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On the Modeling of Periodic Sandwich Structures with the Use of the Broken Line Hypothesis

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In this paper a dynamic analysis of sandwich plate with a certain periodic microstructure is considered. The initial system of governing equations is derived basing on the classic broken line hypothesis. As a result of transformations one can obtain a system of three differential equations of motion with periodic, highly oscillating and non-continuous coefficients. In order to derive a system of equations with constant coefficients tolerance averaging technique is applied. Eventually, in the calculation example a free vibration analysis of certain periodic plate strip is performed with the use of both the derived model and a FEM model. It can be observed that the consistency of obtained results is highly dependent on the calculation assumptions.

Keywords: sandwich structure, periodic microstructure, free vibrations, broken line hypothesis.

1. Introduction

Sandwich structures are certain specific composites made of three layers - outer layers, which are made of materials characterized by high mechanical properties, so as they can be treated as the main bearing part of the structure, and the inner layer, so called *core*. The core is usually made of light-weight materials, which have two functions: it increases the stiffness of the whole structure, by increasing its thickness, and stands for the thermal and acoustic isolation. Due to the fact, that every each layer of such structure is being modeled and optimized to work best in certain specific conditions, sandwich plates can be characterized by physical and mechanical properties, which are unreachable for 'classic', homogeneous materials. Hence, they have many applications in modern engineering.

There are many different approaches to modeling of sandwich structures. First of all, sandwich structures can be treated as a system of two outer layers modeled as beams, membrane systems, plates or even shells, connected with each other with a certain type of elastic material, such as: one- or multiparametric Winkler's type material, Murakami's type material or Pasternak's type material, among others (cf. Chonan [1], Oniszczuk [2], Szcześniak [3,4], Navarro [5]). Secondly, the whole structure can be treated as a single multilayered plate or shell and then analyzed with the use of complicated deformation field hypothesis, such as: broken line hypothesis, Zig-Zag theory or Reissner-Mindlin hypothesis (cf. Magnucki and Ostwald [6], Magnucka-Blandzi et al. [7], Carrera and Brischetto [8]). Eventually, researchers use a vast variety of numerical methods to analyze sandwich structures, such as the finite difference method or the finite element method, which is considered to be the most versatile method of analysis of any type of structures recently.

In this work let us focus on the modeling of vibrations of sandwich plates based on the broken line hypothesis. The considered sandwich plate is assumed to be characterized by certain periodic microstructure, such as: periodically varying thickness and/or material properties of the outer layers and/or the core. As a result of this assumption, the initial system of governing equations contains periodic, highly oscillating and non-continuous coefficients. In order to obtain a system of equations with constant coefficients, which is convenient to solve, the tolerance averaging technique is applied. In literature one can find a detailed description of this technique, cf. Woźniak and Wierzbicki [9], Woźniak et al. [10], Woźniak [11], along with many applications, for example in thermomechanics of laminates, cf. Pazera and Jędrysiak [12], dynamics of medium thickness plates cf. Jędrysiak [13], Baron [14,15] or dynamics and stability of cylindrical shells, cf. Tomczyk and Szczerba [16]. The author also used this technique in his previous papers concerning vibrations of three-layered structures, cf. Marczak and Jędrysiak [17].

Basing on the derived model the free vibration analysis of periodic plate strip will be performed. Eventually, the obtained results will be verified by FEM model and the consistency of results will be discussed.

2. Modeling foundations

Let us introduce $0x_1x_2x_3$ as an orthogonal Cartesian coordinate system, where $\mathbf{x} \equiv (x_1, x_2)$, and t as a time coordinate. The considered three-layered plate is assumed to be rectangular and to have spans L_1 and L_2 along x_1- and x_2-axis directions, respectively. Hence, its midplane can be denoted as: $\Pi \equiv [0, L_1] \times [0, L_2]$. Let us focus on certain specific type of sandwich structures, which are symmetric to its midplane and made of isotropic materials. By introducing $h_c(\mathbf{x})$ and $h_f(\mathbf{x})$ as thicknesses of the core and the outer layers, respectively, the whole region occupied by the plate can be denoted as: $\Omega \equiv \{(\mathbf{x}, x_3) : -h_c(\mathbf{x})/2 - h_f(\mathbf{x}) \leq x_3 \leq h_c(\mathbf{x})/2 + h_f(\mathbf{x}), \mathbf{x} \in \Pi\}$, cf. Fig. 1. Since the whole structure is assumed to be made of isotropic materials, let us introduce $E_f(\mathbf{x})$, $\nu_f(\mathbf{x})$, $G_f(\mathbf{x})$ and $\rho_f(\mathbf{x})$ as Young's modulus, Poisson's ratio, shear modulus and mass density of the outer layers, respectively, and $E_c(\mathbf{x})$, $\nu_c(\mathbf{x})$, $G_c(\mathbf{x})$ and $\rho_c(\mathbf{x})$ as Young's modulus, Poisson's ratio, shear modulus and mass density of the core, respectively.

