

Transport of Temperature Fluctuations Across a Two-Phased Laminate Conductor

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In the periodic composite materials temperature or displacement fluctuations suppressed in directions perpendicular to the periodicity surfaces should expect a damping reaction from the composite. This phenomenon, known as the *boundary effect behavior* has been investigated only in the framework of approximated models. In this paper extended tolerance model of heat transfer in periodic composites is used as a tool allows analytical investigations of highly oscillating boundary thermal loadings. It has been shown that mentioned reaction is dual – different for even and odd fluctuations.

Keywords: even temperature fluctuations, two-phased laminated conductor, boundary value problem.

1. Introduction

The existence of a series of open questions about using the periodic layer illustrated in Fig. 1. as a building barrier is a crucial motivation for the topic. The investigation some of these questions is consider as the *aim of the paper*. We assume that the temperature in the region occupied by these septum is caused by two fixed temperatures: one in the interior and the other inside the room surrounded by that dividing wall. The layer is thoughtfully cut out from the front wall as shown in Fig. 2. and this wall is large enough to be assumed that the temperature in this bulkhead does not change in the x^2 direction. Hence the layer can be treated as a fragment of the another laminated layer infinite along the x^2 - direction axis. For the layer debarked from this wall a new coordinate system will be introduced.

Hence, the first type boundary value problem for two-phase and infinite along the x^2 axis direction laminated layer, cf. Fig. 3. will be concerned as a subject of considerations. The x^1 -variable, corresponding to the periodicity direction, will be

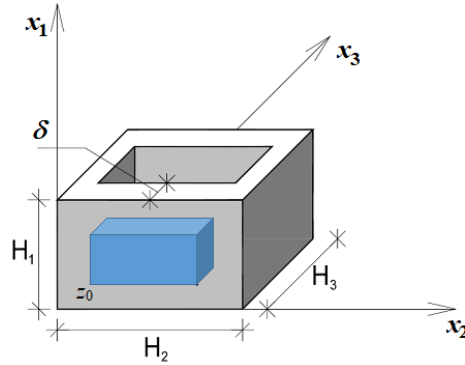


Figure 1 The considered room and the way leading to the debarking of the layer from a dividing wall

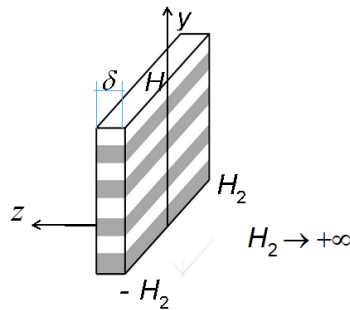


Figure 2 Considered two-phased periodically layered composite

replaced with the y letter and x^3 -variable, perpendicular to the periodicity direction, will be replaced by the z letter in new coordinate system:

$$(x^1, x^3) = (x_0^1, x_0^3) + (y, z) \tag{1}$$

In the formula (1), the variable x^2 was dropped due the assumption that was made and the sign $x_0 \equiv (x_0^1, x_0^2, x_0^3)$ for the coordinates of a fixed point x_0 of the front wall was assumed. Figure 1. illustrates how the layer presented in Figure 2. is mentally cut from the dividing wall 1.

The subject of considerations is a stationary boundary value problem for the parabolic heat conductivity equation. Thus the temperature field θ is not a function of time. It should be a function of the variables y and z , $\theta = \theta(y, z)$, and at the same time it is known on the boundary $\partial\Omega$ of the region Ω occupied by the conductor:

$$\theta(y, z)|_{(y,z) \in \partial\Omega} = \theta_{\partial} \tag{2}$$

We shall assume that the region Ω in which the temperature field is defined has the form of rectangle:

$$\Omega = \Xi \times \Phi \tag{3}$$

wherein $y \in \Xi$ and $z \in \Phi$. The parabolic heat conductivity equation for isotropic constituents and for which the boundary problem has been formulated will be rewritten in the form:

$$\rho c \dot{\theta} - \nabla^T(k \nabla \theta) = -b \tag{4}$$

where:

$$\nabla \equiv [\partial_1, \partial_2, \partial_3]^T \tag{5}$$

and the symbols k, ρ , and c , are: the heat conductivity field, the mass density field and the specific heat field of the considered layer. These fields take constant values in homogeneity parts of the layer. Symbol b stands for the heat sources. The heat flux component normal to the planes separating homogeneity regions will be assumed to be continuous.

