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Asymptotic Behavior of Magnetostrictive Elasticity for Microstructured Materials

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According to the classical theory of Weiss, Landau, and Lifshitz, in a ferromagnetic body there is a spontaneous magnetization field m, such that $\|m\| = \tau^0 = const$ in all points of this material Ω. In any stationary configuration, this ferromagnetic body consists of areas (Weiss domains) in which the magnetization is uniform (i.e. $m = const$) separated by thin transition layers (Bloch walls). Such stationary configuration corresponds to the minimum point of the magnetostrictive free energy E. We are considering an elastic magnetostrictive body in our paper. The elastic magnetostrictive free energy E_δ depends on a small parameter δ such that $\delta \to 0$. As usual, the displacement field is denoted by u. We will show that each sequence of minimizers (u_i, m_i) contains a subsequence that converges to a couple of fields (u_0, m_0). By means of a Γ-limit procedure we will show that this couple (u_0, m_0) is a minimizer of the new functional E_0. This new functional E_0 describes the magnetic-elastic properties of the body with microstructure.

Keywords: homogenization, Γ-limit procedure, existence of a minimum for free energy, elasticity with microstructure.

1. Introduction

Ferromagnetic behavior, in particular the presence of spontaneous magnetization, can be explained by the theory of Weiss, Landau, and Lifshitz. This states that, in any stationary configuration, a ferromagnetic body breaks up into uniformly magnetized regions (Weiss domains) separated by thin transition layers (Bloch walls). The magnetization field m has fixed modulus and variable orientation $\|m(x)\|_{\mathrm{R}^3} = \tau^0 = const$ in all points x of this ferromagnetic body Ω. Such stationary configuration corresponds to the minimum point of the magnetostrictive free energy E. The model of free energy E is equivalent to the model considered in [1] (formulae (2.6), (2.7)).

The asymptotic behavior of magnetostrictive materials has been considered in [2] in the case of a rigid body in which elastic energy has been omitted. In our paper, the elastic magnetostrictive free energy E_δ depends on a small parameter δ such that $\delta \to 0$. As usual, the displacement field is denoted by u. We will show that each sequence of minimizers (u_n, m_n) contains a subsequence that converges to a couple of fields (u_0, m_0). By means of a Γ-limit procedure we will show that this couple (u_0, m_0) is a minimizer of the new functional E_0. This new functional E_0 is the Γ-limit of the sequence E_δ. This Γ-limit E_0 describes the magnetic-elastic properties of the body with microstructure, equivalently as described in [3].

The main purpose of the work is to show that the asymptotic form of the magnetic-elastic model (formulated in the paper [1]) is a model of an elastic material with a microstructure (given in the presentation [3]).

2. The physical model

Let us consider a ferromagnetic body occupying a bounded domain Ω of the Euclidean space R^3. Assume also that the body has a uniform temperature below its Curie point. We do not assume that the considered body is homogeneous. Under such conditions, on a sufficiently small, but macroscopic scale, the magnetization field m has a prescribed constant modulus, i.e.:

$$\|m(x)\| = \tau^0 = const \tag{1}$$

in all $x \in \Omega$, and variable orientation. This scale is still larger than that of the lattice structure, so the field m can be assumed to vary smoothly in space.

The ferromagnetic behavior is essentially due to the occurrence of a force which tends to align the magnetic field. This corresponds to an energy contribution depending on the space derivatives of m. In a simplified form so called exchange energy is given by:

$$a \int_\Omega \|\nabla m(x)\|^2_{\mathrm{R}^{3\times3}}\, dx \tag{2}$$

where $a > 0$. The non-convex anisotropy density $\varphi(m)$ defines the anisotropic energy:

$$\int_\Omega \varphi(m(x))dx \tag{3}$$

Exterior magnetic field f defines the energy:

$$-\int_\Omega f(x) \circ m(x)dx \tag{4}$$

where $\circ$ is the scalar multiplication in R^3.