It should be emphasized, that the three-layered structure under consideration is characterized by a certain periodic microstructure, connected with periodically varying thicknesses and/or material properties of the layers. Basing on this microstructure, one can distinguish a small, repeatable part called *periodicity cell* Δ. Let us assume, that the basic periodicity cell takes a rectangular shape and has dimensions l_1 and l_2 along x_1- and x_2-axis directions, respectively. The diameter of the considered periodicity cell will be referred to as *microstructure parameter* l. In all subsequent formulas let us denote a spatial derivative as $\partial_i \equiv \frac{\partial}{\partial x_i}$, $i = 1,2,3$, and a time derivative as an overdot.

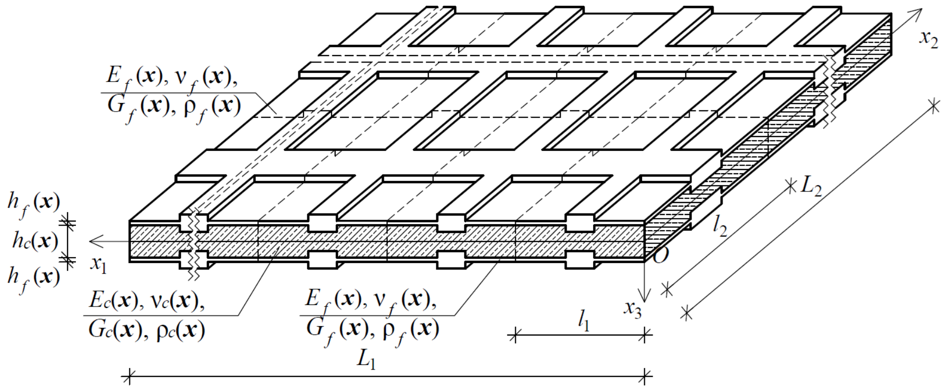

Figure 1 Sandwich three-layered plate with a certain periodic microstructure

The modeling of sandwich structure is based on the well-known broken line hypothesis, cf. Magnucki and Ostwald [6], according to which the displacements along specific direction can be described as follows:

$$
\begin{aligned}
&u_1(\mathbf{x}, x_3, t) = \begin{cases} -x_3\partial_1 w(\mathbf{x}, t) - h_c(\mathbf{x})\psi_1(\mathbf{x}, t) & for \quad x_3 \in H_f^- \\ -x_3\partial_1 w(\mathbf{x}, t) + 2x_3\psi_1(\mathbf{x}, t) & for \quad x_3 \in H_c \\ -x_3\partial_1 w(\mathbf{x}, t) + h_c(\mathbf{x})\psi_1(\mathbf{x}, t) & for \quad x_3 \in H_f^+ \end{cases} \\
&u_2(\mathbf{x}, x_3, t) = \begin{cases} -x_3\partial_2 w(\mathbf{x}, t) - h_c(\mathbf{x})\psi_2(\mathbf{x}, t) & for \quad x_3 \in H_f^- \\ -x_3\partial_2 w(\mathbf{x}, t) + 2x_3\psi_2(\mathbf{x}, t) & for \quad x_3 \in H_c \\ -x_3\partial_2 w(\mathbf{x}, t) + h_c(\mathbf{x})\psi_2(\mathbf{x}, t) & for \quad x_3 \in H_f^+ \end{cases} \\
&u_3(\mathbf{x}, x_3, t) = w(\mathbf{x}, t) \\
&H_f^- \equiv \{(\mathbf{x}, x_3) : -h_c(\mathbf{x})/2 - h_f(\mathbf{x}) \leq x_3 < -h_c(\mathbf{x})/2, \mathbf{x} \in \Pi\} \\
&H_c \equiv \{(\mathbf{x}, x_3) : -h_c(\mathbf{x})/2 \leq x_3 \leq h_c(\mathbf{x})/2, \mathbf{x} \in \Pi\} \\
&H_f^+ \equiv \{(\mathbf{x}, x_3) : h_c(\mathbf{x})/2 < x_3 \leq h_c(\mathbf{x})/2 + h_f(\mathbf{x}), \mathbf{x} \in \Pi\}
\end{aligned}
\tag{1}
$$

where ψ_α, $\alpha = 1,2$ are certain dimensionless displacements along x_α-axis direction, cf. Fig. 2. It should be emphasized, that according to assumptions (1) displacements along x_3-axis direction $u_3(\mathbf{x},x_3,t)$ are equal to the displacements of the midplane of the structure $w(\mathbf{x},t)$.

