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Micro-Vibrations and Wave Propagation in Biperiodic Cylindrical Shells

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The objects of consideration are thin linearly elastic Kirchhoff-Love-type circular cylindrical shells having a periodically microheterogeneous structure in circumferential and axial directions (biperiodic shells). The aim of this contribution is to study a certain long wave propagation problem related to micro-fluctuations of displacement field caused by a periodic structure of the shells. This micro-dynamic problem will be analysed in the framework of a certain mathematical averaged model derived by means of *the combined modelling procedure*. The combined modelling applied here includes two techniques: the asymptotic modelling procedure and a certain extended version of the known tolerance non-asymptotic modelling technique based on a new notion of *weakly slowly-varying function.* Both these procedures are conjugated with themselves under special conditions. Contrary to the starting exact shell equations with highly oscillating, non-continuous and periodic coefficients, governing equations of the averaged combined model have constant coefficients depending also on a cell size. It will be shown that the micro-periodic heterogeneity of the shells leads to exponential micro-vibrations and to exponential waves as well as to dispersion effects, which cannot be analysed in the framework of the asymptotic models commonly used for investigations of vibrations and wave propagation in the periodic structures.

Keywords: biperiodic shells, asymptotic and tolerance modelling, length-scale effect, micro-vibrations, wave propagation.

1. Introduction

Thin linearly elastic Kirchhoff-Love-type circular cylindrical shells with a periodically micro-inhomogeneous structure in circumferential and axial directions are analysed. In the general case, by periodic inhomogeneity we shall mean here periodically variable shell thickness and periodically variable inertial and elastic properties of the shell material. Shells of this kind are termed *biperiodic*. Cylindrical shells with periodically spaced families of stiffeners as shown in Fig. 1 are typical example of such shells.

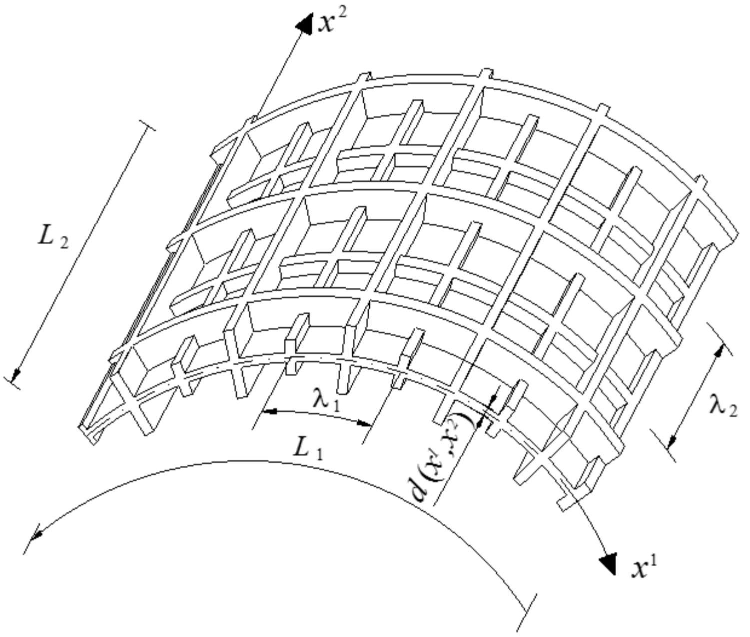

Figure 1 A fragment of a shell with two families of biperiodically spaced ribs

The dynamic problems of periodic shells are described by partial differential equations with highly oscillating, non-continuous, periodic coefficients. Hence, the direct application of these equations to investigations of engineering problems is noneffective. That is why there exists a number of various modelling methods leading to simplified averaged equations with constant coefficients. Periodic shells (plates) are usually described using *homogenized models* derived by means of *asymptotic methods*, cf. Lewiński and Telega [1]. It should be also mentioned homogenized models formulated by means of microlocal parameters, cf. Matysiak and Nagórko [2], Unfortunately, in the models of this kind *the effect of a periodicity cell length dimensions* (called *the length-scale effect*) on the overall shell behaviour is neglected.

The length-scale effect can be taken into account using *the non-asymptotic tolerance averaging technique*, cf. Woźniak and Wierzbicki [3, 4], Woźniak et al [5, 6]. This technique is based on the concept of *tolerance relations* between points and real numbers related to the accuracy of the performed measurements and calculations. The tolerance relations are determined by *the tolerance parameters*. Contrary to exact equations of theories of microheterogeneous structures (partial differential equations with functional, highly oscillating, non-continuous coefficients), *governing equations of the averaged tolerance models have coefficients which are constant or slowly-varying and depend on the period lengths of inhomogeneity*. Hence, these equations make it possible to analyse the length-scale effect.

Some applications of this method to the modelling of mechanical and thermomechanical problems for various periodic structures are shown in many works. The extended list of papers and books on this topic can be found in Woźniak and Wierzbicki [3], Woźniak et al. [5, 6]. We mention here monograph by Tomczyk [7], where the length-scale effect in dynamics and stability of periodic cylindrical shells is investigated, paper by Marczak and Jędrysiak [8], where vibrations of periodic three-layered plates with inert core are studied. In the last years the tolerance modelling was adopted for mechanical and thermomechanical problems of functionally graded structures, e.g. for heat conduction in functionally graded composites by Nagórko and Woźniak [9], Ostrowski and Michalak [10], for thermoelasticity of transversally graded laminates by Pazera and Jędrysiak [11], for vibrations of annular plates with longitudinally graded structure by Wirowski [12], for dynamics and stability of functionally graded cylindrical shells by Tomczyk and Szczerba [13, 14, 15].

A certain extended version of the tolerance modelling technique has been proposed by Tomczyk and Woźniak in [16]. This version is based on a new notion of *weakly slowly-varying functions* which is a certain extension of the well known concept of *slowly-varying functions*, cf. [3-6]. New *mathematical averaged general tolerance and combined asymptotic-tolerance models* of dynamic problems for thin shells with either one- or two-directional periodic microstructure in directions tangent to the shell midsurface, derived by means of the concept of *weakly slowly-varying functions*, have been proposed by Tomczyk and Litawska in [17, 18, 19, 20, 21]. The models mentioned above are certain generalizations of the corresponding *standard tolerance and combined asymptotic-tolerance models* proposed in [7], which have been obtained by using the classical concept of *slowly-varying functions*. Note, that following [6] and [16], the concepts of *weakly slowly-varying and slowly-varying functions* are recalled in Section 3 of this paper.

The aim of this note is to study certain problems of micro-vibrations and of long wave propagation related to micro-fluctuations of displacement field caused by a periodic structure of the shells. Note, that we deal with long waves if condition $\lambda/L << 1$ holds, where λ is the characteristic length dimension of the cell and L is the wavelength. These micro-dynamic problems will be analysed in the framework of *the general combined asymptotic-tolerance model* proposed in [20]. An important advantage of this model is that *it makes it possible to separate the macroscopic description of the modelling problem from its microscopic description*. It will be shown that the periodic microheterogeneity of the shells leads to exponential micro-vibrations and to exponential waves as well as to dispersion effects, which cannot be analysed in the framework of the asymptotic models commonly used for investigations of vibrations and wave propagation in the periodic shells under consideration. The new wave propagation speed depending on a cell size will be obtained. It has to be emphasized that these micro-dynamic problems cannot be also studied within *the standard asymptotic-tolerance model for biperiodic shells* presented in [7].

