon_{33}])}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2}$$

$$\breve{E}_{11} = \tilde{\varepsilon}_{11}$$
$$\breve{E}_{33} = \tilde{\varepsilon}_{33} - \frac{[e_{33}]\,(\hat{\varepsilon}_{33}\,[e_{31}] + \hat{e}_{31}\,[\varepsilon_{33}]) + [\varepsilon_{33}]\,(\hat{e}_{31}\,[e_{31}] - \hat{c}_{33}\,[\varepsilon_{33}])}{\hat{c}_{33}\,\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.7}$$

$$K_1 = \tilde{\beta}_1 - \frac{[c_{13}]\,(\hat{\varepsilon}_{33}\,[\beta_3] - \hat{e}_{31}\,[p_3]) + [e_{31}]\,(\hat{e}_{31}\,[\beta_3] - \hat{c}_{33}\,[p_3])}{\hat{c}_{33}\,\hat{\varepsilon}_{33} + \hat{e}_{31}^2}$$
$$K_3 = \tilde{\beta}_3 - \frac{[c_{33}]\,(\hat{\varepsilon}_{33}\,[c_{33}] + \hat{e}_{31}\,[e_{33}]) + [e_{31}]\,(\hat{e}_{31}\,[c_{33}] - \hat{c}_{33}\,[e_{33}])}{\hat{c}_{33}\,\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.8}$$

$$P_3 = \tilde{p}_3 - \frac{[c_{33}]\,(\hat{\varepsilon}_{33}\,[\beta_3] - \hat{e}_{31}\,[p_3]) + [e_{31}]\,(\hat{e}_{31}\,[\beta_3] + \hat{c}_{33}\,[p_3])}{\hat{c}_{33}\,\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.9}$$

$$d_{11}^{(r)} = c_{11}^{(r)} - c_{13}^{(r)}\, h' \frac{\hat{\varepsilon}_{33}\,[c_{13}] + \hat{e}_{31}\,[e_{13}]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} - e_{31}^{(r)}\, h' \frac{\hat{e}_{31}\,[c_{13}] - \hat{c}_{33}\,[e_{13}]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.10}$$

$$d_{12}^{(r)} = c_{12}^{(r)} - c_{13}^{(r)}\, h' \frac{\hat{\varepsilon}_{33}\,[c_{13}] + \hat{e}_{31}\,[e_{13}]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} - e_{31}^{(r)}\, h' \frac{\hat{e}_{31}\,[c_{13}] - \hat{c}_{33}\,[e_{13}]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.11}$$

$$d_{13}^{(r)} = c_{13}^{(r)} - c_{13}^{(r)}\, h' \frac{\hat{\varepsilon}_{33}\,[c_{33}] + \hat{e}_{31}\,[e_{33}]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} - e_{31}^{(r)}\, h' \frac{\hat{e}_{31}\,[c_{33}] - \hat{c}_{33}\,[e_{33}]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.12}$$

$$E_{31}^{(r)} = e_{31}^{(r)} - e_{31}^{(r)}\, h' \frac{\hat{\varepsilon}_{33}\,[e_{31}] + \hat{e}_{31}\,[\varepsilon_{33}]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} - e_{31}^{(r)}\, h' \frac{\hat{e}_{31}\,[c_{13}] - \hat{c}_{33}\,[\varepsilon_{33}]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.13}$$

$$K_1^{(r)} = \beta_1^{(r)} - c_{13}^{(r)}\, h' \frac{-\hat{\varepsilon}_{33}\,[\beta_3] + \hat{e}_{31}\,[p_3]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} - e_{31}^{(r)}\, h' \frac{-\hat{e}_{31}\,[\beta_3] - \hat{c}_{33}\,[p_3]}{\hat{c}_{33}\hat{\varepsilon}_{33} + \hat{e}_{31}^2} \tag{A.14}$$

Appendix B:

Here, the constants involved in the solution of anticrack perturbed problem developed by Kaczyński and Kaczyński [7] are presented within the homogenized model.