Basing on the presented deformation hypothesis (1), one can calculate proper strain tensor elements ε_{ij}, $i,j = 1,2,3$. By assuming stress-strain relation according to Hooke's law it is possible to obtain initial governing equations in the form of equations of equilibrium:

$$\begin{gathered}
C_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma\delta}w(\mathbf{x},t) - \hat{C}_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma}\psi_\delta(\mathbf{x},t) - A_{11}(\rho_f,\rho_c)\partial_{\alpha\alpha}\ddot{w}(\mathbf{x},t) \\
+A_{12}(\rho_f,\rho_c)\partial_\alpha\ddot{\psi}_\alpha(\mathbf{x},t) + B_1\ddot{w}(\mathbf{x},t) = p(\mathbf{x},t)/[h_c(\mathbf{x})]^3 \\
C_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma}w(\mathbf{x},t) - \hat{C}_{\alpha\beta\gamma\delta}\partial_{\alpha\gamma}\psi_\beta(\mathbf{x},t) - A_{11}(\rho_f,\rho_c)\partial_\delta\ddot{w}(\mathbf{x},t) \\
+A_{12}(\rho_f,\rho_c)\ddot{\psi}_\delta(\mathbf{x},t) + B_2\psi_\delta(\mathbf{x},t) = 0
\end{gathered} \tag{2}$$

where α, β, γ, $\delta = 1$, 2, $p(\mathbf{x},t)$ is an external loading along x_3-axis direction and:

$$\begin{array}{lll}
C_{1111} = C_{2222} = A_{11}(E_f,E_c), & C_{1122} = C_{2211} = A_{11}(E_f v_f, E_c v_c) & \\
C_{1212} = C_{2121} = C_{1221} = C_{2112} = A_{11}(G_f,G_c), & & \\
\hat{C}_{1111} = \hat{C}_{2222} = A_{12}(E_f,E_c), & \hat{C}_{1122} = \hat{C}_{2211} = A_{12}(E_f v_f, E_c v_c) & \\
\hat{C}_{1212} = \hat{C}_{2121} = \hat{C}_{1221} = \hat{C}_{2112} = A_{12}(G_f,G_c) & & \\
B_1 = (2\rho_f h_f + \rho_c h_c)/h_c^3, & B_2 = 2G_c/h_c^2 & \\
A_{11}(Y,Z) = Ya_1 + \frac{1}{12}Z, & A_{12}(Y,Z) = Ya_2 + \frac{1}{6}Z & \\
a_1 = (\frac{2}{3}X^2 + X + \frac{1}{2})X, & a_2 = X^2 + X, & X = h_f/h_c
\end{array} \tag{3}$$

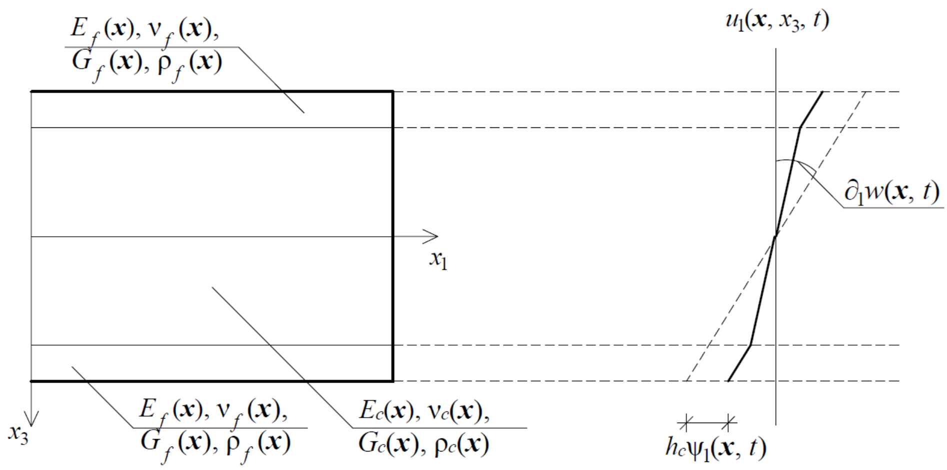

Figure 2 Assumed deflection along x1-axis direction

Let us remind, that all denotations in (3) describing thickness and material properties of the certain layer of the structure can be a function of coordinate $\mathbf{x}$. Hence, as a result we arrive at a system of three equations of equilibrium (there is no summation over δ in $(2)_2$) with periodic, highly oscillating and non-continuous coefficients, which is difficult to solve using well-known mathematical methods. In order to obtain a system of equations with constant coefficients, the tolerance averaging technique will be used.

It should be emphasized, that the correctness of the shown broken line hypothesis in the analysis of sandwich plates has been already proved in the literature, for example in: Magnucki and Ostwald [6], where the consistency of results of stability analysis between the broken line hypothesis, the FEM analysis and the experimental results was presented. Moreover, the Author also used the broken line hypothesis in the vibrations analysis of sandwich plates without periodic microstructure, cf. [18], also obtaining a satisfactory consistency of results with FEM analysis.

3. Basics of the Tolerance Averaging Technique

The Tolerance Averaging Technique was developed by Woźniak and base on several basic concepts such as: a tolerance parameter δ, a basic periodicity cell $\Delta(\mathbf{x})$, a tolerance periodic function $TP^k_\delta(\Delta)$, a slowly varying function $SV^k_\delta(\Delta)$, a highly oscillating function $HO^k_\delta(\Delta)$ or a fluctuation shape function $FS^k_\delta(\Delta)$. In literature one can find many publications, where those basic concepts are described in details, cf. Woźniak and Wierzbicki [9], Woźniak et al. [10], Woźniak [11], hence let us remind only several most important assumptions of modeling.