2. Thermal boundary loads

Impulses which will be investigated in the framework of the scope of the paper as loading perturbations imposed on the average temperature field are λ - periodic under the repetitive layer with the thickness equal to λ . These impulses are divided onto packages indicated by the positive integer v . Every package consists of three impulses defined below by formulas (7). Hence the choice of the repetitive cell plays here important role and will be used as the parameter controlling considerations. Figure 3 illustrates the construction of the first package of impulses. Impulses with support included in the one component regions are referred to as even impulses. Even fluctuations are considered as linear combinations of even impulses. In accordance of the Fourier series theory we shall support these impulses to the orthogonalization procedure.

In the subsequent considerations saturation η_I of the first constituent will be treated as a certain parameter. We have $\eta = \eta_I = l^I/\lambda$ where l^I is referred to the thickness of the first lamina and hence the second constituent saturation $\eta_{II} = 1 - \eta = l^{II}/\lambda$ for l^{II} taken as the thickness of the second lamina. That is why:

$$\eta_I = \frac{l^I}{l^I + l^{II}}, \quad \eta_{II} = \frac{l^{II}}{l^I + l^{II}} \tag{6}$$

Formulas:

$$\begin{aligned}
 f_L(v; y) &= \begin{cases} \frac{\lambda}{2} \{1 - \alpha_1 [1 + \cos 2\pi v (\frac{y}{\lambda \eta^I} + 1)]\} & \text{for } -\lambda \eta^I \leq y \leq 0 \\ \frac{\lambda}{2} \{1 - \alpha_1 [1 + \cos 2\pi v (\frac{y}{\lambda \eta^{II}} + 1)]\} & \text{for } 0 \leq y \leq \lambda \eta^{II}, \bar{y} = 0 \end{cases} \\
 f_P(v; y) &= \begin{cases} \frac{\lambda}{2} \{1 - \alpha_2 [1 + \cos 2\pi v (\frac{y}{\lambda \eta^{II}} - 1)]\} & \text{for } -\lambda \eta^I \leq y \leq 0, \bar{y} = 0 \\ \frac{\lambda}{2} \{1 - \alpha_2 [1 + \cos 2\pi v (\frac{y}{\lambda \eta^{II}} - 1)]\} & \text{for } 0 \leq y \leq \lambda \eta^{II} \end{cases} \quad (7) \\
 f_{NP}(v; y) &= \begin{cases} -\frac{\lambda}{2} \cos(2v-1)\pi(\frac{y}{l^I} + 1) & \text{for } -l^I \leq y \leq 0 \\ -\frac{\lambda}{2} \cos(2v-1)\pi(\frac{y}{l^{II}} - 1) & \text{for } 0 \leq y \leq l^{II} \end{cases}
 \end{aligned}$$

define mentioned impulses of the v -th package. For any positive integer v impulses $f_L(v; y)$, $f_P(v; y)$, $f_{NP}(v; y)$ will be referred to as v -th left even, v -th right even and v -th odd fluctuations, respectively. Coefficients:

$$\alpha_L = \frac{1}{\eta_I + 2\eta_{II}}, \quad \alpha_R = \frac{1}{2\eta_I + \eta_{II}} \quad (8)$$

provide the fulfillment of oscillation conditions $\langle f_1 \rangle = \langle f_2 \rangle = 0$. Used above and subsequently *averaging operation* is a typical integral averaging formula defined by:

$$\langle f \rangle = \frac{1}{l^I + l^{II}} \int_{-l^I}^{l^{II}} f(y) dy \quad (9)$$

Averaged value $\langle f \rangle$ of f is constant provided that f is λ -periodic function.

Orthogonalization procedure is understood here as a procedure assigning to impulses (7) the sequence of λ -periodic fluctuations forming the orthogonal Fourier basis with respect to the scalar product:

$$\varphi_1 \circ \varphi_2 = \langle \varphi_1 \varphi_2 \rangle \quad (10)$$

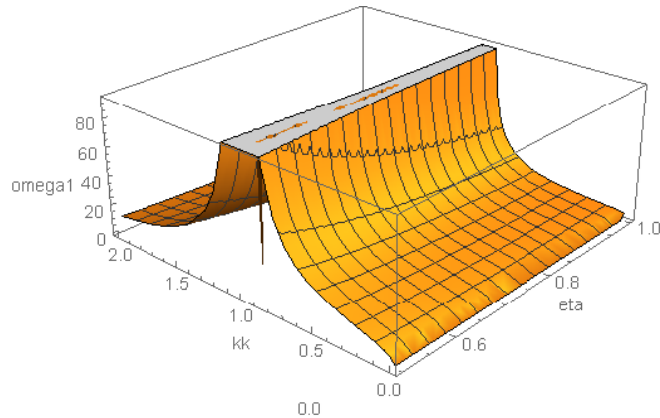


Figure 3 The first package of three one directional temperature impulses for a two-phase laminate