2. Starting equations

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface M in circumferential and axial directions, respectively. We denote $\mathbf{x} \equiv (x^1, x^2) \in \Omega \equiv (0, L_1) \times (0, L_2)$, where L_1, L_2 are length dimensions of M, cf. Fig. 1. Let $O\,\bar{x}^1\bar{x}^2\bar{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space R^3 and denote $\bar{\mathbf{x}} \equiv (\bar{x}^1, \bar{x}^2, \bar{x}^3)$. A cylindrical shell midsurface M is given by $M \equiv \left\{ \bar{\mathbf{x}} \in R^3 : \bar{\mathbf{x}} = \bar{\mathbf{r}}\left(x^1, x^2\right), \left(x^1, x^2\right) \in \Omega \right\}$, where $\bar{\mathbf{r}}(\cdot)$ is the smooth function such that $\partial\,\bar{\mathbf{r}}/\partial x^1 \cdot \partial\,\bar{\mathbf{r}}/\partial x^2 = 0$, $\partial\,\bar{\mathbf{r}}/\partial x^1 \cdot \partial\,\bar{\mathbf{r}}/\partial x^1 = 1$, $\partial\,\bar{\mathbf{r}}/\partial x^2 \cdot \partial\,\bar{\mathbf{r}}/\partial x^2 = 1$. It means that on M the orthonormal parametrization is introduced. Sub- and superscripts $\alpha, \beta, \ldots$ run over 1,2 and are related to x^1, x^2, summation convention holds. Partial differentiation related to x^α is represented by ∂_α. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha \ldots \partial_\delta$. Let $a^{\alpha\beta}$ stand for the midsurface first metric tensor. The time coordinate is denoted by $t \in \mathrm{I} = [t_0, t_1]$. Let $d(\mathbf{x})$, r stand for the shell thickness and the midsurface curvature radius, respectively.

Let λ_1 and λ_2 be the period lengths of the shell structure respectively in x^1- and x^2-directions. *The basic cell* Δ and an arbitrary cell $\Delta(\mathbf{x})$ with the centre at point $\mathbf{x} \in \Omega_\Delta$ are defined by means of: $\Delta \equiv [-\lambda_1/2,\ \lambda_1/2] \times [-\lambda_2/2,\ \lambda_2/2] \subset \Omega$, $\Delta(\mathbf{x}) \equiv \mathbf{x} + \Delta$, $\mathbf{x} \in \Omega_\Delta$, $\Omega_\Delta \equiv \{\mathbf{x} \in \Omega : \Delta(\mathbf{x}) \subset \Omega_\Delta\}$. The diameter $\lambda \equiv [(\lambda_1)^2 + (\lambda_2)^2]^{1/2}$ of Δ is assumed to satisfy conditions: $\lambda/d_{\max} >> 1$, $\lambda/r << 1$ and $\lambda/\min(L_1, L_2) << 1$. Hence, the diameter will be called *the microstructure length parameter.*

Setting $\mathbf{z} \equiv (z^1, z^2) \in [-\lambda_1/2,\ \lambda_1/2] \times [-\lambda_2/2,\ \lambda_2/2]$, we assume that the cell Δ has two symmetry axes: for $z^1 = 0$ and $z^2 = 0$. It is also assumed that inside the cell not only the geometrical but also elastic and inertial properties of the shell are described by symmetric (i.e. even) functions of $\mathrm{z} \equiv (z^1, z^2)$.

Denote by $u_\alpha = u_\alpha(\mathbf{x}, t)$, $w = w(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, $t \in \mathrm{I}$, the shell displacements in directions tangent and normal to M , respectively. Elastic properties of the shell are described by shell stiffness tensors $D^{\alpha\beta\gamma\delta}(\mathbf{x})$, $B^{\alpha\beta\gamma\delta}(\mathbf{x})$. Let $\mu(\mathbf{x})$ stand for a shell mass density per midsurface unit area. The external forces will be neglected.

The considerations are based on the well-known Kirchhoff-Love theory of thin elastic shells, cf. Kaliski [22].

It is assumed that the behaviour of the shell under consideration is described by the action functional determined by lagrangian L being a highly oscillating function with respect to $\mathbf{x}$ and having the well-known form:

$$\begin{aligned} L = -\tfrac{1}{2}(&D^{\alpha\beta\gamma\delta}\partial_\beta u_\alpha \partial_\delta u_\gamma + \tfrac{2}{r} D^{\alpha\beta 11} w \partial_\beta u_\alpha + \tfrac{1}{r^2} D^{1111} w w \\ &+ B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta} w \partial_{\gamma\delta} w - \mu a^{\alpha\beta}\dot{u}_\alpha \dot{u}_\beta - \mu \dot{w}^2) \end{aligned} \tag{1}$$

Applying the principle of stationary action we arrive at the system of Euler-Lagrange equations, which can be written in an explicit form as:

$$\begin{aligned} &\partial_\beta(D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma) + r^{-1}\partial_\beta(D^{\alpha\beta 11} w) - \mu a^{\alpha\beta}\ddot{u}_\beta = 0 \\ &r^{-1} D^{\alpha\beta 11}\partial_\beta u_\alpha + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta} w) + r^{-2} D^{1111} w + \mu \ddot{w} = 0 \end{aligned} \tag{2}$$

It can observed that equations (2) coincide with the well-known governing equations of Kirchhoff-Love theory of thin elastic shells, cf. [22]. For periodic shells, coefficients $D^{\alpha\beta\gamma\delta}(\mathbf{x})$, $B^{\alpha\beta\gamma\delta}(\mathbf{x})$, $\mu(\mathbf{x})$ of (1) and (2) are highly oscillating, non-continuous and λ-periodic functions. Applying the combined asymptotic-tolerance modelling technique to lagrangian (1), the averaged model equations with constant coefficients depending also on a cell size were derived in [20]. The combined modelling under consideration includes two techniques: *the consistent asymptotic modelling procedure* given in [6] and *an extended version of the known tolerance non-asymptotic modelling technique* based on a new notion of *weakly slowly-varying function* proposed by Tomczyk and Woźniak in [16]. Here, the model equations formulated in [20] will be used to investigation of the long wave propagation in biperiodic shells under consideration. To make the analysis more clear, in the next section this model will be reminded, following [20]. Moreover, the basic concepts and assumptions of the extended tolerance modelling technique and of the consistent asymptotic approach will be outlined, following [6, 16].

3. Modelling procedure. General asymptotic-tolerance model

The combined modelling technique under consideration is realized in two steps. The first step is based on *the consistent asymptotic procedure* [6]. The second one is realized by means of *the extended version of the tolerance non-asymptotic technique* [16].

3.1. *Step 1. Consistent asymptotic model equations*

The fundamental concepts of the consistent asymptotic procedure are those of *an averaging operation* and *fluctuation shape functions.* Below, the mentioned above concepts will be specified with respect to two-dimensional region $\Omega \equiv (0, L_1) \times (0, L_2)$ defined in this contribution.

Let $f(\mathbf{x})$ be a function defined in $\bar{\Omega} \equiv [0, L_1] \times [0, L_2]$, which is integrable and bounded in every cell $\Delta(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$. *The averaging operation of* $f(\cdot)$ is defined by:

$$< f > (\mathbf{x}) \equiv \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} f(\mathbf{z}) d\mathbf{z} \quad \mathbf{z} \in \Delta(\mathbf{x}) \quad \mathbf{x} \in \Omega_\Delta \tag{3}$$

It can be seen that if $f(\cdot)$ is Δ-periodic then $< f >$ is constant.

