Mechanics and Mechanical Engineering
Vol. 22, No. 4 (2018) 871–874

https://doi.org/10.2478/mme-2018-0067

A Second Order Secular J–S Planetary Theory
Part I : Lemma

Osman M. Kamel
Astronomy and Space Sciences Department, Faculty of Sciences, Cairo University, Giza, Egypt
kamel_osman@yahoo.com

Adel S. Soliman
Theoretical Physics Department, National Research Centre, Dokki, Giza, Egypt
adelssoliman7@yahoo.com

Received (3 October 2016)
Revised (9 March 2017)
Accepted (16 August 2018)

A concise lemma is given for the construction of a semi – analytic Hamiltonian second order secular J–S planetary theory using the Jacobi – Radau system of origins and in terms of the non-singular variables of H. Poincaré. We truncate our expansions at the desired power in the eccentricities and the sines of the inclinations.

Keywords: dynamics of the solar system, Hamiltonian planetary theory, celestial mechanics.

1. Methods and Results

The second order secular Hamiltonian equations of motion for the Jupiter – Saturn subsystem is given by:

$$\begin{aligned} \frac{dL_u}{dt} = \frac{\partial F_s}{\partial \lambda_u} \frac{d\lambda_u}{dt} = -\frac{\partial F_s}{\partial L_u} \\ \frac{dH_u}{dt} = \frac{\partial F_s}{\partial K_u} \frac{dK_u}{dt} = -\frac{\partial F_s}{\partial H_u} \\ \frac{dP_u}{dt} = \frac{\partial F_s}{\partial Q_u} \frac{dQ_u}{dt} = -\frac{\partial F_s}{\partial P_u}, \quad u = 1, 2 \end{aligned} \tag{1}$$

Script 1 refer to Jupiter and 2 refer to Saturn where:

$$F_s = F_0 + (F_{1s})_{P+I} + (F_{2s})_{P+I}\,. \tag{2}$$

Scripts $_{1s}$, $_{2}s$ refer to first order and second order secular parts, whilst scripts $_{P}$, $_{I}$, refer to principal and indirect part of the J–S planetary Hamiltonian respectively.

F_0 denotes the zero order part of the Hamiltonian.

Using the Jacobi–Radau referential, we may write

$$(F_{1s})_{P+I} = \left[k^2\beta_1\beta_2\left(\Delta_{12}^{-1} - m_0 r_{01} r_{02}\cos\theta_{1,2} r_{02}^{-3}\right)\right]_s \tag{3}$$

$$(F_{2s})_{P+I} = \left[k^2\beta_1^2\beta_2\left(\Delta_{12}^{-3}\left(r_1^2 - r_1 r_2\cos\theta_{1,2}\right) + \right.\right. \tag{4}$$

$$\left.\left. m_0\left(r_{01} r_{02}\cos\theta_{1,2} r_{02}^{-3} - \frac{3}{2} r_{01}^2 r_{02}^2\cos^2\theta_{1,2} r_{02}^{-5}\right)\right)\right]_s \tag{5}$$

The J–S Hamiltonian in terms of the Jacobian coordinates is given by:

$$F = \frac{k^2 m_0\beta_1}{2a_1} + \frac{k^2 m_0\beta_2}{2a_2} + k^2 m_0\beta_2\left(\frac{1}{r_{02}} - \frac{1}{r_2}\right) + \sigma\frac{k^2\beta_1\beta_2}{r_{12}} \tag{6}$$

We should mention from the beginning that the Jacobi set of origins give rise to only one perturbing function, a quite important advantage.

We adopt in our treatise the Poincare' variables defined by

$$\begin{aligned}
L_u &= \beta_u\sqrt{k^2 m_0 m_{0u} a_u}\,; \quad \lambda_u = l_u + \varpi_u \\
H_u &= \sqrt{2L_u\left(1-\sqrt{1-e_u^2}\right)} cos\varpi_u \\
K_u &= -\sqrt{2L_u\left(1-\sqrt{1-e_u^2}\right)} sin\varpi_u \\
P_u &= \sqrt{2L_u\sqrt{1-e_u^2}\,(1-\cos i_u\,)}\ \cos\Omega_u \\
Q_u &= -\sqrt{2L_u\sqrt{1-e_u^2}\,(1-\cos i_u)}\sin\Omega_u\,, \quad u = 1,2\,.
\end{aligned} \tag{7}$$

These variables are consistent with the above definition of the Hamiltonian, and we denote:

k – the Gaussian constant

m_0 – mass of the Sun

a_u – Semi major axis of planet; $u = 1,2$

σ – small parameter, of the order of planetary masses. In this study it will be taken equal to 10^{-3}.

$\sigma\beta_u = m_u$ – mass of planet; $u = 1,2$

r_{ou} – distance of planet u from the Sun; $u = 1,2$

r_u – distance of planet u from the origin of coordinates associated with planet; $u = 1,2$

$m_{0u} = (m_0 + m_1 + \ldots + m_{u-1})/(\mathrm{m}_0 + m_1 + \ldots + m_{u-1} + m_u)$

$\Delta_{uv} = r_{uv}$ – mutual distance between planets u and v in Eq. (8); for a first order J–S theory.

