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# A Two Dimensional Axisymmetric Thermoelastic Diffusion Problem of Micropolar Porous Circular Plate with Dual Phase Lag Model

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In the present work, we consider a two dimensional axisymmetric problem of micropolar porous circular plate with thermal and chemical potential sources in the context of the theory of dual phase lag generalized thermoelastic diffusion. The potential functions are used to analyze the problem. The Laplace and Hankel transforms techniques are used to find the expressions of displacements, microrotation, volume fraction field, temperature distribution, concentration and stresses in the transformed domain. The inversion of transforms based on Fourier expansion techniques is applied to obtain the results in the physical domain. The numerical results for resulting quantities are obtained and depicted graphically. Effect of porosity, LS theory and phase lag are presented on the resulting quantities. Some particular cases are also deduced.

*Keywords*: micropolar porous, thermoelastic diffusion, circular plate, thermal and chemical potential sources, Laplace and Hankel transforms.

## 1. Introduction

Nowacki [1] and Eringen [2] developed the theory of micropolar thermoelasticity. The linear theory of miccropolar coupled thermoelasticity was examined by Tauchert, Claus and Ariman [3]. Boschi and Iesan [4] derived the governing equations of the linear theory of thermoelasticity investigated with two relaxation times. Many researchers Tauchert [5], Nowacki and Olszak [6], Dost and Tabarrok [7],

Chandrasekharaiah [8] and Dhaliwal and Singh [9] also worked on the micropolar thermoelasticity theory.
Ciarletta [10] established a theory of micropolar thermoelasticity that permits propagation of thermal waves at a finite speed. He also presented a solution of Galerkin type for homogeneous and isotropic bodies and this solution is used to study the effect of concentrated heat sources. Sherief, Hamza and Sayed [11] derived the general solution for a homogeneous and anisotropic medium with a microstructure considering the effects of heat and allowing for second sound effects. They also proved uniqueness theorem for a half space whose boundary surface is rigidly fixed and subjected to an axisymmetric thermal shock. Passarella and Zampoli [12] obtained reciprocal and variational principle of convolution type for micropolar thermoelastic materials with centre of symmetry by using the thermoelasticity theory of type II. Marin and Beleanu [13] proved some differential relations for certain cross sectional integrals by using the initial boundary value problem of micropolar bodies in thermoelasticity without energy dissipation. Othman, Tantawi and Hilal [14] presented effect of initial stress and gravity field on micropolar thermoelastic medium with microtemperatures.
Iesan [15] developed the linear theory of micropolar materials with voids and studied the propagation of shock waves in homogeneous and isotropic micropolar elastic medium with voids. Marin [16, 17] presented the generalized solutions for boundary value problems in micropolar elastic bodies with voids. Many researchers (Kumar and Choudhary [18,19], Kumar and Deswal [20], Passarella, Tibullo and Zampoli [21] worked on micropolar elastic medium with voids. Marin Abd-Alla, Raducanu and Abo-Dahab [22] proved the solution of mixed initial boundary value problem for porous micropolar bodies depends continuously on coefficients which couple the micropolar deformation equations with the equations that model the evolution of voids. Yong Ai and Wu [23] presented precise integration solution with anisotropic thermal diffusivity and permeability for thermal consolidation problems of a multilayered porous thermoelastic medium subjected to a heat source.
The theory of thermoelastic diffusion using coupled thermoelastic model was developed by Nowacki [24, 25, 26, 27]. Sherief and Saleh [28] studied the problem of generalized thermoelastic diffusion with one relaxation time having a permeating substance in contact with the bounding plane due to time dependent thermal shock.
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Kumar and Kansal [29] constructed the fundamental solution of the system of differential equations in the theory of micropolar thermoelastic diffusion with voids. El-Sayed [30] applied the theory of generalized thermoelastic diffusion with one relaxation time to study the two dimensional problem of a thermoelastic half space with a permitting substance and with the bounding plane. Abbas, Kumar and Kaushal [31] used finite element method to study the deformation in a micropolar thermoelastic diffusion medium subjected to thermal source within the context of Lord Shulman theory [32]. El-Karamany and Ezzat [33] derived the constitutive equations for thermoelastic diffusion in anisotropic and isotropic solids using generalized thermoelasticity theory with two time delays and kernel functions.
The dual phase lag model was investigated by Tzou [34, 35] and also introduced the universal constitutive equations in the heat flux vector and temperature gradient with dual phase lag model for the study of fundamental behaviors of wave, diffusion,