Let ∂ stand for gradient operator in Ω; $\partial = (\partial_1, \partial_2)$, $\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2$. Denote by ∂^k the k-th gradient in Ω. Let $h(\cdot)$ be a λ-periodic rapidly oscillating function defined in $\bar{\Omega}$, which is continuous together with gradients $\partial^k h$, $k = 1, ..., R-1$, and has a piecewise continuous (or in special cases continuous) bounded gradient $\partial^R h$. Nonnegative integer R is assumed to be specified in every problem under consideration. Periodic function $h(\mathbf{x})$ will be called *the fluctuation shape function of the R-th kind*, $h(\mathbf{x}) \in FS^R(\Omega, \Delta)$, if it depends on λ as a parameter and satisfies conditions: $h \in O(\lambda^R)$, $\partial^k h \in O(\lambda^{R-k})$, $k = 1, 2, ..., R$, $< \mu h >= 0$, where $\mu(\mathbf{x})$ is a shell mass density.

The asymptotic modelling is based on assumption called *the consistent asymptotic decomposition.* It states that the displacement fields occurring in the starting lagrangian have to be replaced by families of fields depending on small parameter

$\varepsilon = 1/m,\ m = 1, 2, ...$and defined in an arbitrary cell. These families of displacements are decomposed into averaged part independent of ε and highly-oscillating part depending on ε.

We start with *the consistent asymptotic averaging* of lagrangian (1) under *the consistent asymptotic decomposition* of families of displacements $u_{\varepsilon\alpha}(\mathbf{z}, t)$, $w_\varepsilon(\mathbf{z}, t)$, $(\mathbf{z}, t) \in \Delta_\varepsilon \times I$:

$$\begin{aligned} u_{\varepsilon\alpha}(\mathbf{z}, t) &\equiv u_\alpha(\mathbf{z}/\varepsilon, t) = u^0_\alpha(\mathbf{z}, t) + \varepsilon h_\varepsilon(\mathbf{z})\, U_\alpha(\mathbf{z}, t) \\ w_\varepsilon(\mathbf{z}, t) &\equiv w(\mathbf{z}/\varepsilon, t) = w^0(\mathbf{z}, t) + \varepsilon^2 g_\varepsilon(\mathbf{z})\, W(\mathbf{z}, t) \end{aligned} \tag{4}$$

where $\varepsilon = 1/m, \quad m = 1, 2, ...,\ \mathbf{z} \in \Delta_\varepsilon(\mathbf{x}),\ \Delta_\varepsilon \equiv (-\varepsilon\lambda_1/2, \varepsilon\lambda_1/2) \times (-\varepsilon\lambda_2/2, \varepsilon\lambda_2/2)$ (scaled cell), $\Delta_\varepsilon(\mathbf{x}) \equiv \mathbf{x} + \Delta_\varepsilon\ ,\ \ \mathbf{x} \in \Omega_\Delta$ (scaled cell with a centre at $\mathbf{x} \in \Omega_\Delta$).

Unknown functions u^0_α, U_α in (4) are assumed to be continuous and bounded in Ω together with their first derivatives. Unknown functions w^0, W in (4) are assumed to be continuous and bounded in Ω together with their derivatives up to the second order. Unknowns $u^0_\alpha, w^0$and U_α, W are called *macrodisplacements* and *fluctuation amplitudes*, respectively. They are independent of ε. This is the main difference between the asymptotic approach under consideration and approach which is used in the known homogenization theory, cf. Bensoussan et al. [23], Jikov et al. [24].

By functions $h_\varepsilon(\mathrm{z}) \equiv h(\mathbf{z}/\varepsilon) \in FS^1(\Omega, \Delta)$, $g_\varepsilon(\mathbf{z}) \equiv g(\mathbf{z}/\varepsilon) \in FS^2(\Omega, \Delta)$ in (4) are denoted λ-periodic highly oscillating *fluctuation shape functions* depending on ε. The *fluctuation shape functions* are assumed to be known in every problem under consideration. They have to satisfy conditions: $h \in O(\lambda)$, $\lambda\partial_\alpha h \in O(\lambda)$, $g \in O(\lambda^2)$, $\lambda\partial_\alpha g \in O(\lambda^2)$, $\lambda^2\partial_{\alpha\beta} g \in O(\lambda^2)$, $< \mu\, h > = < \mu\, g > = 0$. It has to be emphasized that $\partial_\alpha h_\varepsilon(\mathbf{z}) \equiv \frac{1}{\varepsilon}\partial_\alpha h(\mathbf{z}/\varepsilon)$, $\partial_\alpha g_\varepsilon(\mathbf{z}) \equiv \frac{1}{\varepsilon}\partial_\alpha g(\mathbf{z}/\varepsilon)$, $\partial_{\alpha\beta} g_\varepsilon(\mathbf{z}) \equiv \frac{1}{\varepsilon^2}\partial_{\alpha\beta} g(\mathbf{z}/\varepsilon)$.

We substitute the right-hand sides of (4) into (1) and take into account that under limit passage $\varepsilon \to 0$, terms depending on ε can be neglected and every continuous and bounded function of argument $\mathbf{z} \in \Delta_\varepsilon(\mathbf{x})$ tends to function of argument $\mathbf{x} \in \bar{\Omega}$. Moreover, if $\varepsilon \to 0$ then by means of a property of the mean value, cf. [24], the obtained result tends weakly to the function being *the averaged form of starting lagrangian* (1) *under consistent asymptotic decomposition* (4), cf. [7, 20, 21]. Next, applying the principle of stationary action we arrive at *the governing equations of consistent asymptotic model for the periodic shells under consideration.* These equations consist of partial differential equations for macrodisplacements u^0_α, w^0 coupled with linear algebraic equations for fluctuation amplitudes U_α, W. After eliminating fluctuation amplitudes from the governing equations by means of:

$$\begin{aligned} &U_\gamma = -(G^{-1})_{\gamma\eta}[< \partial_\beta h D^{\beta\eta\mu\vartheta} > \ \partial_\vartheta u^0_\mu + r^{-1} < \partial_\beta h D^{\beta\eta 11} > w^0] \\ &W = -E^{-1} < \partial_{\alpha\beta} g B^{\alpha\beta\gamma\delta} > \partial_{\gamma\delta} w^0 \end{aligned} \tag{5}$$

where $G_{\alpha\gamma} = < \partial_\beta h\, D^{\alpha\beta\gamma\delta} \partial_\delta h >$, $E = < \partial_{\alpha\beta} B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} g >$, we arrive finally at *the asymptotic model equations expressed only in macrodisplacements* u^0_α, w^0:

$$\begin{aligned} &D_h^{\alpha\beta\gamma\delta} \partial_{\beta\delta} u^0_\gamma + r^{-1} D_h^{\alpha\beta 11} \partial_\beta w^0 - < \mu > a^{\alpha\beta} \ddot{u}^0_\beta = 0 \\ &B_g^{\alpha\beta\gamma\delta} \partial_{\alpha\beta\gamma\delta} w^0 + r^{-1} D_h^{11\gamma\delta} \partial_\delta u^0_\gamma + r^{-2} D_h^{1111} w^0 + < \mu > \ddot{w}^0 = 0 \end{aligned} \tag{6}$$

where:

$$\begin{aligned} &D_h^{\alpha\beta\gamma\delta} \equiv < D^{\alpha\beta\gamma\delta} > - < D^{\alpha\beta\eta\chi}\, \partial_\chi h > (G^{-1})_{\eta\zeta} < \partial_\chi h\, D^{\chi\zeta\gamma\delta} > \\ &B_g^{\alpha\beta\gamma\delta} \equiv < B^{\alpha\beta\gamma\delta} > - < B^{\alpha\beta\mu\zeta} \partial_{\mu\zeta} g > E^{-1}\ \ < \partial_{\mu\zeta} g\, B^{\mu\zeta\gamma\delta} > \end{aligned} \tag{7}$$