The orthogonal procedure used in this paper coincides with that used in [8,9,10] and hence the orthogonal even fluctuations ("left" and "right") are defined by the formulas:

$$\varphi_L(y) = \frac{f_L(v; y) + \alpha f_P(v; y)}{\lambda}, \quad \varphi_R(y) = \frac{f_L(v; y) - \alpha f_P(v; y)}{\lambda} \tag{11}$$

Parameter α is defined for the pair of λ -periodic fluctuations φ_1 and φ_2 by the orthogonalization condition $\langle \varphi_1 k \varphi_2 \rangle = 0$:

$$\alpha = \frac{\langle f_L(v; y) k f_L(v; y) \rangle}{\langle f_L(v; y) k f_P(v; y) \rangle} \tag{12}$$

For different parameters v_1 and v_2 all pairs made up of left even $\varphi_L(v_1; y)$, $\varphi_L(v_2; y)$, and right even fluctuations $\varphi_P(v_1; y)$, $\varphi_P(v_2; y)$ are orthogonal. Similarly, pairs made up of even $\varphi_L(v_1; y)$ and odd fluctuation $\varphi_{NP}(v_2; y)$ are orthogonal for different parameters and . For any parameters and , not necessary different, pairs made up of two odd fluctuations $\varphi_{NP}(v_1; y)$, $\varphi_{NP}(v_2; y)$ are orthogonal.

3. Model equations

In order to answer the question, how the impulses (7) are transported across the considered laminated layer, we will use the *Extended Tolerance Model* of heat conduction, cf. [8 [8,9,10] as a tool for finding solutions of the parabolic heat transfer equation, which in the regions of the composite homogeneity can be represented by the Fourier expansion:

$$\theta(y, z) = u(z) + \lambda a_p(z) \varphi^p(y) \quad \text{in } \Omega \setminus \Gamma \tag{13}$$

with respect to a certain orthogonal basis $\varphi^p(y)$ of λ -periodic temperature fluctuations.

Index p runs over finite or infinite subset of positive integers in according to the class of fluctuations we are to investigated and which are formed a corresponding Fourier basis. Coefficients $a_p(z)$ used in (13), usually referred to as *Fourier Amplitudes*, not used in the analysis, are assumed to be equal to zero. Here and subsequently summation convention with respect to any repetitive indices holds.

The coefficients $a_p(z)$ of expansion (13) and the average temperature should satisfy the infinite system of equations:

$$\begin{aligned} \nabla_{\Phi}^T [\langle k \rangle \nabla_{\Phi} u - \langle k \nabla g^A \rangle \psi_A + \langle k \nabla_{\Xi} \varphi^p \rangle a_p] &= \langle b \rangle \\ \lambda^2 \nabla_{\Phi}^T \langle k \varphi^p \varphi^q \rangle \nabla_{\Phi} a_q + \lambda (\langle k \varphi^q \nabla_{\Xi}^T \varphi^p \rangle - \langle k \varphi^p \nabla_{\Xi}^T \varphi^q \rangle) \nabla_{\Phi} a_q & \\ + \langle k \nabla_{\Xi}^T \varphi^p \nabla_{\Xi} \varphi^q \rangle a_q + \langle k \nabla_{\Xi}^T \varphi^p \rangle \nabla u &= \lambda \langle \varphi^p b \rangle \end{aligned} \tag{14}$$

together with an additional equation:

$$H^{AB} \psi_A + \langle k \nabla_{\Xi}^T g^A \rangle_{\Gamma} \nabla u = 0 \tag{15}$$

allowing to determine amplitudes ψ_A , $A = 1, \dots, S$, where S is the number of components of the periodic composite, related to the fluctuations g^A , $A = 1, \dots, S$, called as tolerance fluctuations describing behaviors strictly related to the surfaces separating components.

Equations (14) and (15), are special case of model equations, used for isotropic conductors, derived in [8], [9] and [10] from the heat transfer equation under the technique based on the *Tolerance Averaging Technique* approach, cf. [3-7]. In the second from equations (14), coefficients in terms including *Fourier Amplitudes* a_p are the square 3×3 matrices. These coefficients are linear combinations of the conductivity matrices K_I and K_{II} of the composite components. Equations (14) and (15) can be considered as an alternative to the other averaged models of heat conduction, cf. [1], [2].

Note that for even fluctuations φ^p :

$$\langle \nabla_{\Xi}^T \varphi^p K \rangle = 0 \quad (16)$$

while:

$$\langle \nabla_{\Xi}^T \varphi^p K \rangle \neq 0 \quad (17)$$

for the odd ones. It means that equations (14) for even fluctuations amplitudes do not depend on the average temperature.