pure phonon scattering and phonons electrons interactions and phonon scattering on macroscopic level. The dual phase lag model is used for the investigation of the microstructural effect on the behavior of heat transfer for macroscopic formulation. The applicability and physical meanings of dual phase lag model have been introduced by the experimental results of Tzou [36]. Liu and Chang [37] investigated the transient heat conduction in an infinitely long solid cylinder with dual phase lag heat transfer for an exponentially decaying pulse boundary heat flux and for a short pulse boundary heat flux. Kumar and Gupta [38] introduced the generalized form of mass diffusion equation instead of classical Fick's diffusion and thermal phase lag on the propagation of waves in thermoelastic diffusion medium with different symmetry. Abbas and Zenkour [39] studied the dual phase lag model on thermoelastic interactions in a semi-infinite medium due to a ramp type heating by using finite element method.

Ezzat, El-Karamany and El-Bary [40] applied the governing equations of a new mathematical model of generalized thermoelasticity with memory dependent derivative and dual phase lag model to a half space due to ramp type heating. Othman, Atwa and Elwan [41] introduced a three dimensional model of generalized thermoelasticity equations for a homogeneous isotropic elastic half space in a generalized thermoelastic medium with dual phase lag model under the effect of the gravity field. The thermoelastic interactions in a homogeneous and isotropic thick plate in the context of the theory of two temperature thermoelasticity with dual phase lag model due to a ring load has been investigated by Kumar, Sharma and Lata [42].

In the work, we investigate an axisymmetric problem of micropolar porous circular plate with mass diffusion in the context of dual phase lag theory of thermoelasticity due to thermal and chemical potential sources. The potential functions and Laplace and Hankel transforms are proposed to solve the problem. Inversion of transforms is applied to obtain the results in the physical domain. Effect of porosity, LS theory and phase lag are presented on the resulting quantities.

## 2. Basic equations

Following Kumar and Partap [43], Kumar and Kansal [44] and Chandrasekharaiah [45], the field equations and the constitutive relations in a micropolar porous thermodiffusion medium with dual phase lag model in the absence of body forces, body couples, heat sources and extrinsic equilibrated body force are taken as:

$$(\lambda+2\mu+k)\nabla\left(\nabla.\overrightarrow{u}\right)-(\mu+K)\nabla\times\nabla\times\overrightarrow{u}+K\nabla\times\overrightarrow{\phi}+b\nabla\phi^*-\beta_1\nabla T$$
$$-\beta_2\nabla C=\rho\frac{\partial^2\overrightarrow{u}}{\partial t^2} \tag{1}$$

$$(\alpha+\beta+\gamma)\nabla\left(\nabla.\overrightarrow{\phi}\right)-\gamma\nabla\times\left(\nabla\times\overrightarrow{\phi}\right)+\mathrm{K}\nabla\times\overrightarrow{u}-2K\overrightarrow{\phi}=\rho j\frac{\partial^2\overrightarrow{\phi}}{\partial t^2} \tag{2}$$

$$\alpha_1\nabla^2\phi^*-b\left(\nabla.\overrightarrow{u}\right)-\xi_1\phi^*-\omega_0\frac{\partial\phi^*}{\partial t}+\nu_1T+\nu_2C=\rho\chi\frac{\partial^2\phi^*}{\partial t^2} \tag{3}$$