Since displacement fields $u_\alpha(\mathbf{x},t), w(\mathbf{x},t)$ have to be uniquely defined in $\Omega \times \mathrm{I}$, we conclude that $u_\alpha(\mathbf{x},t), w(\mathbf{x},t)$ have to take the form:

$$\begin{aligned} &u_\alpha(\mathbf{x},t) = u_\alpha^0(\mathbf{x},t) + h(\mathbf{x})U_\alpha(\mathbf{x},t) \\ &w(\mathbf{x},t) = w^0(\mathbf{x},t) + g(\mathbf{x})\,W(\mathbf{x},t) \qquad (\mathbf{x},t) \in \Omega \times \mathrm{I} \end{aligned} \tag{8}$$

with U_α, W given by (5). Tensors $D_h^{\alpha\beta\gamma\delta}$, $B_g^{\alpha\beta\gamma\delta}$ given by (7) are *tensors of effective elastic moduli* for the considered composite biperiodic shells.

Equations (6) for macrodisplacements $u_\alpha^0(\mathbf{x},t), w^0(\mathbf{x},t)$ together with expressions (5) for fluctuation amplitudes U_α, W and expressions (7) for effective moduli as well as with decomposition (8) represent *the consistent asymptotic model of selected dynamic problems for the thin biperiodic cylindrical shells under consideration.*

Coefficients of equations (6) are *constant* but they are *independent of the microstructure cell size.* Hence, this model is not able to describe the length-scale effect on the overall shell dynamics and it will be referred to as *the macroscopic model.*

It has to be emphasized that the macroscopic model obtained here by means of *the general combined modelling* coincides with the corresponding macroscopic model derived by applying *the standard combined modelling*, cf. [7]. It follows from the fact that in the first step of both the general and standard combined modelling procedures there are specifications for neither *a weakly slowly-varying function* nor *a slowly-varying function.*

In the first step of combined modelling it is assumed that within the asymptotic model, solutions u_α^0, w^0 to the problem under consideration are known. Hence, there are also known functions:

$$\begin{aligned} &u_{0\alpha}(\mathbf{x},t) = u_\alpha^0(\mathbf{x},t) + h(\mathbf{x})U_\alpha(\mathbf{x},t) \\ &w_0(\mathbf{x},t) = w^0(\mathbf{x},t) + g(\mathbf{x})\,W(\mathbf{x},t) \qquad (\mathbf{x},t) \in \Omega \times \mathrm{I} \end{aligned} \tag{9}$$

where U_α, W are given by means of (5).

3.2. *Step 2. Superimposed general tolerance model*

The second step of the combined modelling is based on *the extended version of the tolerance modelling technique* [16].

The fundamental concepts of the tolerance modelling procedure under consideration are those of *two tolerance relations between points and real numbers determined by tolerance parameters, weakly slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation,* cf. [3-6, 16]. It has to be emphasized that in the classical approach we deal with not *weakly slowly-varying* but with more restrictive *slowly-varying functions.*

Below, the mentioned above concepts and assumptions will be specified with respect to two-dimensional region $\Omega \equiv (0, L_1) \times (0, L_2)$ defined in this contribution.

- *Tolerance between points*:

  Let λ be a positive real number. Points x, y belonging to $\Omega = (0, L_1) \times (0, L_1)$ are said to be in tolerance determined by λ, if and only if the distance between points $\mathbf{x}, \mathbf{y}$ does not exceed λ, i.e. $\|\mathbf{x} - \mathbf{y}\| \le \lambda$.

- *Tolerance between real numbers*:

  Let $\tilde{\delta}$ be a positive real number. Real numbers μ, ν are said to be in tolerance determined by $\tilde{\delta}$, if and only if $|\mu - \nu| \leq \tilde{\delta}$.

  The above relations are denoted by: $x \overset{\lambda}{\approx} y$, $\mu \overset{\tilde{\delta}}{\approx} \nu$. Positive parameters $\lambda, \tilde{\delta}$ are called *tolerance parameters*.

- *Weakly slowly-varying functions*:

  Let $F(\mathbf{x})$ be a function defined in $\bar{\Omega} = [0, L_1] \times [0, L_2]$, which is continuous, bounded and differentiable in $\bar{\Omega}$ together with their derivatives up to the R-th order. Nonnegative integer R is assumed to be specified in every problem under consideration. Note, that function F can also depend on time coordinate t as parameter. Let $\delta \equiv (\lambda, \delta_0, \delta_1, .., \delta_R)$ be the set of tolerance parameters. The first of them is related to the distances between points in Ω, the second one is related to the distances between values of function $F(\cdot)$ and the k-th one to the distances between values of the k-th gradient of $F(\cdot)$, $k = 1, .., R$. We recall that gradient operator in Ω is denoted by ∂; $\partial = (\partial_1, \partial_2)$, $\partial_\alpha = \partial / \partial x_\alpha$, $\alpha = 1, 2$, and that ∂^k stands for the k-th gradient in Ω. A function $F(\cdot)$ is called *weakly slowly-varying of the R-th kind* with respect to cell Δ and tolerance parameters δ, $F \in WSV_\delta^R(\Omega, \Delta)$, if and only if:

$$\begin{aligned} &(\forall (\mathbf{x}, \mathbf{y}) \in \Omega^2)[(\mathbf{x} \overset{\lambda}{\approx} \mathbf{y}) \Rightarrow F(\mathbf{x}) \overset{\delta_0}{\approx} F(\mathbf{y}) \\ \text{and :} \quad & \\ &\partial^k F(\mathbf{x}) \overset{\delta_k}{\approx} \partial^k F(\mathbf{y}) \quad k = 1, 2, ..., R] \end{aligned} \tag{10}$$

  Roughly speaking, *weakly slowly-varying* function $F(\cdot)$ can be treated as constant on an arbitrary cell.

  Let us recall that the known *slowly-varying function* $F(\mathbf{x})$, $F \in SV_\delta^R(\Omega, \Delta) \subset WSV_\delta^R(\Omega, \Delta)$, satisfies not only condition (10) but also the extra restriction:

$$(\forall \mathbf{x} \in \Omega)[\lambda \left| \partial^k F(\mathbf{x}) \right| \overset{\delta_k}{\approx} 0 \, , \quad k = 1, 2, ..., R \,] \tag{11}$$

- *Tolerance-periodic functions*:

  An integrable and bounded function $f(\mathbf{x})$ defined in $\bar{\Omega} = [0, L_1] \times [0, L_2]$, which can also depend on time coordinate t as parameter, is called *tolerance-periodic* with respect to cell Δ and tolerance parameters $\delta \equiv (\lambda, \delta_0)$, if for every $\mathbf{x} \in \Omega_\Delta$ there exist Δ-periodic function $\tilde{f}(\cdot)$ such that $f \,|\Delta(\mathbf{x}) \bigcap Dom\, f$ and $\tilde{f} \left| \Delta(\mathbf{x}) \bigcap Dom\, \tilde{f} \right.$ are indiscernible in tolerance determined by $\delta \equiv (\lambda, \delta_0)$. Function $\tilde{f}$ is a Δ-*periodic approximation* of f in $\Delta(\mathbf{x})$. For function $f(\cdot)$ being tolerance periodic together with its derivatives up to the R-th order, we shall write $f \in TP_\delta^R(\Omega, \Delta)$, $\delta \equiv (\lambda, \delta_0, \delta_1, .., \delta_R)$.