The considered body, occupying Ω, is clamped on a part Γ_2 of the boundary $\partial\Omega$ of Ω. We assume that $\Gamma_2 \neq \emptyset$. Let $\Gamma_1 = \partial\Omega - \Gamma_2$. For a boundary force $G : \Gamma_1 \to \mathrm{R}^3$ its work is defined by the integral:

$$-\int_{\Gamma_1} G(x) \circ u(x) d\mu(x) \tag{5}$$

where μ is the 2-dimensional measure on $\partial\Omega$. For a volume force $F : \Omega \to \mathrm{R}^3$ its work is defined by the formula:

$$-\int_{\Omega} F(x) \circ u(x) dx \tag{6}$$

The elastic potential at the point $x \in \Omega$ is defined by $j(x, \varepsilon)$ for any symmetric matrix $\varepsilon \in \mathrm{R}^{3\times3}$. In particular, we can assume that $2j(x, \varepsilon) = \varepsilon \circ C(x)\varepsilon$, where $C(x)$ is the tensor of elastic moduli in the point x. The potential w (defined in R^3) represents the magnetostatic energy:

$$\frac{1}{2} \int_{\mathrm{R}^3} \|\nabla w(x)\|^2_{\mathrm{R}^3} \, dx \tag{7}$$

The magnetostrictive free energy E_δ consists of seven terms. These terms are known as the exchange energy, the anisotropic energy, the energy of the external field, magnetostatic energy, work of volume and boundary force, and the elastic-magnetic energy. The magnetostrictive free energy, below the Curie point, is defined by:

$$\begin{aligned} E_\delta(u, w, m) &= \delta\, a \int_\Omega \|\nabla m(x)\|^2_{\mathrm{R}^{3\times3}} \, dx + \tfrac{1}{\delta} \int_\Omega \varphi(m(x)) dx \\ &- \int_\Omega f(x) \circ m(x) dx + \tfrac{1}{2} \int_{\mathrm{R}^3} \|\nabla w(x)\|^2_{\mathrm{R}^3} \, dx - \int_\Omega F(x) \circ u(x) dx \\ &- \int_{\Gamma_1} G(x) \circ u(x) d\mu(x) + \int_\Omega j(x, \varepsilon(u(x)) - \varepsilon^0(m(x))) dx \end{aligned} \tag{8}$$

where $\varepsilon^0(m(x))$ is the free or spontaneous *magnetostrain tensor* corresponding to the magnetization m (in the case of cubic crystal lattice the most general quadratic form of $\varepsilon^0(m)$ is given in [1], formula (2.4)). As usual $2\varepsilon(u) = (\nabla u + \nabla u^T)$. The magnetostatic potential w satisfy Maxwell's equation:

$$div(-\nabla w + m_{\chi\Omega}) = 0 \tag{9}$$

in R^3, where $m_{\chi\Omega}(x) = m(x)$ if $x \in \Omega$ and $m_{\chi\Omega}(x) = 0$ if $x \notin \Omega$. Let:

$$S^2_{\tau^0} = \{\xi \in \mathrm{R}^3 : \|\xi\| = \tau^0 > 0\} \tag{10}$$

We consider sequence of free energies E_δ where $\delta \to 0$. Then for the Γ-limit E_0 of the sequence $\{E_\delta\}$, the minimum point m_0 of E_0 is a discontinuous function, see [2], pages 182-183. The field m_0 satisfies:

$$\varphi(m_0(x)) = \inf_m \{\varphi(m) : m \in S^2_{\tau^0}\} \tag{11}$$

for all $x \in \Omega$. That is the minimum point m_0 of E_0 is a spontaneous magnetization field uniform on regions, separated by two-dimensional transitional surfaces.