In general for n planets, not only $u = 1, 2$, the Hamiltonian takes the form

$$F = \sum_{u=1}^{n} \frac{k^2 m_0 \beta_u}{2a_u} + \sum_{u=2}^{n} k^2 m_0 \beta_u \left(\frac{1}{r_{0u}} - \frac{1}{r_u} \right) + \sigma \sum_{u=1}^{n-1} \sum_{v=u+1}^{n} \frac{k^2 \beta_u \beta_v}{r_{uv}} \tag{8}$$

where n is the total number of planets.

For the motion of planet u, take the origin of coordinates to be the center of mass of the Sun and the planets $1, 2, \ldots, u-1$, and a_u, e_u, i_u, ϖ_u, Ω_u are the orbital elements of planet u referred to origin and axes of Jacobi, appearing in the equalities, denoting the Poincaré variables.

Whence, for a second order secular theory we should extract the secular terms of $1/r_2 - 1/r_{02}$ and that of $1/r_{12}$, the indirect and the principal parts of the J–S Hamiltonian respectively, i.e. the secular parts of:

$$\left[\sigma\beta_1 r_{01} r_{02} \cos\theta_{12} r_{02}^{-3} + \sigma^2 \beta_1^2 \left(-r_{01} r_{02} \cos\theta_{12} r_{02}^{-3} - \frac{1}{2} r_{01}^2 r_{02}^{-3} + \frac{3}{2} r_{01}^2 r_{02}^2 \cos^2\theta_{12} r_{02}^{-5} \right)\right] \tag{9}$$

and

$$\left[\Delta_{12}^{-1} + \sigma\beta_1 \Delta_{12}^{-3} \left(r_1^2 - r_1 r_2 \cos\theta_{12} \right)\right] \tag{10}$$

Whence we should find the following three terms: Δ^{-s}, r^P, $r\,r' \cos\theta$. Where $s = 1, 3, 5, \ldots$; p denotes any positive or negative real integer; θ = angle between r, r'. These three expressions should be firstly assigned in terms of the classical orbital elements, and secondly in terms of the non singular variables of H. Poincaré.

By partial differentiation of F_s w.r.t. the Poincaré variables, truncating at the fourth power of H_u, K_u, P_u, Q_u, $u = 1, 2$, for instance, we acquire a rather short Poisson series free from λ_u, in the R.H.S. of the second order secular equations of motion.

All transformation formulas from the classical orbital elements to the Poincaré variables are cited in [1]. A supercomputer is needed for very high degree Poisson series multiplications.

The *Macsyma* programs are recommended. There is no general analytical solution for the equations of motion. But in celestial mechanics there is some special series solutions for the equations, for instance the von Zeiple and the Hori–Lie transformation procedures. When we expand to power > 2 in the Poincaré variables the 12 non-homogeneous non linear differential equations of motion may be solved by numerical integration methods. Our aim is to acquire the values of our variables as $\lambda(t)$, $H(t)$, $K(t)$, $P(t)$, $Q(t)$ at any epoch.

References

[1] **Kamel, O.M.**: MSc. Thesis submitted to the Faculty of Sciences, Cairo University, **1970**.

[2] **Kael, O.M.**: PhD. Thesis submitted to the Faculty of Sciences, Cairo University, **1973**.

[3] **Brouwer, D. and Clemence, G.**: *Methods of Celestial Mechanics*, Academic Press, **1965**.

[4] **Smart, W.M.**: *Celestial Mechanics*, Longmans, **1960**.

[5] **Roy, A.E.**: *Orbital Motion*, Fourth Edition, IOP Publishing Ltd., Bristol and Philadelphia, **2005**.

[6] **Murray, C.D. and Dermott, S.F.**: *Solar System Dynamics*, Cambridge University Press, **1999**.

[7] **Kamel, O.M. and Bakry, A.A**: *Astrophysics and Space Science*, Reidel, 78, 3–26, **1981**.

[8] **Kamel, O.M.**: *The Moon and the Planets*, Reidel, 26, 239–277, **1982**.

[9] **Kamel, O.M. and Soliman, A.S.**: *Earth, Moon and Planets*, Kluwer, 60, 141–175, **1993**.

[10] **Jordan, D. and Smith, P.**: *Nonlinear Ordinary Differential Equations*, Oxford University Press, Fourth Edition, **2007**.

[11] **Jordan, D. and Smith, P.**: *Nonlinear Ordinary Differential Equations: Problems and Solutions*, Oxford University Press, **2007**.

[12] **Elmabsout, B. Kamel, O.M. and Soliman, A.S.**: The expansion of the inverse of the mutual distance between two planets raised to any real integer using Taylor theorem and elliptic expansions, Part I, II, III submitted to *Mechanics and Mechanical Engineering*.