  The concepts of *fluctuation shape functions* and *averaging operation* have been explained in Subsection 3.1.

The tolerance modelling is based on two assumptions. The first assumption is called *the tolerance averaging approximation.* The second one is termed *the micro-macro decomposition.*

- *Tolerance averaging approximation*:

  Let $f(\mathbf{x})$ be an integrable periodic function defined in $\bar{\Omega} = [0, L_1] \times [0, L_2]$ and let $F(\mathbf{x}) \in WSV_\delta^1(\Omega, \Delta)$, $G(\mathbf{x}) \in WSV_\delta^2(\Omega, \Delta)$. *The tolerance averaging approximation has the form*:

$$\begin{array}{lll} < f\,\partial^R F > (\mathbf{x}) =< f > \partial^R F(\mathbf{x}) + O(\delta) & R = 0,1 & \partial_1^0 F \equiv F \\ < f\,\partial^R G > (\mathbf{x}) =< f > \partial^R G(\mathbf{x}) + O(\delta) & R = 0,1,2 & \partial_1^0 G \equiv G \end{array} \qquad (12)$$

  In the course of modelling, terms $O(\delta)$ will be neglected. Let us observe that the weakly slowly-varying functions can be regarded as invariant under averaging.

  We recall that the "classical" *slowly-varying functions* $F(\cdot) \in SV_\delta^1(\Omega, \Delta)$, $G(\cdot) \in SV_\delta^2(\Omega, \Delta)$ satisfy not only approximations (12) but also the extra approximate relations:

$$\begin{array}{l} < f\,\partial(hF) > (\mathbf{x}) =< f\,\partial h > F(\mathbf{x}) + O(\delta) \\ < f\,\partial(gG) > (\mathbf{x}) =< f\,\partial g > G(\mathbf{x}) + O(\delta) \\ < f\,\partial^2(gG) > (\mathbf{x}) =< f\,\partial^2 g > G(\mathbf{x}) + O(\delta) \end{array} \qquad (13)$$

  where $h(\cdot) \in FS^1(\Omega, \Delta)$, $g(\cdot) \in FS^2(\Omega, \Delta)$

  Approximations given above will be applied in the modelling problems discussed in this contribution. For details the reader is referred to [3-6, 16.]

- *Micro-macro decomposition*:

  The second fundamental assumption, called *the micro-macro decomposition*, states that the displacements fields occurring in the starting lagrangian under consideration can be decomposed into *unknown averaged* (*macroscopic*) *displacements* being *weakly slowly-varying functions* in $\mathbf{x} \in \Omega$ and highly oscillating *fluctuations* represented by the known highly oscillating λ-periodic *fluctuation shape functions* multiplied by unknown *fluctuation amplitudes* (*microscopic variables*) *weakly slowly-varying* in x.

  In the second step of combined modelling we introduce *the extra micro-macro decomposition* superimposed on the known solutions $u_{0\alpha}, w_0$ obtained within the macroscopic model:

$$\begin{array}{l} u_{c\alpha}(\mathbf{x}, t) = u_{0\alpha}(\mathbf{x}, t) + c(\mathbf{x}) Q_\alpha(\mathbf{x}, t) \\ w_b(\mathbf{x}, t) = w_0(\mathbf{x}, t) + b(\mathbf{x}) V(\mathbf{x}, t) \end{array} \qquad (14)$$

  where *fluctuation* (*microscopic*) *amplitudes* Q_α, V are *the new weakly slowly-varying unknowns*, i.e. $Q_\alpha \in WSV_\delta^1(\Omega, \Delta)$, $V \in WSV_\delta^2(\Omega, \Delta)$. Functions $c(\mathbf{x})$ and $b(\mathbf{x})$ are *the new* periodic, continuous and highly-oscillating *fluctuation shape functions* which are assumed to be known in every problem under consideration. These functions have to satisfy conditions: $c \in O(\lambda)$, $\lambda \partial_\alpha c \in O(\lambda)$, $b \in O(\lambda^2)$, $\lambda \partial_\alpha b \in O(\lambda^2)$, $\lambda^2 \partial_{\alpha\beta} b \in O(\lambda^2)$, $< \mu\, c > = < \mu\, b > = 0$, where $\mu(\cdot)$ is the shell mass density.

We substitute the right-hand sides of (14) into (1). The resulting lagrangian is denoted by L_{cb}. Then, we average L_{cb} over cell Δ using averaging formula (3) and applying *the tolerance averaging approximation* (12). As a result we obtain function $< L_{cb} >$ called *the tolerance averaging of starting lagrangian* (1) *in* Δ *under superimposed decomposition* (14). Next, applying the principle of stationary action, under the extra approximation $1 + \lambda/r \approx 1$, we arrive at the system of Euler-Lagrange equations for Q_α, V, which can be written in an explicit form as:

$$
\begin{aligned}
&\underline{\underline{< D^{\alpha\beta\gamma\delta}(c)^2 >}}\, \partial_{\beta\gamma} Q_\delta - < \partial_\beta c D^{\alpha\beta\gamma\delta} \partial_\gamma c > Q_\delta - \underline{< \mu(c)^2 >}\, a^{\alpha\beta} \ddot{Q}_\beta \\
&= r^{-1} < D^{\alpha\beta 11} \partial_\beta c\, w_0 > + < D^{\alpha\beta\gamma\delta} \partial_\delta c \partial_\beta u_{0\gamma} >
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\underline{\underline{< B^{\alpha\beta\gamma\delta}(b)^2 >}} \partial_{\alpha\beta\gamma\delta} V + [\underline{\underline{2 < B^{\alpha\beta\gamma\delta} b \partial_{\alpha\beta} b >}} - 4\underline{\underline{< \partial_\alpha b B^{\alpha\beta\gamma\delta} \partial_\beta b >}}] \partial_{\gamma\delta} V \\
&+ < \partial_{\alpha\beta} b\, B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} b > V \underline{+ < \mu(b)^2 >} \ddot{V} = - < B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} b \partial_{\alpha\beta} w_0 >
\end{aligned}
\tag{16}
$$

Equations (15) and (16) together with *the micro-macro decomposition* (14) constitute *the superimposed microscopic model*. Coefficients of the derived model equations are constant and some of them *depend on a cell size* λ (the singly and doubly underlined terms). The right-hand sides of (15) and (16) are known under assumption that $u_{0\alpha}, w_0$ were determined in the first step of modelling. The basic unknowns Q_α, V of the model equations must be *the weakly slowly-varying functions in periodicity directions*, i.e. $Q_\alpha \in WSV^1_\delta(\Omega, \Delta)$, $V \in WSV^2_\delta(\Omega, \Delta)$. This requirement can be verified only *a posteriori and it determines the range of the physical applicability of the model*.

3.3. *General combined asymptotic-tolerance model*

Summarizing results obtained above, we conclude that *the general combined asymptotic-tolerance model of selected dynamic problems for the biperiodic shells under consideration* presented here is represented by:

- *Macroscopic model* defined by equations (6) for *macrodisplacements* u^0_α, w^0 with expressions (5) for *fluctuation amplitudes* U_α, W, formulated by means of *the consistent asymptotic modelling* and being independent of the microstructure length. Unknowns of this model must be continuous and bounded functions in $\mathbf{x}$.

- *Superimposed microscopic model equations* (15), (16) for *micro-fluctuation amplitudes* Q_α, V derived by means of *an extended version of the tolerance (non-asymptotic) modelling* and having constant coefficients depending also on a cell size λ (underlined terms) as well as combined with the macroscopic model equations (6) under assumption that in the framework of the asymptotic model the solutions (9) to the problem under consideration are known. Unknown micro-fluctuation amplitudes of this model must be *weakly slowly-varying functions* in $\mathbf{x}$.