The Γ-convergence method is based on the fact that we are considering the sequence of functionals E_δ (for $\delta \searrow 0$) such that the exchange energy tends to 0 and the anisotropic energy tends to infinity (except for the minimum points of magnetic field). The Γ-limit E_0 of the sequence $\{E_\delta\}$, at the fixed point (u, w, m) is equal to the lower limit of all possible sequences $E_\delta(u_\delta, w_\delta, m_\delta)$ where $(u_\delta, w_\delta, m_\delta)$ converges to (u, w, m). That is, the value of $E_0(u, w, m)$ is equal to infimum for all limits $\liminf_{\delta\searrow 0} E_\delta(u_\delta, w_\delta, m_\delta)$ (so infimum after all convergent sequences $(u_\delta, w_\delta, m_\delta) \to (u, w, m)$ and after all sequences of functionals E_δ, where $\delta \searrow 0$).

3. Main results

At the beginning we will give some mathematical definitions and assumptions, which specify the problem under consideration. It is necessary for the theorems to be scientific in nature, that is, they can be verified.

Let us fix some notations. We assume that the domain Ω is a bounded, connected, not empty, open, Lipschitz set in R^3. Let T^S be the space of symmetric 3×3 tensors ($T^S \subset \mathrm{R}^{3\times 3}$). The fields and potentials, considered in formulae (8) and (9), fulfill the following assumptions.

Assumption 1. The potential $j : \Omega \times T^S \to \mathrm{R}$ is a convex normal integrand, such that

1. The function $\varepsilon \mapsto j(x, \varepsilon)$ is convex and *lower semi-continuous* for almost every $x \in \Omega$,
2. there exists a Borel function $\tilde{j} : \Omega \times T^S \to \mathrm{R}$ such that $\tilde{j}(x, \cdot) = j(x, \cdot)$ for almost every $x \in \Omega$.
3. Moreover, we assume that there exist $\tilde{a}, \tilde{A} \in \mathrm{R}$ and exist $b, B \in \mathrm{R}$, such that $B \geq b > 0$ and:

$$\tilde{a} + b \left\|\varepsilon\right\|^2_{\mathrm{R}^{3\times 3}} \leq j(x, \varepsilon) \leq \tilde{A} + B \left\|\varepsilon\right\|^2_{\mathrm{R}^{3\times 3}} \tag{12}$$

for every $x \in \Omega$ and every $\varepsilon \in T^S$.

Assumption 2. Let $S^2_{\tau^0}$ be defined by the formula (10). Moreover, let $\varphi : S^2_{\tau^0} \to [0, \infty)$ be a given continuous function and assume that there exists $\{\alpha, \beta\} \subset S^2_{\tau^0}$, $\alpha \neq \beta$ such that:

$$\{\xi \in S^2_{\tau^0} : \varphi(\xi) = 0\} = \{\alpha, \beta\} \tag{13}$$

Assumption 3. For all $x \in \Omega$ and all $m \in S^2_{\tau^0}$, $\varepsilon^0(m)(x)$ is a symmetric 3×3 matrix. That is $\varepsilon^0(m)(x) \in T^S$ for all $x \in \Omega$ and all $m \in S^2_{\tau^0}$. The space of L^∞ functions, values in T^S, is denoted by $L^\infty(\Omega, T^S)$. There exists $q \geq 1$, $q \in \mathrm{R}$, such that the function $m(\cdot) \mapsto \varepsilon^0(m)(\cdot)$ is continuous from the space $L^\infty(\Omega)^3$, endowed with $\left\|\cdot\right\|_{L^q}$, to the space $L^\infty(\Omega, T^S)$, endowed with $\left\|\cdot\right\|_{L^2}$. Moreover, let exists $M \in \mathrm{R}$ such that $\left\|\varepsilon^0(m)(x)\right\|_{\mathrm{R}^{3\times 3}} < M$ for every $x \in \Omega$.

Assumption 4. The function $f \in L^q(\Omega)^3$ for some $q \geq 1$ ($q < \infty$).

Assumption 5. $F \in L^2(\Omega)^3$ and $G \in L^2(\Gamma_1)^3$, and $\mu(\Gamma_2) > 0$.