$$K_1^*\left(1+\tau_t\frac{\partial}{\partial t}\right)\nabla^2 T$$
$$=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left[\rho C^*\dot{T}+\beta_1\mathrm{T}_0\left(\nabla.\dot{\overrightarrow{u}}\right)+\nu_1\mathrm{T}_0\dot{\phi}^*+a_0\mathrm{T}_0\dot{\mathrm{C}}\right] \quad (4)$$

$$D\beta_2\nabla^2\left(\nabla.\overrightarrow{u}\right)+\mathrm{D}\nu_2\nabla^2\phi^*+Da_0\nabla^2\mathrm{T}+\dot{\mathrm{C}}-\mathrm{D}b_0\nabla^2\mathrm{C}=0 \quad (5)$$

$$P=-\beta_2\left(\nabla.\overrightarrow{u}\right)+b_0C-a_0T \quad (6)$$

$$t_{ij}=\lambda u_{r,r}\delta_{ij}+\mu\left(u_{i,j}+u_{j,i}\right)+K\left(u_{j,i}-\varepsilon_{ijk}\phi_k\right)-\beta_1T\delta_{ij}-\beta_2C\delta_{ij}$$
$$+b\delta_{ij}\phi^* \quad (7)$$

$$m_{ij}=\alpha\phi_{r,r}\delta_{ij}+\beta\phi_{i,j}+\gamma\phi_{j,i} \quad (8)$$

where: $\overrightarrow{u}$ is the displacement vector, $\overrightarrow{\phi}$ is the microrotation vector, $\rho$ is the density, $j$ is the micro inertia, $\lambda,\ \mu,\ K, \alpha,\ \beta,\ \gamma$ are micropolar constants, $\alpha_1$, b,$\xi_1, \omega_0$, m and $\chi$ are the elastic constants due to the presence of voids, $\phi^*$ is the change in volume fraction field, $K_1^*$ is the coefficient of thermal conductivity, $T$ is the change in temperature of the medium at any time, $C^*$ is the specific heat at constant strain, $C$ is the concentration of the diffusion material in the body, $D$ is the thermoelastic diffusion constant, $a_0,\ b_0$ are respectively, coefficients describing the measure of thermodiffusion and of mass diffusion effects, $\beta_1=(3\lambda+2\mu+k)\,\alpha_{t1}$, $\beta_2=(3\lambda+2\mu+k)\,\alpha_{c1}$, $\nu_1=(3\lambda+2\mu+k)\,\alpha_{t2}$, $\nu_2=(3\lambda+2\mu+k)\,\alpha_{c2}$, $\alpha_{t1},\ \alpha_{t2}$ are coefficients of linear thermal expension and $\alpha_{c1},\ \alpha_{c2}$ are the coefficients of linear diffusion expansion, $\tau_t,\ \tau_q$, are the thermal relaxation times, $t_{ij},\ m_{ij}$ are the stress tensor and couple stress tensor, $\delta_{ij}$ is the kroneckor delta and the Laplacian operator is $\nabla^2=\frac{\partial^2}{\partial \mathrm{r}^2}+\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}+\frac{1}{\mathrm{r}^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial \mathrm{z}^2}$.

## 3. Formulation of the problem

A homogeneous and isotropic micropolar porous thermodiffusion elastic circular plate of thickness 2d is considered and the region $0\le r\le\infty,\quad -d\le z\le d$ is occupied by the plate. Cylindrical polar coordinate system $(r,\ \theta,\ z)$ with symmetry about z-axis is considered. The origin of the coordinate system $(r,\ \theta,\ z)$ is taken as the middle surface of the plate. We assume that the z-axis is normal to the plate along its thickness. $\mathrm{T}_0$ is the initial temperature of the thick circular plate taken as a constant temperature.