- Decomposition:

$$
\begin{aligned}
&u_\alpha(\mathbf{x}, t) = u^0_\alpha(\mathbf{x}, t) + h(\mathbf{x}) U_\alpha(\mathbf{x}, t) + c(\mathbf{x}) Q_\alpha(\mathbf{x}, t) \\
&w(\mathbf{x}, t) = w^0(\mathbf{x}, t) + g(\mathbf{x})\, W(\mathbf{x}, t) + b(\mathbf{x}) V(\mathbf{x}, t) \qquad \mathbf{x} \in \Omega \quad t \in \mathrm{I}
\end{aligned}
\tag{17}
$$

where functions $u^0_\alpha, U_\alpha, w^0, W$ have to be obtained in the first step of combined modelling, i.e. in the framework of *the consistent asymptotic modelling*.

Coefficients of all equations derived in the framework of combined modelling are constant in contrast to coefficients in starting equations (2) which are discontinuous, highly oscillating and periodic in $\mathbf{x}$. Moreover, some of them *depend on a cell size* λ. Thus, *the combined model can be applied to the analysis of many phenomena caused by the length-scale effect*.

It can be shown, cf. [7], that under assumption that fluctuation shape functions $h(\mathbf{x})$, $g(\mathbf{x})$ of macroscopic model coincide with fluctuation shape functions $c(\mathbf{x})$, $b(\mathbf{x})$ of microscopic model, we can obtain *microscopic model equations* (15), (16) in which $c(\cdot)$ and $b(\cdot)$ are replaced by $h(\cdot)$ and $g(\cdot)$, respectively, and in which the right-hand sides are equal to zero:

$$\underline{\underline{< D^{\alpha\beta\gamma\delta}(h)^2 >}}\,\partial_{\beta\gamma}Q_\delta - < \partial_\beta h\, D^{\alpha\beta\gamma\delta}\partial_\gamma h > Q_\delta - \underline{< \mu(h)^2 >}\, a^{\alpha\beta}\ddot{Q}_\beta = 0 \tag{18}$$

$$\begin{aligned}&\underline{\underline{< B^{\alpha\beta\gamma\delta}(g)^2 >}}\partial_{\alpha\beta\gamma\delta}V + [\underline{\underline{2 < B^{\alpha\beta\gamma\delta} g\partial_{\alpha\beta} g >}} - 4\underline{\underline{< \partial_\alpha g\, B^{\alpha\beta\gamma\delta}\partial_\beta g >}}]\partial_{\gamma\delta}V \\ &+ < \partial_{\alpha\beta} g\, B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta} g > V \underline{+ < \mu(g)^2 >}\ddot{V} = 0\end{aligned} \tag{19}$$

Equations (18), (19) are independent of solutions $u_{0\alpha}$, w_0 given by means of (9) and obtained in the framework of *the macroscopic model*. Hence, they *describe selected problems of the shell micro-dynamics* (e.g. the free micro-vibrations, propagation of waves related to the micro-fluctuation amplitudes) *independently of the shell macro-dynamics*.

3.4. *Standard combined asymptotic-tolerance model*

Let us compare *the general combined model* proposed here with the corresponding known *standard combined model* presented and discussed in [7], which was derived under assumption that the unknown fluctuation amplitudes $Q_\alpha(\mathbf{x}, t)$, $V(\mathbf{x}, t)$ in micro-macro decomposition (14) are *slowly-varying*, i.e. $Q_\alpha \in SV^1_\delta(\Omega, \Delta)$, $V \in SV^2_\delta(\Omega, \Delta)$ We recall that the main difference between *the weakly slowly-varying* and the well-known *slowly-varying functions* is that the products of derivatives of *slowly-varying functions* and *microstructure length parameter* λ are treated as negligibly small, cf. (11). Following [7], *the standard combined asymptotic-tolerance model* consists of:

- *Macroscopic model* defined by equations (6) for u^0_α, w^0 with expressions (5) for U_α, W. It is assumed that in the framework of this model the solutions (9) to the problem under consideration are known.

- *Superimposed microscopic model equations* (15), (16) without the doubly underlined terms:

$$\begin{aligned}&< \partial_\beta c\, D^{\alpha\beta\gamma\delta}\partial_\gamma c > Q_\delta + \underline{< \mu(c)^2 >}\, a^{\alpha\beta}\ddot{Q}_\beta = \\ &= r^{-1} < D^{\alpha\beta 11}\partial_\beta c\, w_0 > + < D^{\alpha\beta\gamma\delta}\partial_\delta c\, \partial_\beta u_{0\gamma} >\end{aligned} \tag{20}$$

$$< \partial_{\alpha\beta} b\, B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta} b > V \underline{+ < \mu(b)^2 >}\, \ddot{V} = - < B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta} b\, \partial_{\alpha\beta} w_0 > \tag{21}$$

- Decomposition (17), in which *the weakly slowly-varying functions* $Q_\alpha(\mathbf{x},t) \in WSV_\delta^1(\Omega,\Delta)$, $V(\mathbf{x},t) \in WSV_\delta^2(\Omega,\Delta)$ are replaced by *slowly-varying functions* $Q_\alpha(\mathbf{x},t) \in SV_\delta^1(\Omega,\Delta)$, $V(\mathbf{x},t) \in SV_\delta^2(\Omega,\Delta)$.

From comparison of both the general and the standard combined models it follows that *the general model equations* (15), (16) contain a bigger number of terms depending on the microstructure size than *the standard model equations* (20), (21). Thus, the general model proposed here makes it possible to investigate the length-scale effect in more detail.

It can be observed that within the framework of *the general combined model*, unknown fluctuation (microscopic) amplitudes $Q_\alpha(\mathbf{x},t)$, $V(\mathbf{x},t)$ are governed by *partial differential equations* (15), (16), whereas within the framework of *the standard combined model* these unknowns are governed by *ordinary differential equations* (20), (21) involving only time derivatives. Hence there are no extra boundary conditions for unknowns $Q_\alpha(\mathbf{x},t)$, $V(\mathbf{x},t)$ of the standard combined model and that is why they play the role of kinematic internal variables.

4. Examples of applications

In this section we shall investigate two special micro-dynamic problems. The first of them deals with harmonic micro-vibrations in axial direction. The second one deals with propagation of the long waves related to micro-fluctuations of axial displacements.

It has to be emphasized that these aforementioned special micro-dynamic problems can be studied in the framework of neither the asymptotic models nor standard combined models for the biperiodic shells under consideration.

4.1. *Formulation of the problem*

The object of considerations is a closed biperiodically densely stiffened cylindrical shell with r, $L_1 = 2\pi r$, L_2 as its midsurface curvature radius, circumferential and axial lengths, respectively, cf. Fig. 1. The stiffened shell under consideration is treated as a shell with periodically varying thickness $d(\mathbf{x})$ and periodically varying elastic $D^{\alpha\beta\gamma\delta}(\mathbf{x})$, $B^{\alpha\beta\gamma\delta}(\mathbf{x})$ and inertial $\mu(\mathbf{x})$ properties. It is assumed that both the shell and stiffeners are made of homogeneous isotropic materials.

We recall that *the microstructure length parameter* λ has to satisfy conditions: $\lambda/d_{\max} >> 1$, $\lambda/r << 1$ and $\lambda/\min(L_1, L_2) << 1$.