As is well known, for every $m \in L^2(\Omega)^3$ Maxwell's equation admits a unique solution w_m in $H^1(\mathrm{R}^3)$. The linear mapping $m \mapsto w_m$ is continuous from $L^2(\mathrm{R}^3)^3$ (endowed with $\left\|\cdot\right\|_{L^2}$) to $H^1(\mathrm{R}^3)^3$ (endowed with $\left\|\cdot\right\|_{H^1}$). Then we can consider E_δ only for u_δ and m_δ variables (because w_δ is a unique solution of the Maxwell equation).

We say that the sequence E_δ is Γ-convergent (or, more precisely, Γ(weak in $H^1(\Omega)^3$ and $L^1(\Omega)^3$)-convergent) to E_0, and we then write:

$$E_0 = \Gamma - \lim_{n\to\infty} E_{\delta_n} \tag{14}$$

if both conditions (15) and (16) below hold:

For any sequences $\{u_n\}$ and $\{m_n\}$ such that $u_n \to u_0$ weakly in $H^1(\Omega)^3$ and $m_n \to m_0$ in $\|\cdot\|_{L^1(\Omega)}$, and $\delta_n \searrow 0$ we have:

$$E_0(u_0, m_0) \leq \lim \inf_{n\to\infty} E_{\delta_n}(u_n, m_n) \tag{15}$$

For any functions $u \in H^1(\Omega)^3$ and $m \in L^1(\Omega, S^2_{\tau^0})$ there exist sequences $\{u_n\} \subset H^1(\Omega)^3$ and $\{m_n\} \subset H^1(\Omega, S^2_{\tau^0})$, and $\{\delta_n\} \subset (0,\infty)$ such that $u_n \to u$ weakly in $H^1(\Omega)^3$ and $m_n \to m$ in $\|\cdot\|_{L^1(\Omega)}$, and $\delta_n \searrow 0$, and:

$$\lim_{n\to\infty} E_{\delta_n}(u_n, m_n) = E_0(u, m) \tag{16}$$

The anisotropy density $\varphi(\cdot) : S^2_{\tau^0} \to [0, +\infty)$ is zero only in points $\{\alpha, \beta\} \subset S^2_{\tau^0}$. Let:

$$Z = \{\alpha, \beta\} = \{\xi \in S^2_{\tau^0} : \varphi(\xi) = 0\} \tag{17}$$

Consider the set:

$$U_0 = \{m \in L^1(\Omega, \mathrm{R}^3) : m(x) \in Z\} \tag{18}$$

For any $m \in U_0$ consider the set:

$$A_m = \{x \in \Omega : m(x) = \alpha\} \tag{19}$$

Theorem 1. The Γ-limit of E_δ is given by:

$$\begin{aligned} E_0(u,w,m) &= 2c_0 P_\Omega(A_m) \\ &- \int_\Omega f(x)\circ m(x)dx + \tfrac{1}{2}\int_{\mathrm{R}^3} \|\nabla w(x)\|^2_{\mathrm{R}^3} dx - \int_\Omega F(x)\circ u(x)dx \\ &- \int_{\Gamma_1} G(x)\circ u(x)d\mu(x) + \int_\Omega j(x, \varepsilon(u(x)) - \varepsilon^0(m(x)))dx \end{aligned} \tag{20}$$

where:

$$c_0 = \inf \left\{ \begin{aligned} &\int_0^1 \varphi^{1/2}(\gamma(t)) \|\gamma'(t)\|_{R^3} dt : \gamma \in C^1([0,1], \mathrm{R}^3) \\ &\gamma(t) \in S^2_{\tau^0}, \gamma(0) = \alpha, \gamma(1) = \beta \end{aligned} \right\} \tag{21}$$

In view of Maxwell's equation w is determined by m and $P_\Omega(A)$ is the perimeter of the set $A \subset \Omega$ in the sense of De Giorgi (see [4]).