For two dimensional case, the displacement and microrotation vectors as:

$$\overrightarrow{u}=(u_r,\,0,\,u_z) \qquad \overrightarrow{\phi}=(0,\,\phi_\theta,\,0) \quad (9)$$

The following non-dimensional variables are defined as:

$$r'=\frac{\omega^*r}{c_1}\quad z'=\frac{\omega^*z}{c_1}\quad u_r'=\frac{\rho c_1\omega^*u_r}{\beta_1T_0}\quad u_z'=\frac{\rho c_1\omega^*u_z}{\beta_1T_0}\quad \phi'_\theta=\frac{\rho c_1^2\phi_\theta}{\beta_1T_0}$$
$$\phi^{*\prime}=\frac{\rho c_1^2\phi^*}{\beta_1T_0}\quad T'=\frac{T}{T_0}\quad C'=\frac{\beta_2C}{\beta_1T_0}\quad t'=\omega^*t\quad \tau_t'=\omega^*\tau_t \quad (10)$$
$$\tau_q'=\omega^*\tau_q\quad P'=\frac{P}{\beta_2}\quad t_{ij}'=\frac{t_{ij}}{\beta_1T_0}\quad m_{ij}'=\frac{\omega^*}{c_1\beta_1T_0}m_{ij}$$

where:

$$c_1^2 = \frac{\lambda + 2\mu + K}{\rho} \qquad \omega^* = \frac{K}{\rho j}$$

With the aid of expression relating displacement components $u_r$ and $u_z$ to scalar potentials $\phi$ and $\psi$ as:

$$u_r = \frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial z} \tag{11}$$

$$u_z = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial r} - \frac{\psi}{r} \tag{12}$$

The Laplace and Hankel transforms are given by:

$$\overline{f}\,(r,z,s) = L\{\overline{f}\,(r,z,t)\} = \int_0^\infty f\,(r,z,t)\,e^{-st}dt \tag{13}$$

$$\tilde{f}\,(\xi,z,s) = H\ \overline{f}\,(x,z,s)\ = \int_0^\infty r\overline{f}\,(x,z,s)\,J_n\,(\xi r)\,dr \tag{14}$$

The expressions are obtained by using (9) in (1)-(6), using (10)-(12) and then applying Laplace and Hankel transforms defined by (13)-(14) as:

$$\left[\frac{d^2}{dz^2} - \xi^2 - s^2\right]\widetilde{\phi} + p_0\widetilde{\phi^*} - \tilde{T} - \tilde{C} = 0 \tag{15}$$

$$\left[\frac{d^2}{dz^2} - \xi^2 - \frac{s^2}{\delta^2}\right]\widetilde{\psi} - \frac{p}{\delta^2}\widetilde{\phi_\theta} = 0 \tag{16}$$

$$\delta^{*2}\left(\frac{d^2}{dz^2} - \xi^2\right)\widetilde{\psi} + \left[\frac{d^2}{dz^2} - \xi^2 - 2\delta^{*2} - \frac{s^2}{\delta^2}\right]\widetilde{\phi_\theta} = 0 \tag{17}$$

$$p_0\delta_1^*\left(\frac{d^2}{dz^2} - \xi^2\right)\widetilde{\phi} - \left[\frac{d^2}{dz^2} - \xi^2 - p_1\delta_1^* - \delta_2^* s - \delta_3^* s^2\right]\widetilde{\phi^*}$$
$$-\overline{\beta}_1\delta_1^* T - \overline{\beta}_2\delta_1^*\tilde{C} = 0 \tag{18}$$

$$\epsilon R_2\left(\frac{d^2}{dz^2} - \xi^2\right)\widetilde{\phi} + \epsilon\overline{\beta}_1 R_2\widetilde{\phi^*}\ - \left[R_1\left(\frac{d^2}{dz^2} - \xi^2\right) - R_2 Q^*\right]\widetilde{\mathrm{T}}$$
$$+S^* R_2\tilde{C} = 0 \tag{19}$$