It assumed that the fluctuation shape functions are known in the problem under consideration.

The subsequent analysis will be restricted to general equations of micro-dynamics (18).

Let the investigated problem be rotationally symmetric with a period λ/r; hence $Q_1(\cdot,t)$ in (18) is equal to zero and the remaining unknown $Q_2(\cdot,t)$ of (18) is independent of x^1. Obviously, fluctuation shape functions $h(\cdot)$ are λ-periodic functions of both arguments x^1 and x^2. Hence, the micro-fluctuations $u_{h2}(\mathbf{x},t)$ given by $u_{h2}(\mathbf{x},t) = h(\mathbf{x})Q_2(x^2,t)$ are also functions of x^1 and x^2.

Now, system of equations (18) obtained by means of the general combined modelling reduces to one equation for $Q_2(x^2,t)$ describing the shell's micro-dynamics in axial direction:

$$\begin{aligned} &\underline{\underline{< D^{2222}(h)^2 >}}\partial_{22}Q_2 - (< D^{2112}(\partial_1 h)^2 > + < D^{2222}(\partial_2 h)^2 >)Q_2 + \\ &-\underline{< \mu(h)^2 >}\ddot{Q}_2 = 0 \end{aligned} \tag{22}$$

The singly and doubly underlined terms in (22) depend on a cell size. Note that corresponding equation derived by means of the standard combined modelling has a form of (22) without term including the space derivatives of $Q_2(x^2, t)$ (i.e. without the doubly underlined term).

The subsequent analysis will be based on micro-dynamic equation (22).

4.2. *Micro-vibrations*

In order to investigate the problem of harmonic micro-vibrations in axial direction we assume solution to equation (22) in the form:

$$Q_2(x^2, t) = Q^*(x^2)\cos(\omega t) \tag{23}$$

with ω as a vibration frequency.

Hence, under denotations:

$$\begin{aligned} k^2 &\equiv \frac{<D^{2112}(\partial_1 h)^2> + <D^{2222}(\partial_2 h)^2>}{\lambda^2 <D^{2222}(\bar{h})^2>} \\ \omega_*^2 &\equiv \frac{<D^{2112}(\partial_1 h)^2> + <D^{2222}(\partial_2 h)^2>}{\lambda^2 <\mu(\bar{h})^2>} \end{aligned} \tag{24}$$

where $\bar{h} = \lambda^{-1}h$, $h \in O(\lambda)$, equation (22) yields:

$$\partial_{22}Q^*(x^2) - k^2[1 - (\omega/\omega_*)^2]\, Q^*(x^2) = 0 \tag{25}$$

where ω_* is referred to as *the free micro-vibration frequency* depending on a cell size. It can be shown that averages $< D^{2112}(\partial_1 h)^2 >$, $< D^{2222}(\bar{h})^2 >$, $< \mu(\bar{h})^2 >$ are greater than zero; hence $k^2 > 0$ and $\omega_*^2 > 0$. Function $Q^*(x^2)$ is a slowly-varying function of x^2. The boundary conditions are assumed in the form $Q^*(x^2 = 0) = Q_0$, $Q^*(x^2 = L_2) = 0$, where Q_0 is the known arbitrary constant.

The solutions to equation (25) depend on relations between vibrations frequencies ω and ω_*. It means that micro-fluctuation amplitude (23) also depends on relations between ω and ω_*. Solutions to (25) imply the following special cases of micro-vibrations:

1. If $0 < \omega^2 < \omega_*^2$ and setting $k_\omega^2 \equiv k^2[1 - (\omega/\omega_*)^2]$ then:

$$\begin{aligned} Q_2(x^2, t) = Q_0[&\exp(-k_\omega x^2)(1 - \exp(-2k_\omega L_2))^{-1} \\ &+ \exp(k_\omega x^2)(1 - \exp(2k_\omega L_2))^{-1}]\cos(\omega t) \end{aligned} \tag{26}$$

   In this case *micro-vibrations decay exponentially.* It can be observed that if $0 < \omega^2 << \omega_*^2$ then we can take into account the following approximate form of solution (26):

$$Q_2(x^2, t) = Q_0 \exp(-k_\omega x^2)\cos(\omega t) \tag{27}$$

   From (27) it follows that *micro-vibrations are strongly decaying near the boundary* $x^2 = 0$. It means that they can be treated as equal to zero outside a certain narrow layer near boundary $x^2 = 0$. Thus, equation (22) being a starting point in the micro-dynamic problem under consideration makes it possible to investigate *the boundary layer phenomena.*

2. If $\omega^2 = \omega_*^2$ then:
$$Q_2(x^2,t) = Q_0(1 - x^2/L_2)\cos(\omega t) \tag{28}$$
we deal with a linear decaying of micro-vibrations.

3. If $\omega^2 > \omega_*^2$ and $\kappa^2 \equiv k^2[(\omega/\omega_*)^2 - 1] \neq (n\pi)^2(L_2)^{-2}$ then:
$$Q_2(x^2,t) = Q_0 \sin(\kappa(L_2 - x^2))(\sin(\kappa L_2))^{-1}\cos(\omega t) \tag{29}$$
micro-vibrations are not decaying, they oscillate.

4. If $\omega^2 > \omega_*^2$ and $\kappa^2 \equiv k^2[(\omega/\omega_*)^2 - 1] = (n\pi)^2(L_2)^{-2}$ then the solution to equation (25) does not exist; *we obtain resonance micro-vibrations with resonance frequencies*:
$$\omega_n^2 = \omega_*^2[1 + (n\pi)^2(L_2\kappa)^{-2}]\,, \quad n = 1, 2, ... \tag{30}$$

We recall that these aforementioned special micro-dynamic problem can be studied in the framework of neither the asymptotic models nor standard combined models for the biperiodic shells under consideration. It can be observed that within *the asymptotic model* after neglecting the length-scale terms, equation (22) reduces to equation $(< D^{2112}(\partial_1 h)^2 > + < D^{2222}(\partial_2 h)^2 >)Q_2 = 0$, which has only trivial solution $Q_2 = 0$. It can be also seen that the corresponding equation of *the standard combined model* has a form of (22) without space derivatives of Q_2 (doubly underlined term), i.e. form of ordinary differential equation involving the time derivatives only. The above effects cannot be analysed by using this ordinary differential equation.

4.3. *Wave propagation problem*

Now, let the shells under consideration be unbounded along the axial coordinate x^2, cf. Fig. 1. We shall analyse the wave propagation problem. The waves related to micro-fluctuation amplitude $Q_2(x^2,t)$ are taken into account. Hence, equation (22) describing the shells' micro-dynamics in an axial direction will be applied. We look for solution to equation (22) in the form $Q_2(x^2,t) = F(x^2 - ct)$, where c is the wave propagation velocity. Setting $\bar{h} = \lambda^{-1}h$, from equation (22) we obtain:
$$(c^2 - \tilde{c}^2)\partial_{22}F + \bar{c}^2\lambda^{-2}F = 0 \tag{31}$$
where speeds $\tilde{c}$ and $\bar{c}$ are defined by:
$$\tilde{c}^2 \equiv \frac{< D^{2222}(\bar{h})^2 >}{< \mu(\bar{h})^2 >} \qquad \bar{c}^2 \equiv \frac{< D^{2112}(\partial_1 h)^2 > + < D^{2222}(\partial_2 h)^2 >}{< \mu(\bar{h})^2 >} \tag{32}$$

Function $F(x^2,t)$ is a slowly-varying function of x^2. Equation (31) implies the following special cases of wave propagation in the biperiodic shells under consideration:
(a) *sinusoidal waves* if $c > \tilde{c}$,
(b) *exponential waves* if $c < \tilde{c}$,
(c) *degenerate case* if $c = \tilde{c}$

The above effect cannot be analysed in the framework of asymptotic models and of standard combined models.