We recall that the perimeter of a measurable set A coincides with the surface area of the boundary ∂A, if the boundary is regular (for example, Lipschitz continuous). Otherwise, for more general sets, called of "finite perimeter", we have:

$$P_\Omega(A) = \mu(\partial^* A \cap \Omega) \tag{22}$$

where μ is the 2-dimensional Hausdorff measure and $\partial^* A \subset \partial A$ is the *reduced boundary* of A, kind of "measure theoretical boundary" (see [4]). Of course, the boundary ∂A of A is a two-dimensional surface. Analogously to the proof of theorem 5.1 from [2], we hope to show that the boundary of the set A is the sum of 2-dimensional surfaces of class C^1.

Below we prove the condition (15). Let $\{u_n\} \subset H^1(\Omega)^3$, $u_0 \in H^1(\Omega)^3$, $\{m_n\} \subset H^1(\Omega)^3$ and $m_0 \in L^1(\Omega)^3$. Moreover, let $u_n \to u_0$ weakly in $H^1(\Omega)^3$ and $m_n \to m_0$ in $\|\cdot\|_{L^1(\Omega)}$. We prove the condition (15), for the functional:

$$m \mapsto \delta a \int_\Omega \|\nabla m\|^2_{\mathrm{R}^{3\times 3}}\, dx + \frac{1}{\delta}\int_\Omega \varphi(m) dx + \frac{1}{2}\int_{\mathrm{R}^3} \|\nabla w\|^2_{\mathrm{R}^3}\, dx \tag{23}$$

exactly as in the paper [2]. Note that w is uniquely determined from the Maxwell equation, for given m.

In view of:

$$\begin{aligned} &\textstyle\int_\Omega \|m_n - m_0\|^q_{\mathrm{R}^3}\, dx \le \int_\Omega (2\tau^0)^{q-1} \|m_n - m_0\|_{\mathrm{R}^3}\, dx \\ &\quad \le (2\tau^0)^{q-1} \textstyle\int_\Omega \|m_n - m_0\|_{\mathrm{R}^3}\, dx \to 0 \end{aligned} \tag{24}$$

for all $q \ge 1$, $q < \infty$, we get $m_n \to m_0$ in $\|\cdot\|_{L^q(\Omega)}$ for all $q \ge 1$, $q < \infty$. Due to Assumption 4:

$$-\int_\Omega f(x) \circ m_n(x) dx \to -\int_\Omega f(x) \circ m_0(x) dx \tag{25}$$

We will prove the formula (25) in the case where $f \in L^1(\Omega)^3$. We define $f_k(x) = f(x)$ if $\|f(x)\|_{\mathrm{R}^3} \le k$ and $f_k(x) = 0$ if $\|f(x)\|_{\mathrm{R}^3} > k$, for $k \in \mathrm{N}$. Since $f \in L^1(\Omega)^3$, it follows that the 3-dimensional measure of:

$$B_k = \{x \in \Omega : \|f(x)\| > k\} \tag{26}$$

tends to 0 as $k \to \infty$. Moreover:

$$\int_{B_k} \|f(x)\|_{\mathrm{R}^3}\, dx \to 0 \tag{27}$$

if $k \to \infty$. Since $\|m_n(x)\|_{\mathrm{R}^3} = \tau^0$ for almost every $x \in \Omega$ and $m_n \to m_0$ in L^1, it follows that $\|m_0(x)\|_{\mathrm{R}^3} = \tau^0$ for almost every $x \in \Omega$. The functions $f_k \in L^\infty(\Omega)^3$, for all $k \in \mathrm{N}$, therefore;

$$\int_\Omega f_k(x) \circ m_n dx \to \int_\Omega f_k(x) \circ m_0(x) dx \tag{28}$$

as $n \to +\infty$, for all set $k \in \mathrm{N}$. For any $\tilde{\varepsilon} > 0$ we can find $\tilde{k} \in \mathrm{N}$ such that

$$\begin{aligned} &\textstyle\int_{B_{\tilde{k}}} f(x) \circ m_n(x) dx \le \int_{B_{\tilde{k}}} \|f(x)\|_{\mathrm{R}^3} \|m_n(x)\|_{\mathrm{R}^3}\, dx \\ &= \tau^0 \textstyle\int_{B_{\tilde{k}}} \|f(x)\|_{\mathrm{R}^3}\, dx < \tau^0 \tilde{\varepsilon} \end{aligned} \tag{29}$$