In the paper we are to investigate the transport of single pairs of fluctuations across two-phased periodic composite in two cases: 1^o for the pair formed by two even fluctuations and 2^o for the pair formed by even fluctuation and odd fluctuation. In both cases, we assume the vanishing heat source, $b = 0$. Symbol I denotes unit matrix.

Ad 1^o. For any even fluctuation, condition (16) is satisfied and hence equations (14) separate. For two even fluctuations $\langle \nabla_{\Xi}^T \varphi^2 K \varphi^1 \rangle - \langle \nabla_{\Xi}^T \varphi^1 K \varphi^2 \rangle = 0$ and thus a single equation for the average temperature u should be taken into account:

$$\nabla_{\Phi}^T [\langle K \rangle \nabla_{\Phi} u + \langle K \nabla g^A \rangle \psi_A] = 0 \quad (18)$$

Thus, infinite system of equations (14) reduces to the pair of equations for even amplitudes:

$$\begin{aligned} \lambda^2 \nabla_{\Phi}^T \langle \varphi^1 K \varphi^1 \rangle \nabla_{\Phi} a_1 - \langle \nabla_{\Xi}^T \varphi^1 K \nabla_{\Xi} \varphi^1 \rangle a_1 &= 0 \\ \lambda^2 \nabla_{\Phi}^T \langle \varphi^2 K \varphi^2 \rangle \nabla_{\Phi} a_2 - \langle \nabla_{\Xi}^T \varphi^2 K \nabla_{\Xi} \varphi^2 \rangle a_2 &= 0 \end{aligned} \quad (19)$$

describing their transport across the considered layer irrespectively of all other fields. This homogeneous system of ordinary differential equations can be written in the form:

$$\lambda^2 A_K \frac{d^2 a}{dz^2} - 2\lambda S_K \frac{da}{dz} - \frac{\langle k \rangle_H}{\langle k \rangle} C_K a = 0 \quad (20)$$

for:

$$A_K = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \quad C_K = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \quad (21)$$

for vanishing damping matrix coefficient:

$$S_K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (22)$$

and for the amplitude vector $a \equiv [a_1, a_2]^T$. Constants, introduced in (21) are given by formulas:

$$\gamma_1 = \frac{\langle k \varphi_1 \varphi_1 \rangle}{\langle k \rangle}, \quad \gamma_2 = \frac{\langle k \varphi_2 \varphi_2 \rangle}{\langle k \rangle}, \quad c_1 = \frac{\langle k \left(\frac{d}{dy} \varphi_1 \right)^2 \rangle}{\langle k \rangle}, \quad c_2 = \frac{\langle k \left(\frac{d}{dy} \varphi_2 \right)^2 \rangle}{\langle k \rangle} \quad (23)$$

and

$$d = \frac{\langle k\varphi_1 \frac{d}{dy} \varphi_2 \rangle - \langle k\varphi_2 \frac{d}{dy} \varphi_1 \rangle}{\langle k \rangle} \tag{24}$$

We are to examine how a certain impulse is transported across the considered composite. This impulse is a linear combination:

$$\xi_1\varphi_1(y) + \xi_2\varphi_2(y) \tag{25}$$

with real coefficients ξ_1, ξ_2 and for even fluctuations $\varphi_1 = \varphi_L(v_1, y), \varphi_2 = \varphi_P(v_2, y)$ and arbitrary positive integers v_1, v_2 . In the considered case the temperature field is represented by the reduced form of Fourier expansion (13):

$$\theta(y, z) = u(z) + a_1(z)\varphi_L(v_1, y) + a_2(z)\varphi_P(v_2, y) \tag{26}$$

Equations (20) consist of two independent second order differential equations with constant coefficients.

Ad 2°. Although for an even fluctuation the condition (16) is satisfied, model equations (14) in the Case 2° are not separated. Thereby the single equation applies for the average temperature u :

$$\nabla_{\Phi}^T[\langle K \rangle \nabla_{\Phi} u + \langle K \nabla g^A \rangle \psi_A] = 0 \tag{27}$$