The modeling process is based on an *averaging operation*, which for 2D issue can be described with the use of following formula:

$$< \frac{\partial^k}{\partial x_i^k} f > (\mathbf{x}) = \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} \tilde{f}^{(k)}(\mathbf{x}, y) dy, \qquad k = 0,\, 1,\, 2... \tag{4}$$

where $\Delta(\mathbf{x}) \equiv \mathbf{x} + \Delta$ is a basic periodicity cell with a center at $\mathbf{x}$, and $\tilde{f}^{(k)}(\mathbf{x}, y)$ is a periodic approximation of k^{th} derivative of function $f(\mathbf{x})$. As a result of the averaging operation of certain function $f(\mathbf{x})$ one obtain a constant, averaged value of this function.

There are two main assumptions in the tolerance averaging technique. The first of them is a heuristic *micro-macro decomposition*, according to which certain field can be presented as a sum of the averaged *macrofield* of certain physical property, being a slowly varying function, and a sum of products of certain assumed fluctuation shape functions and fluctuation amplitudes, which are also slowly varying functions:

$$\begin{array}{lll} w(\cdot,t) = W(\cdot,t) + g^A(\cdot)Q^A(\cdot,t), & A = 1,2,...,N, & \\ W(\cdot,t) \in SV^k_\delta(\Delta), & Q^A(\cdot,t) \in SV^k_\delta(\Delta), & g^A(\cdot) \in FS^k_\delta(\Delta) \end{array} \tag{5}$$

The second assumption is a set of *tolerance averaging approximations*, according to which certain terms can be treated as equal with a respect to the tolerance parameter δ. Several of such approximations are presented below:

$$\begin{array}{l} < \varphi > (\mathbf{x}) = < \tilde{\varphi} > (\mathbf{x}) + O(\delta), \\ < \varphi F > (\mathbf{x}) = < \varphi > (\mathbf{x}) F(\mathbf{x}) + O(\delta), \\ < \varphi \partial_\alpha (gF) > (\mathbf{x}) = < \varphi \partial_\alpha g > (\mathbf{x}) F(\mathbf{x}) + O(\delta), \\ < g \partial_\alpha (\varphi \Phi) > (\mathbf{x}) = \; - < \varphi \Phi \partial_\alpha g > (\mathbf{x}) + O(\delta), \\ \mathbf{x} \in \Pi, \qquad \alpha = 1,2, \qquad 0 < \delta << 1, \\ \varphi, \Phi \in TP^k_\delta(\Delta), \qquad F \in SV^k_\delta(\Delta), \qquad g \in FS^k_\delta(\Delta) \end{array} \tag{6}$$

4. Tolerance modeling of periodic sandwich plate

The whole modeling procedure consists of several transformations of initial governing equations of considered sandwich plate (2). In the first step, the whole system of equations is averaged with the use of the averaging operator (4). Then, micro-macro decomposition of displacement fields is applied in the following form:

$$\begin{aligned} w(\mathbf{x},t) &= W(\mathbf{x},t) + g^A(\mathbf{x})Q^A(\mathbf{x},t), \\ \psi_\alpha(\mathbf{x},t) &= \Theta_\alpha(\mathbf{x},t) + h^B_\alpha(\mathbf{x})\Phi^B_\alpha(\mathbf{x},t) \end{aligned} \tag{7}$$

where:
– $W(\mathbf{x},t)$ and $\Theta_\alpha(\mathbf{x},t)$ are macroscale displacements, $W(\mathbf{x},t)$, $\Theta_\alpha(\mathbf{x},t) \in SV^4_\delta(\Delta)$, $\alpha = 1,2$, $Q^A(\mathbf{x},t)$ and ${\Phi_\alpha}^B(\mathbf{x},t)$ are fluctuation amplitudes,
– $Q^A(\mathbf{x},t), \Phi^B_\alpha(\mathbf{x},t) \in SV^4_\delta(\Delta)$, $A = 1,2\ldots,N$, $B = 1,2,\ldots,M$, and $g^A(\mathbf{x})$ and ${h_\alpha}^B(\mathbf{x})$ are assumed fluctuation shape functions,
– $g^A(\mathbf{x},t), h^B_\alpha(\mathbf{x},t) \in FS^4_\delta(\Delta)$, which satisfy the following normalizing conditions: $< B_1 g^A >= 0$ and $< A_{12}(\rho_f,\rho_c)h^B_\alpha >= 0$.
In the next step the orthogonalization condition of the obtained equations and assumed fluctuation-shape functions is formulated and eventually, several transformations with the use of the tolerance averaging approximations (6) are performed in order to obtain a convenient form of equations. As a result of the modeling procedure, one can obtain the system of governing equations of the tolerance model (TM) of sandwich three-layered plate in the form:

$$\begin{aligned}
&< C_{\alpha\beta\gamma\delta} > \partial_{\alpha\beta\gamma\delta} W + < C_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma\delta} g^A > Q^A - < \hat{C}_{\alpha\beta\gamma\delta} > \partial_{\alpha\beta\gamma}\Theta_\delta \\
&- < \hat{C}_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma} h^B_\delta > \Phi^B_\delta - < A_{11}(\rho_f,\rho_c) > \partial_{\alpha\alpha}\ddot{W} \\
&- < A_{11}(\rho_f,\rho_c)\partial_{\alpha\alpha} g^A > \ddot{Q}^A + < A_{12}(\rho_f,\rho_c) > \partial_\alpha \ddot{\Theta}_\alpha \\
&+ < A_{12}(\rho_f,\rho_c)\partial_\alpha h^B_\alpha > \ddot{\Phi}^B_\alpha + < B_1 > \ddot{W} =< p/h^3_c > \\
\\
&< C_{\alpha\beta\gamma\delta} > \partial_{\alpha\beta\gamma} W + < C_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma} g^A > Q^A - < \hat{C}_{\alpha\beta\gamma\delta} > \partial_{\alpha\gamma}\Theta_\beta \\
&- < \hat{C}_{\alpha\beta\gamma\delta}\partial_{\alpha\gamma} h^B_\beta > \Phi^B_\beta - < A_{11}(\rho_f,\rho_c) > \partial_\delta \ddot{W} - < A_{11}(\rho_f,\rho_c)\partial_\delta g^A > \ddot{Q}^A \\
&+ < A_{12}(\rho_f,\rho_c) > \ddot{\Theta}_\delta + < B_2 > \Theta_\delta + < B_2 h^B_\delta > \Phi^B_\delta = 0 \\
\\
&< C_{\alpha\beta\gamma\delta} g^K > \partial_{\alpha\beta\gamma\delta} W + < C_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma\delta} g^A g^K > Q^A - < \hat{C}_{\alpha\beta\gamma\delta} g^K > \partial_{\alpha\beta\gamma}\Theta_\delta \\
&- < \hat{C}_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma} h^B_\delta g^K > \Phi^B_\delta - < A_{11}(\rho_f,\rho_c) g^K > \partial_{\alpha\alpha}\ddot{W} \\
&- < A_{11}(\rho_f,\rho_c)\partial_{\alpha\alpha} g^A g^K > \ddot{Q}^A + < A_{12}(\rho_f,\rho_c) g^K > \partial_\alpha \ddot{\Theta}_\alpha \\
&+ < A_{12}(\rho_f,\rho_c)\partial_\alpha h^B_\alpha g^K > \ddot{\Phi}^B_\alpha + < B_1 g^A g^K > \ddot{Q}^A =< p g^K/h^3_c > \\
\\
&< C_{\alpha\beta\gamma\delta} h^L_\delta > \partial_{\alpha\beta\gamma} W + < C_{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma} g^A h^L_\delta > Q^A - < \hat{C}_{\alpha\beta\gamma\delta} h^L_\delta > \partial_{\alpha\gamma}\Theta_\beta \\
&- < \hat{C}_{\alpha\beta\gamma\delta}\partial_{\alpha\gamma} h^B_\beta h^L_\delta > \Phi^B_\beta - < A_{11}(\rho_f,\rho_c) h^L_\delta > \partial_\delta \ddot{W} \\
&- < A_{11}(\rho_f,\rho_c)\partial_\delta g^A h^L_\delta > \ddot{Q}^A + < A_{12}(\rho_f,\rho_c) h^B_\delta h^L_\delta > \ddot{\Phi}^B_\delta + < B_2 h^L_\delta > \Theta_\delta \\
&+ < B_2 h^B_\delta h^L_\delta > \Phi^B_\delta = 0 \\
&A, K = 1,2,...,N, \qquad B, L = 1,2,...,M
\end{aligned} \tag{8}$$

The system of equations (8) is a system of differential equations with constant coefficients, which describes the vibrations of the three-layered sandwich plate with periodic microstructure. However, unlike for example the asymptotic homogenization method, it still allows us to investigate the effect of the microstructure on its macroscopic behavior. System of equations (8) consists of $N + 2M + 3$ equations, depending on the number of assumed fluctuation-shape functions. It must be emphasized, that in equations $(8)_{2,4}$ there is no summation over δ.

System of equations (8) should be followed by four boundary conditions for deflection $W(\mathbf{x},t)$ and three boundary conditions for every each deflection $\Theta_1(\mathbf{x},t)$ and $\Theta_2(\mathbf{x},t)$. One can notice, that there is no need to formulate any boundary conditions for any fluctuation amplitude function $Q^A(\mathbf{x},t)$, ${\Phi_1}^B(\mathbf{x},t)$ and ${\Phi_2}^B(\mathbf{x},t)$. Additionally, two initial conditions should be given for every each of the unknown functions.