Moreover:

$$\int_{B_{\tilde{k}}} f(x) \circ m_0(x) dx \le \int_{B_{\tilde{k}}} \|f(x)\|_{\mathrm{R}^3} \|m_0(x)\|_{\mathrm{R}^3}\, dx \le \tau^0 \tilde{\varepsilon} \tag{30}$$

Then:

$$\begin{aligned}
&\left|\int_\Omega f(x)\circ m_n(x)dx - \int_\Omega f(x)\circ m_0(x)dx\right| \\
&\leq \int_{B_{\tilde k}} \|f(x)\|_{\mathrm{R}^3} \|m_n(x)-m_0(x)\|_{\mathrm{R}^3}\, dx \\
&+\left|\int_{\Omega-B_{\tilde k}} f(x)\circ m_n(x)dx - \int_{\Omega-B_{\tilde k}} f(x)\circ m_0(x)dx\right| \\
&\leq \int_{B_{\tilde k}} \|f(x)\|_{\mathrm{R}^3} \left(\|m_n(x)\|_{\mathrm{R}^3}+\|m_0(x)\|_{\mathrm{R}^3}\right) dx \\
&+\left|\int_{\Omega-B_{\tilde k}} f(x)\circ m_n(x)dx - \int_{\Omega-B_{\tilde k}} f(x)\circ m_0(x)dx\right|
\end{aligned} \tag{31}$$

Due to (29), (30) and (28), for given $\tilde k \in \mathrm{N}$ we can take $n \in \mathrm{N}$ so large that the inequality (31) can be estimated from above by $2\tau^0\tilde\varepsilon + \tilde\varepsilon$ (because (29) holds for all $n \in \mathrm{N}$). Since $\tilde\varepsilon > 0$ can be taken as small as we want, the formula (25) is proved in the case of $f \in L^1$.

We return to the proof of the theorem 1. Since $u_n \to u_0$ weakly in $H^1(\Omega)^3$, we get:

$$-\int_\Omega F(x)\circ u_n(x)dx \to -\int_\Omega F(x)\circ u_0(x)dx \tag{32}$$

see Assumption 5. We can prove that:

$$-\int_{\Gamma_1} G(x)\circ u_n(x)d\mu(x) \to -\int_{\Gamma_1} G(x)\circ u_0(x)d\mu(x) \tag{33}$$

if $u_n \to u_0$ weakly in $H^1(\Omega)^3$, where $u_{|\Gamma_1}$ is the trace of $u \in H^1(\Omega)^3$ (see [5, 6]) and $G \in L^2(\Gamma_1)^3$. Due to Assumption 3, $\varepsilon^0(m_n) \to \varepsilon^0(m_0)$ in $\|\cdot\|_{L^2}$. Since the functional:

$$\varepsilon \mapsto \int_\Omega j(x,\varepsilon)dx \tag{34}$$

is lower semi continuous in weak L^2 topology (see [7]), it follows that:

$$\lim_{n\to\infty} \int_\Omega j(x,\varepsilon(u_n)-\varepsilon^0(m_n))dx \geq \int_\Omega j(x,\varepsilon(u_0)-\varepsilon^0(m_0))dx \tag{35}$$

Therefore, the condition (15) holds.