For $A = 1, 2$ together with Equation (15) and the infinite system of equations (14) reducing in the considered case to the equations for the pair of even $\varphi^1 = \varphi_L(v_1, y)$ and odd $\varphi^2 = \varphi_{NP}(v_2, y)$ fluctuations:

$$\begin{aligned} \lambda^2 \nabla_{\Phi}^T \langle k(\varphi^1)^2 \rangle \nabla_{\Phi} a_1 - \lambda \langle \langle k\varphi^1 \nabla_{\Xi}^T \varphi^2 \rangle - \langle k\varphi^2 \nabla_{\Xi}^T \varphi^1 \rangle \rangle \nabla_{\Phi} a_2 - \langle k \nabla_{\Xi}^T \varphi^1 \nabla_{\Xi} \varphi^1 \rangle a_1 = 0 \\ \lambda^2 \nabla_{\Phi}^T \langle k(\varphi^2)^2 \rangle \nabla_{\Phi} a_2 - \langle k \nabla_{\Xi}^T \varphi^2 \nabla_{\Xi} \varphi^2 \rangle a_2 \\ + \lambda \langle \langle k\varphi^1 \nabla_{\Xi}^T \varphi^2 \rangle - \langle k\varphi^2 \nabla_{\Xi}^T \varphi^1 \rangle \rangle \nabla_{\Phi} a_1 + \langle k \nabla_{\Xi}^T \varphi^2 \rangle \nabla u = 0 \end{aligned} \tag{28}$$

with amplitudes a_1 and a_2 as basic unknowns. Equations (28), written in the absence of heat sources, reduce to the non-homogeneous system of second order ordinary differential equations:

$$\lambda^2 A_K \frac{d^2 a}{dz^2} - 2\lambda S_K \frac{da}{dz} - \frac{\langle k \rangle_H}{\langle k \rangle} C_K a = - \left[\begin{array}{c} 0 \\ \langle \nabla_{\Xi}^T \varphi^1 K \rangle \end{array} \right] \frac{du}{dz} \tag{29}$$

for:

$$\langle k \rangle = \eta_I k^I + \eta_{II} k^{II}, \quad \langle k \rangle_H = \frac{k^I k^{II}}{\eta_I k^{II} + \eta_{II} k^I} \tag{30}$$

In (29) coefficients A_K and C_K are diagonal 2×2 square matrices:

$$A_K = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \quad C_K = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \tag{31}$$

and the S factor is an antisymmetric square matrix:

$$S_K = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix} \tag{32}$$

and the amplitude vector $a \equiv [a_1, a_2]^T$. Constants $\gamma_1, \gamma_2, c_1, c_2, d$ introduced in (31) and (32) are given by the formulas:

$$\gamma_1 = \frac{\langle k\varphi_1\varphi_1 \rangle}{\langle k \rangle}, \quad \gamma_2 = \frac{\langle k\varphi_2\varphi_2 \rangle}{\langle k \rangle} \quad (33)$$

$$c_1 = \frac{\langle k \left(\frac{d}{dy}\varphi_1 \right)^2 \rangle}{\langle k \rangle_H}, \quad c_2 = \frac{\langle k \left(\frac{d}{dy}\varphi_2 \right)^2 \rangle}{\langle k \rangle_H} \quad (34)$$

$$d = \frac{\langle k\varphi_1 \frac{d}{dy}\varphi_2 \rangle - \langle k\varphi_2 \frac{d}{dy}\varphi_1 \rangle}{\langle k \rangle} \quad (35)$$

for $\varphi_1 = \varphi_L(v_1, y)$, $\varphi_2 = \varphi_{NP}(v_2, y)$ and given pair of integer positive indicators v_1, v_2 . We are to examine how the impulse being a linear combination:

$$M_1\varphi_1(y) + M_2\varphi_2(y) \quad (36)$$

with real coefficients M_1, M_2 is transported in the considered composite. Indicators v_1 and v_2 may be equal here. In this case, the temperature field is represented by a two-element Fourier expansion (13):

$$\theta(y, z) = u(z) + a_1(z)\varphi_L(v_1, y) + a_2(z)\varphi_2(v_1, y) \quad (37)$$

for $\varphi_1 \equiv \varphi_L(v_1, y)$ and $\varphi_2 \equiv \varphi_{NP}(v_2, y)$. Parameters v_1 and v_2 need not be different here.

The amplitude vector $a \equiv [a_1, a_2]^T$, as a solution of the non-homogeneous system of equations (29), is presented as the sum of the amplitude vector $b \equiv [b_1, b_2]^T$, which satisfies the system of equations:

$$\lambda^2 A_K \frac{d^2 a}{dz^2} - 2\lambda S_K \frac{da}{dz} - \frac{\langle k \rangle_H}{\langle k \rangle} C_K a = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (38)$$

homogeneous for (29) and the special solution $\tilde{a} \equiv [\tilde{a}_1, \tilde{a}_2]^T$ to the inhomogeneous system (29). Thereby:

$$a = b + \tilde{a} \quad (39)$$

For the considered special case of two-phase laminated layer and in the absence of heat sources, the boundary conditions are reduced to the boundary conditions for two amplitudes a_1 and a_2 :

$$\begin{aligned} a(0) &= a^0 \equiv [a_1(0), a_2(0)]^T \\ a(\delta) &= a^\delta \equiv [a_1(\delta), a_2(\delta)]^T \end{aligned} \quad (40)$$

In the next section we are to discuss the solutions to mentioned boundary problem in both *Case 1^o* and *Case 2^o*.