5. Calculation example – free vibration analysis

In this section free vibration analysis of plate strip with a certain periodic microstructure is presented and discussed. Let us consider a sandwich structure, with dimensions L_1 and L_2 along x_1- and x_2-axis, respectively, simply supported on edges $x_1 = 0$, $x_1 = L_1$, cf. Fig. 3. Material properties and detailed information about dimensions of the structure and the periodicity cell are presented below.

$$\begin{array}{lll} E_1 = 210\,GPa, & E_2 = 105\,GPa, & E_c = 5\,GPa, \\ \nu_1 = 0.3, & \nu_2 = 0.3, & \nu_c = 0.3, \\ G_1 = 80.8\,GPa, & G_2 = 40.4\,GPa, & G_c = 1.9\,GPa, \\ \rho_1 = 7850\,kg/m^3, & \rho_2 = 785\,kg/m^3, & \rho_c = 500\,kg/m^3, \\ L_1 = 1200\,mm, & L_2 = 100\,mm, & l = l_1 = 30\,mm, \\ h_f = 5\,mm, & h_c = 50\,mm. & \end{array} \tag{9}$$

The considered structure can be treated as a one dimensional issue. In such case, governing equations of the tolerance model can be rewritten into a simplified form, which depends on quantity and quality of assumed fluctuation shape functions. Let us formulate two different sets of assumptions.

- **Case I:**

  In this case, let us assume only one fluctuation shape function for displacement $w(\mathbf{x},t)$ and only one fluctuation shape function for displacement $\psi_1(\mathbf{x},t)$:

$$\begin{array}{ll} g \equiv g(y_1) = g^1(y_1) = l^4 \cos(2\pi y_1/l) + c_1, & Q = Q^1(x_1,t) \\ h \equiv h(y_1) = h_1^1(y_1) = l^3 \sin(2\pi y_1/l) + c_2, & \Phi = \Phi_1^1(x_1,t) \end{array} \tag{10}$$

  One can notice, that function g is an even function, while function h – an odd function. Constant c_1 and c_2 can be derived from normalizing conditions $< B_1 g >= 0$ and $< A_{12}(\rho_f,\rho_c)h >= 0$. As a result of all above assumptions, governing equations of the tolerance model of sandwich plate strip can be simplified into the form:

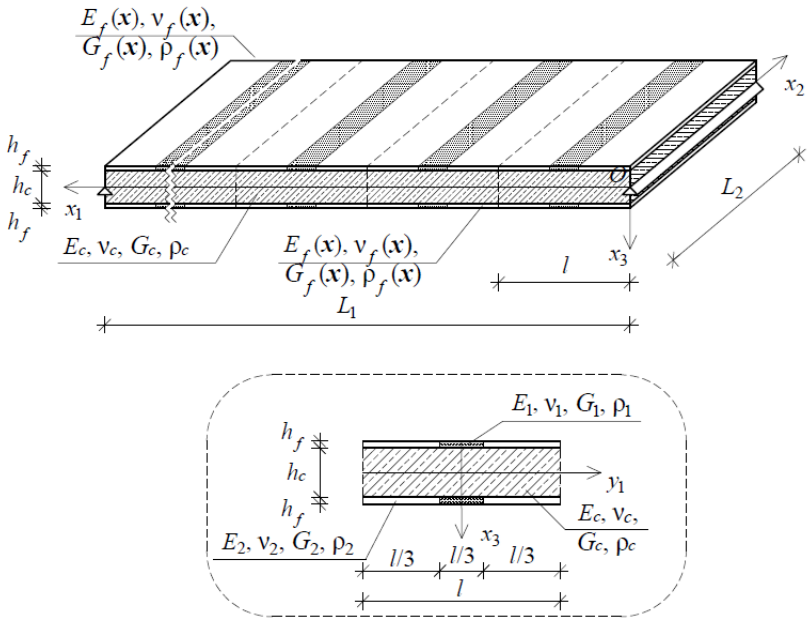

Figure 3 A sketch of considered sandwich plate strip with a details of a basic periodicity cell