Now we will show that the condition (16) holds. According to [2] we show that for any function $m \in L^1(\Omega, S^2_{\tau^0})$ there exists a sequence $\{m_n\} \subset H^1(\Omega)^3$ such that $m_n \to m$ in $\|\cdot\|_{L^1(\Omega)}$. In view of the formula (24), we have $m_n \to m$ in $\|\cdot\|_{L^q(\Omega)}$ for all $q \geq 1$, $q < \infty$. There exists a sequence solutions $\{w_n\}$ of Maxwell's equations, for the parameters $\{m_n\}$. Moreover, $w_n \to w$ in $\|\cdot\|_{H^1}$ and w is the solution of Maxwell's equation for m. Due to [2] (proof of Theorem 2.4), the found sequences $\{m_n\}$ and $\{w_n\}$ satisfy the condition:

$$\begin{aligned}
&\lim_{n\to\infty} \delta_n a \int_\Omega \|\nabla m_n\|^2_{\mathrm{R}^{3\times 3}}\, dx + \tfrac{1}{\delta_n}\int_\Omega \varphi(m_n)dx + \tfrac12 \int_{\mathrm{R}^3} \|\nabla w_n\|^2_{\mathrm{R}^3}\, dx \\
&= 2c_0 P_\Omega(A_m) + \tfrac12 \int_{\mathrm{R}^3} \|\nabla w(x)\|^2_{\mathrm{R}^3}\, dx
\end{aligned} \tag{36}$$

For given $u \in H^1(\Omega)^3$ we take sequence $\{u_n\}$ such that $u_n = u$ for all $n \in \mathrm{N}$. Since:

$$\tilde{\varepsilon} \mapsto \int_\Omega j(x, \varepsilon(u(x)) - \tilde{\varepsilon})dx \tag{37}$$

is lower semi continuous in weak L^2 topology (see [6]), it follows that the functional (37) is lower semi continuous in $\|\cdot\|_{L^2}$. Due to (12), the functional (37) reaches finite values for every $\tilde{\varepsilon} \in L^2(\Omega, T^S)$ (i.e. the effective domain of (37) is equal to the space $L^2(\Omega, T^S)$). Since the functional (37) is convex, it follows that (37) is continuous in the interior of the effective domain of (37), see [8]. So, (37) is continuous in $\|\cdot\|_{L^2}$. Therefore the condition (16) holds for the functional E. Theorem 1 is proven.

We can show that for every $\delta > 0$ there exists minimum point for E_δ. Let for E_δ the minimum point be equal to $(u_\delta, w_\delta, m_\delta) \in (H^1(\Omega)^3, H^1(\mathrm{R}^3), H^1(\Omega, T^S))$. Now we will show the existence of a minimum point for E_0.

Theorem 2. Let $E_0 = \Gamma - \lim_{n\to\infty} E_{\delta_n}$ and $\{(u_{\delta_n}, w_{\delta_n}, m_{\delta_n})\}$ be the sequence of minimum points for the sequence of functionals $\{E_{\delta_n}\}$. Moreover, let $\delta_n \searrow 0$ as $n \to \infty$. Then there exists a subsequence $\{(u_{\delta_{n_k}}, w_{\delta_{n_k}}, m_{\delta_{n_k}})\}$ of $\{(u_{\delta_n}, w_{\delta_n}, m_{\delta_n})\}$ and exists $(u_0, w_0, m_0) \in H^1(\Omega)^3 \times H^1(\mathrm{R}^3) \times L^1(\Omega, \mathrm{R}^3)$, such that $u_{\delta_{n_k}} \to u_0$ weakly in $H^1(\Omega)^3$, $m_{\delta_{n_k}} \to m_0$ in $\|\cdot\|_{L^1(\Omega)}$, $w_{\delta_{n_k}} \to w_0$ in $\|\cdot\|_{H^1}$. Moreover:

$$\lim_{n\to\infty} E_{\delta_{n_k}}(u_{\delta_{n_k}}, w_{\delta_{n_k}}, m_{\delta_{n_k}}) = E_0(u_0, w_0, m_0) \tag{38}$$

and (u_0, w_0, m_0) is a minimum point for E_0.