4. Solution to BVP

Case 1. In this case solution to the formulated boundary value problem can be written as:

$$\begin{aligned} a_1 &= -\frac{\sinh \omega_{exp}(\varphi^1) \frac{z-\delta}{\lambda}}{\sinh \omega_{exp}(\varphi^1) \frac{\delta}{\lambda}} a_1(0) + \frac{\sinh \omega_{exp}(\varphi^1) \frac{z}{\lambda}}{\sinh \omega_{exp}(\varphi^1) \frac{\delta}{\lambda}} a_1(\delta) \\ a_2 &= -\frac{\sinh \omega_{exp}(\varphi^2) \frac{z-\delta}{\lambda}}{\sinh \omega_{exp}(\varphi^2) \frac{\delta}{\lambda}} a_2(0) + \frac{\sinh \omega_{exp}(\varphi^2) \frac{z}{\lambda}}{\sinh \omega_{exp}(\varphi^2) \frac{\delta}{\lambda}} a_2(\delta) \end{aligned} \tag{41}$$

for exponential attenuation:

$$\omega_{exp}(\varphi^1) = \sqrt{\frac{\langle k \rangle_H c_1}{\langle k \rangle \gamma_1}} \tag{42}$$

identical for both even fluctuations $\varphi^1 = \varphi_L$ and $\varphi^2 = \varphi_P$.

Case 2. The boundary problem (40) for the system of two equations (29) is reduced to the pair of two identical fourth order ordinary equations with even derivatives and with constant coefficients. The fourth order differential equation:

$$\frac{d^4}{dz^4} \chi + \left[\frac{4d^2}{\gamma_1 \gamma_2} - \frac{\langle k \rangle_H}{\langle k \rangle} \left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) \right] \frac{d^2}{dz^2} \chi + \frac{\langle k \rangle_H^2}{\langle k \rangle^2} \frac{c_1 c_2}{\gamma_1 \gamma_2} \chi = 0 \tag{43}$$

is satisfied as well for $\chi(z) = a_1(\lambda z)$ as for $\chi(z) = a_2(\lambda z)$. Here (c_1, c_2) , (γ_1, γ_2) and $(-d^2, -d^2)$ are diagonal elements in quadratic matrices A_K , C_K and S^2 , respectively. For equation (43) we assign:

(i) biquadratic characteristic equation:

$$R^4 - \left[\frac{\langle k \rangle_H}{\langle k \rangle} \left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) - \frac{4d^2}{\gamma_1 \gamma_2} \right] R^2 + \frac{\langle k \rangle_H^2}{\langle k \rangle^2} \frac{c_1 c_2}{\gamma_1 \gamma_2} = 0 \tag{44}$$

(ii) quadratic equation (obtained from (44) for $r = R^2$):

$$r^2 - \left[\frac{\langle k \rangle_H}{\langle k \rangle} \left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) - \frac{4d^2}{\gamma_1 \gamma_2} \right] r + \frac{\langle k \rangle_H^2}{\langle k \rangle^2} \frac{c_1 c_2}{\gamma_1 \gamma_2} = 0, \quad r = R^2 \tag{45}$$

(iii) discriminant of a quadratic characteristic equation:

$$\Delta \equiv \left[\frac{\langle k \rangle_H}{\langle k \rangle} \left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) - \frac{4d^2}{\gamma_1 \gamma_2} \right]^2 - 4 \frac{\langle k \rangle_H^2}{\langle k \rangle^2} \frac{c_1 c_2}{\gamma_1 \gamma_2} \tag{46}$$

(iv) roots r_1, r_2, r_3, r_4 of the quadratic equation (ii):

$$r = R e r_i + j I m r_i, \quad j^2 = -1, \quad i = 1, 2, 3, 4 \tag{47}$$

The characteristic equation (i) has four complex roots (47). The minimum absolute values of the real part and the minimum absolute values of the imaginary part of these roots are referred to as *the exponential (damping) intensity* (ω_{exp}) and *the rotational intensity* (ω_{rot}) of the amplitudes a_1 and a_2 respectively. The exponential and rotational intensity are valued as:

$$\omega_{exp} = \sqrt{\frac{1}{2} \left\{ \frac{\langle k \rangle_H}{\langle k \rangle} \left[\left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) - \frac{4d^2}{\gamma_1 \gamma_2} \right] - \sqrt{\Delta} \right\}} \quad \text{and} \quad \omega_{rot} = 0 \quad \text{for} \quad \Delta \geq 0 \tag{48}$$