$$
\begin{aligned}
& < A_{11}(E_f, E_c) > \partial_{1111} W + < A_{11}(E_f, E_c)\partial_{1111} g > Q \\
& - < A_{12}(E_f, E_c) > \partial_{111}\Theta - < A_{12}(E_f, E_c)\partial_{111} h > \Phi \\
& - < A_{11}(\rho_f, \rho_c) > \partial_{11}\ddot{W} - < A_{11}(\rho_f, \rho_c)\partial_{11} g > \ddot{Q} + < A_{12}(\rho_f, \rho_c) > \partial_1\ddot{\Theta} \\
& + < A_{12}(\rho_f, \rho_c)\partial_1 h > \ddot{\Phi} + < B_1 > \ddot{W} = 0 \\
& \\
& < A_{11}(E_f, E_c) > \partial_{111} W - < A_{12}(E_f, E_c) > \partial_{11}\Theta - < A_{11}(\rho_f, \rho_c) > \partial_1\ddot{W} \\
& \qquad + < A_{12}(\rho_f, \rho_c) > \ddot{\Theta} + < B_2 > \Theta = 0 \\
& \\
& < A_{11}(E_f, E_c) g > \partial_{1111} W + < A_{11}(E_f, E_c) g \partial_{1111} g > Q \\
& - < A_{12}(E_f, E_c) g > \partial_{111}\Theta - < A_{12}(E_f, E_c) g \partial_{111} h > \Phi \\
& - < A_{11}(\rho_f, \rho_c) g > \partial_{11}\ddot{W} - < A_{11}(\rho_f, \rho_c) g \partial_{11} g > \ddot{Q} \\
& + < A_{12}(\rho_f, \rho_c) g > \partial_1\ddot{\Theta} + < A_{12}(\rho_f, \rho_c) g \partial_1 h > \ddot{\Phi} + < B_1 g g > \ddot{Q} = 0 \\
& \\
& < A_{11}(E_f, E_c) h \partial_{111} g > Q - < A_{12}(E_f, E_c) h \partial_{11} h > \Phi \\
& - < A_{11}(\rho_f, \rho_c) h \partial_1 g > \ddot{Q} + < A_{12}(\rho_f, \rho_c) h h > \ddot{\Phi} + < B_2 h h > \Phi = 0
\end{aligned}
\tag{11}
$$

- **Case II:**

  In this case, let us assume that displacements of sandwich plate strip along x_3-axis directions are sufficiently well approximated only by macrodeflection $W(x_1,t)$, hence, there is no need to introduce any fluctuation shape function $g^A(y_1)$. As a result only one fluctuation shape function for displacement $\psi_1(\mathbf{x},t)$ is assumed in the form of an odd function:

$$h \equiv h(y_1) = h_1^1(y_1) = l^3 \sin(2\pi y_1/l) + c_2, \qquad \Phi = \Phi_1^1(x_1,t) \tag{12}$$

  where constant c_2 can be derived from normalizing condition:

$$< A_{12}(\rho_f,\rho_c)h >= 0$$

  As a result of such assumptions, governing equations of the tolerance model of sandwich plate strip can be simplified into the form:

$$\begin{aligned}
&< A_{11}(E_f,E_c) > \partial_{1111}W- < A_{12}(E_f,E_c) > \partial_{111}\Theta \\
&- < A_{12}(E_f,E_c)\partial_{111}h > \Phi- < A_{11}(\rho_f,\rho_c) > \partial_{11}\ddot{W} \\
&+ < A_{12}(\rho_f,\rho_c) > \partial_1\ddot{\Theta}+ < A_{12}(\rho_f,\rho_c)\partial_1 h > \ddot{\Phi}+ < B_1 > \ddot{W} = 0 \\
&< A_{11}(E_f,E_c) > \partial_{111}W- < A_{12}(E_f,E_c) > \partial_{11}\Theta \\
&- < A_{11}(\rho_f,\rho_c) > \partial_1\ddot{W}+ < A_{12}(\rho_f,\rho_c) > \ddot{\Theta}+ < B_2 > \Theta = 0 \\
&- < A_{12}(E_f,E_c)h\partial_{11}h > \Phi+ < A_{12}(\rho_f,\rho_c)hh > \ddot{\Phi}+ < B_2 hh > \Phi = 0
\end{aligned} \tag{13}$$

For both of the above cases solutions to systems of equations (11) and (13) are assumed in the forms, which satisfy boundary conditions:

$$\begin{aligned}
W(x_1,t) &= A_W \sin(n\pi x_1/L_1)\sin(\omega t) \\
Q(x_1,t) &= A_Q \sin(n\pi x_1/L_1)\sin(\omega t) \\
\Theta(x_1,t) &= A_\Theta \cos(n\pi x_1/L_1)\sin(\omega t) \\
\Phi(x_1,t) &= A_\Phi \sin(n\pi x_1/L_1)\sin(\omega t)
\end{aligned} \tag{14}$$

where A_W, A_Q, A_Θ, A_Φ are displacement amplitudes, n is a wave number and ω is a frequency of vibrations. By substituting solutions (14) into both system of equations (11) and (13) one can derive two systems of algebraic equations, which can be easily solved in order to find free vibration frequencies of the considered structure.

Apart from the tolerance model of sandwich plate, the FEM model of the considered structure was developed. The whole structure was modelled as 3D structure with the use of eight-node brick elements with reduced integration (C3D8R). Calculations were performed using ABAQUS calculation environment and can be treated as a benchmark for results obtained within the tolerance model.

The obtained free vibration frequencies of first 10 modes for all the considered models are presented in Table 1 and on Fig. 4.

By analyzing data in Table 1 and in Fig. 4. one can observe a significant difference in results derived from Case I and Case II. Free vibration frequencies derived from tolerance model are up to 9% higher, when micro-macro decomposition of vertical displacements is taken into consideration. Consequently, the differences between the results of the tolerance models and FEM model are also highly dependent on the assumed fluctuation shape functions. The relative error between TM – Case I and FEM can reach up to 13% for the lowest free vibration frequencies, while the relative error between TM – Case II and FEM do not exceed 3% (apart from the first mode of vibrations, where it is slightly higher than 4%). Hence, it can be stated, that choosing a proper set of fluctuation shape functions during a modeling process is a crucial part of modeling with the use of the tolerance averaging technique.