$$\left(\frac{d^2}{dz^2} - \xi^2\right)\left(\frac{d^2}{dz^2} - \xi^2\right)\widetilde{\phi} + \overline{\beta}_2\left(\frac{d^2}{dz^2} - \xi^2\right)\widetilde{\phi^*}$$
$$+A^*\left(\frac{d^2}{dz^2} - \xi^2\right)\widetilde{\mathrm{T}} + B^* s - D^*\left(\frac{d^2}{dz^2} - \xi^2\right)\tilde{C} = 0 \tag{20}$$

$$P = -E^*\left(\frac{d^2}{dz^2} - \xi^2\right)\widetilde{\phi} + F^*\tilde{C} - G^*\widetilde{\mathrm{T}} \tag{21}$$

where:

$$c_2^2 = \frac{\mu + K}{\rho} \quad \delta^2 = \frac{c_2^2}{c_1^2} \quad p = \frac{K}{\rho c_1^2} \quad p_0 = \frac{b}{\rho c_1^2} \quad \delta^{*2} = \frac{K c_1^2}{\gamma \omega^{*2}} \quad \delta_1^2 = \frac{c_3^2}{c_1^2}$$

$$c_3^2 = \frac{\gamma}{\rho j} \quad \delta_1^* = \frac{\rho c_1^4}{\alpha_1 \omega^{*2}} \quad \overline{\beta}_1 = \frac{\nu_1}{\beta_1} \quad \overline{\beta}_2 = \frac{\nu_2}{\beta_2} \quad p_1 = \frac{\xi_1}{\rho c_1^2} \quad \delta_2^* = \frac{\omega_0 c_1^2}{\alpha_1 \omega^*}$$

$$\delta_3^* = \frac{\rho\chi c_1^2}{\alpha_1} \quad Q^* = \frac{\rho C^* c_1^2}{K_1^* \omega^*} \quad \epsilon = \frac{\beta_1^2 T_0}{\rho K_1^* \omega^*} \quad S^* = \frac{a_0 \beta_1 T_0 c_1^2}{\beta_2 K_1^* \omega^*} \quad A^* = \frac{a_0 \rho c_1^2}{\beta_1 \beta_2}$$

$$B^* = \frac{\rho c_1^4}{D\omega^* \beta_2^2} \quad D^* = \frac{b_0 \rho c_1^2}{\beta_2^2} \quad E^* = \frac{\beta_1 T_0}{\rho c_1^2} \quad F^* = \frac{b_0 \beta_1 T_0}{\beta_2^2} \quad G^* = \frac{a_0 T_0}{\beta_2}$$

$$R_1 = (1 + \tau_t s) \quad R_2 = \left(1 + \tau_q s + \frac{\tau_q^2}{2} s^2\right) s$$

Solving (15)-(20), we get:

$$\left(\widetilde{\phi},\ \widetilde{\phi^*},\ \widetilde{\mathrm{T}},\ \widetilde{\mathrm{C}}\right) = \int_{i=1}^{4} (1,\ a_i,\ b_i,\ d_i)\, A_i coshm_i z \tag{22}$$

$$\left(\widetilde{\psi},\ \widetilde{\phi_\theta}\right) = \int_{i=5}^{6} (1,\ e_i)\, B_i sinhm_i z \tag{23}$$

where: $a_i,\ b_i,\ d_i,\ e_i$ and $\mathcal{H}$ are given in appendix I.
$m_i,\ (i = 1,\ 2,\ 3,\ 4)$ and $m_i,\ (i = 5,\ 6)$ are respectively roots of:

$$\left[D^8 + P_1 D^6 + P_2 D^4 + P_3 D^2 + P_4\right] = 0 \tag{24}$$

$$\left[D^4 + Q_1 D^2 + Q_2\right] = 0 \tag{25}$$

where $P_1,\ P_2,\ P_3,\ P_4,\ Q_1$ and $Q_2$ are given on appendix II.
Equations (11)-(12), with the help of (22)-(23) and applying Laplace and Hankel transforms defined by (13)-(14) are expressed as:

$$\widetilde{u_r} = -\xi \int_{i=1}^{4} A_i coshm_i z + \int_{i=5}^{6} B_i m_i coshm_i z \tag{26}$$

$$\widetilde{u_z} = \int_{i=1}^{4} A_i m_i sinhm_i z - \xi \int_{i=5}^{6} B_i sinhm_i z \tag{27}$$