and to:

$$\begin{aligned} \omega_{exp} &= \frac{1}{2} \sqrt{\frac{\langle k \rangle_H}{\langle k \rangle} \left[\left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) - \frac{4d^2}{\gamma_1 \gamma_2} \right] + \sqrt{\frac{\langle k \rangle_H^2}{\langle k \rangle^2} \left[\left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) - \frac{4d^2}{\gamma_1 \gamma_2} \right]^2 - \Delta}} \\ \omega_{rot} &= \sqrt{\frac{-\Delta}{\frac{\langle k \rangle_H}{\langle k \rangle} \left[\left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) - \frac{4d^2}{\gamma_1 \gamma_2} \right] + \sqrt{\frac{\langle k \rangle_H^2}{\langle k \rangle^2} \left[\left(\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} \right) - \frac{4d^2}{\gamma_1 \gamma_2} \right]^2 - \Delta}}} \end{aligned} \quad (49)$$

for $\Delta \leq 0$

Open analytical formulas for amplitudes a_1 and a_2 were obtained in [8]:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = e^{\omega_{exp} \frac{z}{\lambda}} Trig_+ \left(\frac{z}{\lambda} \right) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + e^{-\omega_{exp} \frac{z}{\lambda}} Trig_- \left(\frac{z}{\lambda} \right) \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \quad (50)$$

where:

$$\begin{aligned} Trig_+ \left(\frac{z}{\lambda} \right) &= A_{++} \cos \omega_{rot} \frac{z}{\lambda} + A_{+-} \sin \omega_{rot} \frac{z}{\lambda}, \\ Trig_- \left(\frac{z}{\lambda} \right) &= A_{-+} \cos \omega_{rot} \frac{z}{\lambda} + A_{--} \sin \omega_{rot} \frac{z}{\lambda} \end{aligned} \quad (51)$$

Columns $[C_1, C_2]^T, [D_1, D_2]^T$ depend on the boundary values of the amplitudes whereas $A_{++}, A_{+-}, A_{-+}, A_{--}$ are known 2×2 square matrices, cf. [8]. These matrices depend on the conductive and geometrical properties of the laminate. The solution (50) is a certain extended version of the solutions obtained in [11] for the approximate tolerance model of heat conduction.

5. Numerical results

We are to construct graphs illustrating amplitudes a_1 and a_2 as a function of two variables. The first variable is the quotient of the conductivity k^{II}/k^I . The second variable is the saturation (volume fraction) of the first component. In the numerical part the following denotations are used: *conductivity factor* $k = k^{II}/k^I$, volume share of the second component (*saturation factor*) $\eta = \eta_I$, damping intensity: ω_{exp} , rotational intensity ω_{rot} .

The graphs of *damping intensity* ω_{exp} defined in (42), according to the parameters $\eta = \eta_I$ and $k = k_{II}/k_I$ for the fixed values $v = v_1 = v_2$ for the two even fluctuations $\varphi_1 = \varphi_L$ and $\varphi_1 = \varphi_P$ in Figures 4,5,6 are presented. In Figure 4. the ranges $0.5 \leq \eta \leq 1$ and $0 \leq k \leq 2$ was deliberately selected to expose the spine of the graph in the vicinity of $k = 1$. The dependence of the exponential intensity ω_{exp} defined in (48) and (49) on the parameters $\eta = \eta_I$ and $k = k_{II}/k_I$ for the fixed values $v = v_1 = v_2$ for the two even fluctuations (φ_1, b_1) and (φ_2, b_2) while $\varphi_2 = \varphi_L$ and $\varphi_2 = \varphi_{NP}$ is a typical dependence formed by the exponential relation, root relation and the monotonic rational relations, and that is why the corresponding graphs will not be included here. Graphs in Fig. 4. and Fig. 5. show that the plane $k = 1$ (non-distinguishable components of the two-phased laminate) separates two parts of the graph of exponential intensity. These values asymptotically tend to infinity, if the arguments v and η determine point identified by (v, η) placed close to the straight line $k = 1$ in the (v, η) -plane. Together with the changing of thermal properties of the layer (k increasing to infinity or decreasing to zero) the exponential intensity decreases to zero. At the same time also saturation η tends to the number zero or to the number one (corresponding to very thin sheet occupied by one of the components) and occurs together with the exponential intensity decreasing to zero.