The positive, real and distinct $s_i$are three roots of the material characteristic equation

$$W(s) \equiv a_0\, s^6 - b_0\, s^4 + c_0 s^2 - d_0 = 0 \tag{B.1}$$

where the coefficients are:

$$
\begin{aligned}
a_0 &= C_{44}\left(C_{33}\,\breve{\mathrm{E}}_{33} + E_{33}^2\right)\\
b_0 &= C_{33}\left(C_{44}\,\breve{\mathrm{E}}_{11} + E^2\right) + \breve{E}_{33}C^2 + E_{33}\left(2C_{44}E_{15} + C_{11}E_{33} - 2\,CE\right)\\
c_0 &= C_{44}\left(C_{11}\,\breve{\mathrm{E}}_{33} + E^2\right) + \breve{\mathrm{E}}_{11}C^2 + E_{15}\left(2C_{11}E_{33} + C_{44}E_{15} - 2\,CE\right)\\
d_0 &= C_{11}\left(C_{44}\,\breve{\mathrm{E}}_{11} + E_{15}^2\right)
\end{aligned}
\tag{B.2}
$$

provided:

$$
C^2 = C_{11}C_{33} - C_{13}\left(C_{13} + 2C_{44}\right), \quad C = C_{13} + C_{44}, \quad E = E_{15} + E_{31} \tag{B.3}
$$

The corresponding constants m_i and l_i are expressed as:

$$
\begin{aligned}
m_i &= \frac{-\,C_{44}\,E_{33}\,s_i^4 + (C_{11}\,E_{33} + C_{44}E_{15} - C\,E)\,s_i^2 - C_{11}\,E_{15}}{s_i^2\left[(C\,E_{33} - E\,C_{33})s_i^2 - (C\,E_{15} - E\,C_{44})\right]}\\
l_i &= \frac{C_{44}\,C_{33}\,s_i^4 - \left(C^2 - 2C_{44}^2\right)s_i^2 + C_{11}C_{44}}{s_i^2\left[(CE_{33} - EC_{33})s_i^2 - (C\,E_{15} - E\,C_{44})\right]}
\end{aligned}
\tag{B.4}
$$

The constants c_1, c_2, c_3 are the solution of the following linear system:

$$
\begin{bmatrix}
C_{11} - C_{44}K_0^2 & C\,K_0^2 & E\,K_0^2\\
C & C_{33}K_0^2 - C_{44} & E_{33}K_0^2 - E_{15}\\
E & E_{33}k_0^2 - E_{15} & \breve{E}_{11} - \breve{E}_{33}K_0^2
\end{bmatrix}
\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix}
=
\begin{bmatrix} K_1\\ K_3\\ -P_3 \end{bmatrix}
\tag{B.5}
$$

The remaining material coefficients occurring in the representation (45)–(51) are as follows:

$$
\begin{aligned}
& a_i = C_{44}\left(1 + m_i\right) + E_{15}\,l_i\,, \quad d_i = E_{15}\left(1 + m_i\right) - \breve{E}_{11}l_i\\
& b_1 = d_3 - d_2,\;\; b_2 = d_1 - d_3,\;\; b_3 = d_2 - d_1\\
& \delta_1 = K_0\left[C_{44}\left(c_1 - c_2\right) - E_{15}\,c_3\right], \quad \delta_3 = C_{13}\,c_1 + K_0^2\left(C_{33}c_2 + E_{33}c_3\right) - K_3\\
& \tau_1 = K_0\left[E_{15}\left(c_1 - c_2\right) + \breve{E}_{11}\,c_3\right], \quad \tau_3 = E_{31}\,c_1 + K_0^2\left(E_{33}c_2 - \breve{E}_{33}c_3\right) + P_3
\end{aligned}
\tag{B.6}
$$

and the constants $A_1,\ A_2,\ A_3$ are the solution of the following linear system:

$$
\begin{bmatrix}
1 & 1 & 1\\
d_1 & d_2 & d_3\\
a_1 & a_2 & a_3
\end{bmatrix}
\begin{bmatrix} A_1\\ A_2\\ A_3 \end{bmatrix}
=
\begin{bmatrix} -\,c_1\\ \tau_3\\ \delta_3 \end{bmatrix}
\tag{B.7}
$$