By using (22) into (21), yield:

$$\widetilde{P} = \int_{i=1}^{4} K_i A_i coshm_i z \tag{28}$$

$$K_i = -E^* \left(m_i^2 - \xi^2\right) + F^* d_i - G^* b_i,\ i = 1,\ 2,\ 3,\ 4$$

## 4. Boundary conditions

The boundary conditions may be defined at the surface $z = \pm d$ of the plate as:

$$\frac{dT}{dz} = \pm g_0 F(r,\ z) \tag{29}$$

$$t_{zz} = 0 \tag{30}$$

$$t_{zr} = 0 \tag{31}$$

$$m_{z\theta} = 0 \tag{32}$$

$$\frac{d\phi^*}{dz} = 0 \tag{33}$$

$$P = \delta\,(r)\,\delta\,(t) \tag{34}$$

where: $F(r,\ z) = z^2 e^{-\omega r}$, $\omega > 0$, $\delta()$ is the Dirac delta function and $H()$ is the Heavyside unit step function.
The stress components $t_{zz}$, $t_{zr}$ and $m_{z\theta}$ are given by:

$$t_{zz} = (\lambda + 2\mu + K)\frac{\partial u_z}{\partial z} + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + b\phi^* - \beta_1 T - \beta_1 C \tag{35}$$

$$t_{zr} = (\mu + K)\frac{\partial u_r}{\partial z} + \mu\frac{\partial u_z}{\partial r} - K\phi_\theta \tag{36}$$

$$m_{z\theta} = \gamma\frac{\partial \phi_\theta}{\partial z} \tag{37}$$

The expressions of displacements, microrotation, volume fraction field, temperature distribution, concentration and stresses are obtained in the transformed domain, with the use of (9), (13)-(14), (22)-(23), (26)-(28) and (29)-(37) as:

$$\widetilde{u_r} = -\frac{1}{\Delta}\int_{i=1}^{4}\xi\Delta_i coshm_i z + \frac{1}{\Delta}\int_{i=5}^{6} m_i\Delta_i coshm_i z \tag{38}$$

$$\widetilde{u_z} = \frac{1}{\Delta}\int_{i=1}^{4} m_i\Delta_i sinhm_i z - \frac{1}{\Delta}\int_{i=5}^{6}\xi\Delta_i sinhm_i z \tag{39}$$

$$\left(\widetilde{\phi^*},\ \widetilde{\mathrm{T}},\ \widetilde{\mathrm{C}}\right) = \frac{1}{\Delta}\int_{i=1}^{4}(a_i,\ b_i,\ d_i)\,\Delta_i coshm_i z \tag{40}$$

$$\widetilde{\phi_\theta} = \frac{1}{\Delta}\int_{i=5}^{6} e_i\Delta_i sinhm_i z \tag{41}$$

$$\widetilde{t_{zz}} = \frac{1}{\Delta}\int_{i=1}^{4} L_i\Delta_i coshm_i z + \frac{1}{\Delta}\int_{i=5}^{6} M_i\Delta_i coshm_i z \tag{42}$$

$$\widetilde{t_{zr}} = \frac{1}{\Delta}\int_{i=1}^{4} N_i\Delta_i sinhm_i z + \frac{1}{\Delta}\int_{i=5}^{6} S_i\Delta_i sinhm_i z \tag{43}$$