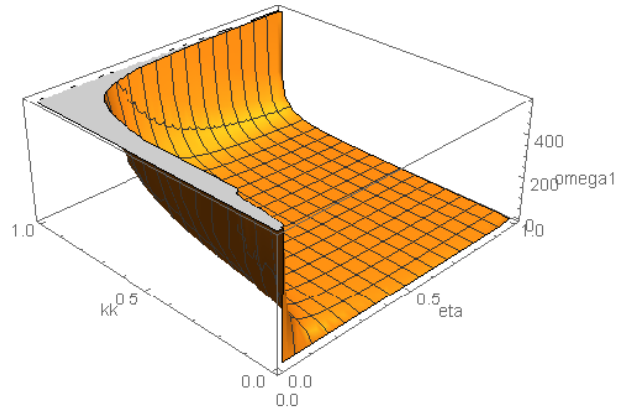


Figure 4 Graph of ω_{exp} for $0 \leq \eta \leq 1$ and $0 \leq k \leq 2$

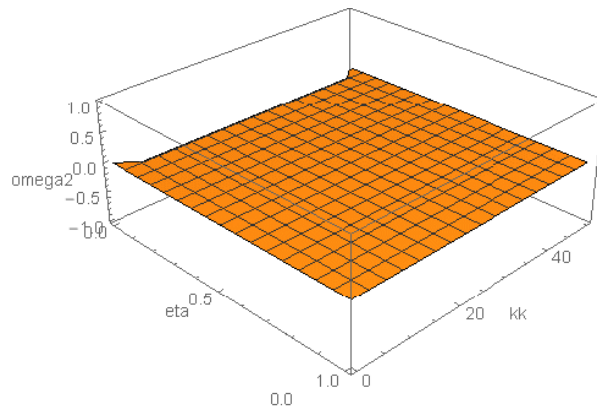


Figure 5 Graph of ω_{exp} for $0 \leq \eta \leq 1$ and $0 \leq k \leq 1$

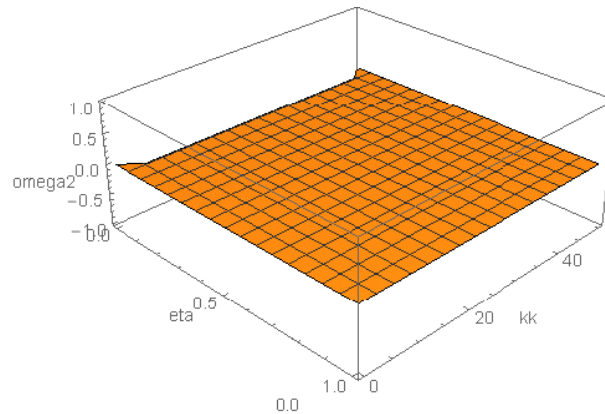


Figure 6 Graph of ω_{exp} for $0 \leq \eta \leq 1$ and $0 \leq k \leq 50$

We have not been able to catch the line at which the exponential intensity takes a local maxima which corresponds to relative large values. The graph in Fig.6 shows the dependence *the rotational intensity* on mentioned parameters.

6. Final remarks

In agreement to the aim of the study the transport of even fluctuations, do not affecting the form of the average temperature and hence do not modifying the effective conductivity matrix were illustrated. Since during transport these fluctuations are coupled (cooperate with each other), the paper is limited to the analysis of transport of pairs of even fluctuations, which in fact can be identified as pairs (φ_1, b_1) and (φ_2, b_2) . Even fluctuations are twofold. The first type can be considered as (φ_L, a_L) or (φ_P, a_P) , where a_L is the amplitude of fluctuation φ_L and a_p is the amplitude of fluctuations φ_P . The second type of even fluctuations are considered as pairs (φ_p, a_p) , $p = 1, 2, \dots$, satisfying (38) for $\varphi_p = \varphi_{NP}(v, y)$ and for $v = 1, 2, \dots$. The first temperature impulses vanish on the surfaces separating the composite phases, and the latter are the impulses of the whole repetitive cell. They seem to be the most representative to the paper considerations. Due to mathematical intricacies, considerations were limited to the analysis of the exponential (damping) intensity ω_{exp} and rotational intensity ω_{rot} of such pairs of fluctuations. These parameters were defined as in the Dissertation [8].

Graphs of these intensities were made as the functions of the two saturation parameters of the first component η and the conductivity quotient $k = k_{II}/k_I$ of the isotropic components of two-phased layered composite.

Mentioned intensities increases as the thermal properties of both components of the composite are similar. After a deeper reflection, this result, seemingly surprising, turns out to be very natural.

Considerations use a model that takes into account the scale effect. Hence obtained solutions are thus controlled by the size λ of the repeatable layer and hence there are not required restrictions on the thickness of this layer. Such constraints are

necessary only when the elimination of Fourier amplitudes is realized. Such eliminations usually lead to the creation of a single equation for the average temperature and thus to the formulation the effective conductivity constant.

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