In order to determine the dispersion relation for the case (a), let us substitute to equation (22) solution of the form $Q_2(x^2,t) = A\sin(k(x^2-ct))$, $k = 2\pi/L$, where L and k are the wavelength and the wave number, respectively, A is an arbitrary constant. It is assumed that $L >> \lambda$. The nontrivial solution ($A \neq 0$) exists only if:

$$[(k\lambda)^2c^2 - (k\lambda)^2\tilde{c}^2 - \bar{c}^2] = 0 \tag{33}$$

where under assumption that $L >> \lambda$ the following condition holds:
$k\lambda = 2\pi\lambda/L << 1$.

The above equation describes the effect of dispersion. It can be seen that for $k\lambda \to 0$ the dispersion effect disappears. From equation (33) it follows that the dispersive long waves related to micro-fluctuation amplitude $Q_2(x^2,t)$ can propagate across the unbounded beperiodic shells under consideration with propagation speed:

$$c^2 = \tilde{c}^2 + \bar{c}^2(k\lambda)^{-2} \tag{34}$$

depending on microstructure size λ. Note, that for homogeneous isotropic shells expression (34) leads to the well-known results $c^2 = D/\mu$, $D = E\delta/(1-\nu^2)$, where E, ν, δ, μ are Young's modulus, Poisson's ratio, the shell thickness and mass density of the shell material, respectively, cf. Kaliski [22].

4.4. *Discussion of results*

Analysing results obtained in this section the following remarks can be formulated:

1. The *general combined asymptotic-tolerance model for biperiodic shells under consideration* presented here makes it possible to analyse selected problems of the shells' micro-dynamics.

2. Harmonic micro-vibrations with micro-vibration frequency ω were analysed. It was shown that the micro-dynamic behaviour of the shell is different for different values of vibration frequency ω. *The micro-vibrations can decay exponentially, they can decay linearly, certain values of ω cause a non-decayed form of micro-vibrations (micro-vibrations oscillate), for certain values of ω we deal with resonance micro-vibrations.* Moreover, *the new higher free vibration frequency ω_* dependent on a cell size λ has been obtained.* It can be observed that if $0 \leq \omega^2 << \omega_*^2$ then micro-fluctuation amplitude Q_2 is strongly decaying near the boundary $x^2 = 0$. It means that the displacement micro-fluctuations can be treated as equal to zero outside a certain narrow layer near boundary $x^2 = 0$. Thus, we have shown that equation (22) of the microscopic tolerance model obtained in the second step of combined modelling describes the space-boundary layer phenomena strictly related to the specific form of boundary conditions imposed on Q_2. All the effects mentioned above can be analysed in the framework of neither the asymptotic models which neglect the length-scale effect nor the standard combined models for the biperiodic shells in which unknown fluctuation amplitudes are governed by ordinary differential equations involving only time derivatives.

3. The possible applications of the combined model proposed here were also illustrated by the analysis of the problem of wave propagation in the shells

unbounded in an axial direction. The long waves, related to micro-fluctuation amplitude $Q_2(x^2, t)$ being unknown of microscopic model equation (22), were studied. It was shown that *the micro-periodic heterogeneity of the shells leads to exponential waves and to dispersion effects, which cannot be analysed in the framework of the asymptotic models and the standard combined models for biperiodic shells*. Moreover, *the new wave propagation speed depending on the microstructure size has been obtained*, cf. formula (34).

5. Final remarks and conclusions

The following remarks and conclusions can be formulated:

- Thin linearly elastic Kirchhoff-Love-type circular cylindrical shells having a periodic microstructure in circumferential and axial directions (*biperiodic shells*) are objects of consideration, cf. Fig.1.

  The new averaged general combined asymptotic-tolerance model for the analysis of selected dynamic problems for the biperiodic cylindrical shells under consideration was derived in Tomczyk and Litawska [20]. Here, the governing equations of this model are recalled and applied for investigations of certain micro-dynamic problems for the shells under consideration. The aforementioned model equations consist of *macroscopic* (*asymptotic*) *model equations* (6) for *macrodisplacements* $u^0_\alpha(\mathbf{x}, t), w^0(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, $t \in \mathrm{I}$, derived by means of *the consistent asymptotic procedure*, cf. [6], and of *general microscopic tolerance* (*non-asymptotic*) *model equations* (15), (16) for *fluctuation amplitudes* (*microscopic variables*) $Q_\alpha(\mathbf{x}, t)$, $V(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, $t \in \mathrm{I}$, formulated by applying *an extended version of the tolerance modelling technique*, cf. [16]. This extended version is based on a new notion of *weakly slowly-varying functions.* Macro- and microscopic models are combined together under assumption that in the framework of the asymptotic model the solutions (9) to the problem under consideration are known. Contrary to the well-known governing equations (2) of Kirchhoff-Love theory with highly oscillating, non-continuous and periodic coefficients, *equations of the asymptotic-tolerance model have constant coefficients depending also on a microstructure size.* Hence, this model allows us to describe the effect of a length scale on the dynamic shell behaviour. Moreover, *the general combined model equations* contain a bigger number of terms depending on a cell size than *the known standard combined model equations* presented in [7] and recalled here by means of equations (20), (21), which were derived applying "classical" concept of *the slowly-varying functions.* Thus, from the theoretical results it follows that the general model allow us to investigate the length-scale effect in more detail. The main advantage of the proposed model is that it makes it possible to separate the macroscopic description of some special problems from their microscopic description. Micro-dynamic behaviour of the shells are described by equations (18), (19) being independent of solutions (9) obtained in the framework of the macroscopic model.

- The main aim of this contribution was to study two special micro-dynamic problems for a certain closed, biperiodically and densely stiffened cylindrical shell, cf. Fig. 1. The first of them deals with harmonic micro-vibrations in axial direction. The second one deals with propagation of the long waves related to micro-fluctuations of axial displacements. In order to analyse these problems, the micro-dynamic equations (18) of the general superimposed microscopic model were applied.

- Some new important results have been obtained analysing the harmonic micro-vibrations with vibration frequency ω. It was shown that the shape of these micro-vibrations depends on relations between values of vibration frequency ω and *a certain new additional higher-order free vibration frequency* ω_* *depending on a cell size* λ. *The micro-vibrations can decay exponentially and very strongly near the boundary* $x^2 = 0$, *they can decay linearly, certain values of* ω *cause a non-decayed form of micro-vibrations (micro-vibrations oscillate), for certain values of* ω *we deal with resonance micro-vibrations.* The problem with strongly decaying micro-vibrations near the boundary $x^2 = 0$ is referred to the space-boundary-layer phenomena.

- Some new important results have been obtained analysing the long wave propagation problem related to micro-fluctuations in axial direction. It was shown that *the tolerance-periodic heterogeneity of the shells leads to exponential waves and to dispersion effects, which cannot be analysed in the framework of the asymptotic models for periodic shells*. Moreover, *the new wave propagation speed depending on the microstructure size has been obtained*, cf. formula (34).

- All the effects mentioned above can be analysed in the framework of neither the asymptotic models which neglect the length-scale effect nor the standard combined models for the biperiodic shells in which unknown fluctuation amplitudes are governed by ordinary differential equations involving only time derivatives.

Some other applications of the general combined asymptotic-tolerance model will be shown in forthcoming papers.

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