Table 1 Comparison of free vibration frequencies obtained within all considered models

Mode	Free vibration frequencies [Hz]		
	TM – Case I	TM – Case II	FEM Model
1	160,0	147,8	141,7
2	582,6	537,7	522,2
3	1158,0	1067,5	1051,2
4	1802,5	1659,9	1653,6
5	2471,9	2274,3	2285,3
6	3146,7	2893,1	2924,3
7	3818,8	3509,4	3560,9
8	4485,8	4121,0	4190,5
9	5147,0	4727,6	4811,4
10	5802,7	5329,7	5423,1

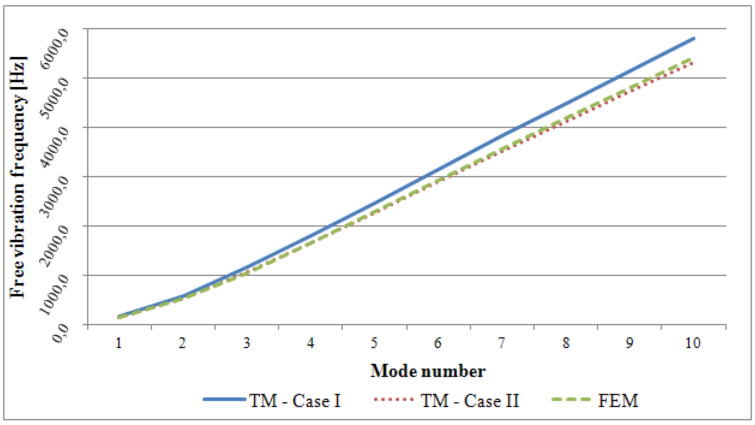

Figure 4 Comparison of free vibration frequencies obtained within all considered models

6. Conclusions

In this article the vibration analysis of periodic three-layered sandwich structure has been performed. Basing on the classic broken line hypothesis the initial governing equations of motion of considered structure were derived. The solution to such system of equations is difficult to obtain, as its coefficients are periodic, highly oscillating and non-continuous functions. The greatest finding of this article is the derivation of the tolerance model of the periodic sandwich structure, which consists of system of equations with constant coefficients. Such systems of governing equations are not only simple to solve but also allows us to investigate the microscale fluctuations connected with the periodic microstructure. Let us mention, that such analysis cannot be performed with the use of other techniques of analysis microperiodic structures, like for example asymptotic homogenization method.

By analyzing the results of free vibration frequencies derived from all presented models one can conclude, that the derived tolerance model of periodic sandwich plate can be perceived as a convenient tool for dynamic analysis of microheterogeneous structures. Unfortunately, the derived results are highly dependent on the correctness of arbitrarily assumed fluctuation shape functions, which stands for an unquestionable drawback of this model. One can observe, that, contrary to Case I, the results of Case II where the fluctuations of vertical displacements were omitted are much closer to benchmark results derived from FEM analysis. It is connected with the specific dimensions of the considered structure, where the microstructure parameter l is of an order of the thickness of the plate. In such case, in literature one can find an alternative tolerance modeling procedure, which leads to more universal form of averaged governing equations, cf. Baron [14,15]. However, by analyzing the results presented in this article, it can be stated that satisfactory results in such case can be also obtained by following a classic tolerance modeling procedure dedicated to structures, which thickness is much smaller than microstructure parameter l, by omitting certain fluctuation amplitudes, in our case - fluctuations of vertical displacements.

Moreover, one can observe, that the relative errors between the results of averaged models and FEM analysis are increasing for higher modes of vibrations. It can be caused by the fact, that for such modes of vibrations the assumption $W(\mathbf{x},t), \Theta_\alpha(\mathbf{x},t) \in SV_\delta^4(\Delta)$, $\alpha = 1{,}2$, becomes much more difficult to fulfill. Consequently, the tolerance parameter assumed for every each of calculation cases increases, while the accuracy of results decreases. Moreover, also the quality of chosen fluctuation shape functions has an influence on the obtained results. In order to avoid producing faulty results, one should consider a possible use of an exact fluctuation shape functions derived from an eigenvalue analysis of a single periodicity cell. Even if such derived function is not convenient in calculations, still, it can give a hint of how a properly assumed fluctuation shape function should look like.

It should be emphasized, that the derived model can be considered correct only for structures, which are symmetric to their midplane. In case of analysis of structure, which do not fulfill such condition, neither the presented model, nor classic broken line hypothesis is able to describe properly its dynamic behavior.

Moreover, in modern engineering one can find many examples of sandwich structures made of materials, which cannot be considered isotropic. As a result, several further adjustments should be made in order to obtain a complex model of considered plates.

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