According to [2] we show that there exists a subsequence $\{m_{\delta_{n_t}}\}$ of $\{m_{\delta_n}\}$ and exists m_0 such that $m_{\delta_{n_t}} \to m_0$ in $\|\cdot\|_{L^1(\Omega)}$. In view of the formula (24), we have $m_{\delta_{n_t}} \to m_0$ in $\|\cdot\|_{L^q(\Omega)}$ for all $q \geq 1$, $q < \infty$. Then $\varepsilon^0(m_{\delta_{n_t}}) \to \varepsilon^0(m_0)$ in $\|\cdot\|_{L^2(\Omega)}$, see Assumption 3. Due to Maxwell's equation, $w_{\delta_{n_t}} \to w_0 \in H^1(\mathrm{R}^3)$ in $\|\cdot\|_{H^1}$ where $w_{\delta_{n_t}}$ (respectively w_0) is the solution of the Maxwell's equation with the parameter $m_{\delta_{n_t}}$ (respectively m_0). Since the set $\left\|\varepsilon^0(m_{\delta_{n_t}})\right\|_{L^2(\Omega)}$ is bounded (see Assumption 3) and since the condition (12) holds, it follows that the sequence $\left\|\varepsilon(u_{\delta_{n_t}})\right\|_{L^2(\Omega)}$ is bounded (because $u \mapsto \int_\Omega F(x) \circ u(x)dx + \int_{\Gamma_1} G(x) \circ u(x)d\mu(x)$ is a linear, continuous functional and the sequence $\int_{\mathrm{R}^3} \left\|\nabla w_{\delta_{n_t}}\right\|^2 dx$ is bounded). Then there exists subsequence $\{(u_{\delta_{n_k}}, w_{\delta_{n_k}}, m_{\delta_{n_k}})\}$ of the sequence $\{(u_{\delta_{n_t}}, w_{\delta_{n_t}}, m_{\delta_{n_t}})\}$ such that $u_{\delta_{n_k}} \to u_0$ weakly in $H^1(\Omega)^3$, because $\varepsilon(u_{\delta_{n_t}})$ is bounded in $L^2(\Omega)^{3\times 3}$ and Ω is clamped on Γ_2. By virtue of the lower semi continuity of the considered functionals, we obtain:

$$\liminf_{k\to\infty} E_{\delta_{n_k}}(u_{\delta_{n_k}}, w_{\delta_{n_k}}, m_{\delta_{n_k}}) \geq E_0(u_0, w_0, m_0) \tag{39}$$

Due to the condition (16) (of the definition of Γ-convergence) in (39) we have equality, because $(u_{\delta_{n_k}}, w_{\delta_{n_k}}, m_{\delta_{n_k}})$ is a minimum point for $E_{\delta_{n_k}}$, for all $k \in \mathrm{N}$. So we've proved Theorem 2.

4. Final remarks

The obtained functional E_0 describes the magnetic-elastic properties of the ferromagnetic material with microstructure. Region occupied by considered body consists of regions in which the magnetization is uniform (i.e. $\varepsilon^0(m(x)) = const$ in these regions) separated by transitional surfaces, represented by $2c_0 P_\Omega(A_m)$ in E_0. For any stationary configuration and for a fixed magnetic field m investigated functional E_0 describes a ferromagnetic body in a way which is equivalent to the description given in [3]. From the other hand side the model investigated in [3] is formulated as differential equations for physical fields corresponding to which are minimizers of the variational problem investigated in this paper. That is why results obtained in this article can be treated as a proof of the existence of solutions for the problem formulated in [3]. Hence model equations taken from [3] together with variational problem investigated in this paper consist a certain physically complete description of ferromagnetic solid made of the considered microstructural material superimposed by the locally uniform magnetization (i.e. $\varepsilon^0(m(x)) = const$ in regions separated by transitional surfaces represented by $2c_0 P_\Omega(A_m)$ in E_0).

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