$$\widetilde{m_{z\theta}} = \frac{1}{\Delta}\int_{i=5}^{6} T_i\Delta_i coshm_i z \tag{44}$$

where:

$$\Delta = \begin{vmatrix} U_1 & U_2 & U_3 & U_4 & 0 & 0 \\ V_1 & V_2 & V_3 & V_4 & Y_5 & Y_6 \\ W_1 & W_2 & W_3 & W_4 & Z_5 & Z_6 \\ 0 & 0 & 0 & 0 & O_5 & O_6 \\ X_1 & X_2 & X_3 & X_4 & 0 & 0 \\ P_1 & P_2 & P_3 & P_4 & 0 & 0 \end{vmatrix}$$

and $\Delta_i (i = 1, 2, 3, 4, 5,\ 6)$ are obtained from $\Delta$ by replacing i$^{th}$ column of $\Delta$ with $|Q,\ 0,\ 0,\ 0,\ R|^{tr}$, $U_i = b_i m_i sinhm_i d$, $V_i = L_i coshm_i d$, $W_i = N_i sinhm_i d$, $X_i = a_i m_i sinhm_i d$, $i = 1,\ 2,\ 3,\ 4$, $O_i = T_i coshm_i d$, $Y_i = M_i coshm_i d$, $Z_i = S_i sinhm_i d$, $P_i = K_i coshm_i d$, $i = 5,\ 6$, $Q = \pm g_0 \frac{z^2\omega}{(z^2+\omega^2)^{3/2}}$, $R = J_0(\xi)$, $L_i = m_i^2 - \frac{\lambda\xi^2}{\rho c_1^2} + p_0 a_i - b_i - d_i$, $i = 1,\ 2,\ 3,\ 4$, $M_i = \xi\left(\frac{\lambda}{\rho c_1^2} - 1\right)m_i$, $i = 5,\ 6$, $N_i = -\left(\frac{2\mu}{\rho c_1^2} + p\right)\xi m_i$, $i = 1,\ 2,\ 3,\ 4$, $S_i = \frac{(\mu+K)m_i^2+\mu\xi^2}{\rho c_1^2} - pd_i$, $i = 5,\ 6$, $T_i = \frac{\gamma\omega^{*2}}{\rho c_1^4} d_i m_i$, $i = 5,\ 6$.

## 5. Particular cases

- If we take $\tau_t = \tau_q^2 = 0, \ \tau_q = \tau_0$, then we obtain the corresponding results for Lord Shulman theory (1967) in micropolar porous thermodiffusion medium.
- If we take $C = 0$, then we obtain the corresponding results for micropolar porous thermoelastic with dual phase lag model.
- Neglecting the porous effect i.e. $\alpha_1, \ \ b, \ \ \xi_1, \ \ \omega_0, \ \ \chi$ and $\phi^*$ tend to zero. Then, the boundary conditions for two temperature micropolar thermoelastic solid with three phase lag model are given by:

$$\frac{dT}{dz} = \pm g_0 F\left(r, \ z\right)$$

$$t_{zz} = 0 \qquad t_{zr} = 0 \qquad m_{z\theta} = 0$$

$$P = \delta\left(r\right)\delta\left(t\right)$$

and the corresponding expressions are given by:

$$\widetilde{u_r} = -\frac{1}{\Delta}\int_{i=1}^{3} \xi \Delta_i coshm_i z + \frac{1}{\Delta}\int_{i=4}^{5} m_i \Delta_i coshm_i z$$

$$\widetilde{u_z} = \frac{1}{\Delta}\int_{i=1}^{3} m_i \Delta_i sinhm_i z - \frac{1}{\Delta}\int_{i=4}^{5} \xi \Delta_i sinhm_i z$$

$$\left(\widetilde{\mathrm{T}}, \ \widetilde{\mathrm{C}}\right) = \frac{1}{\Delta}\int_{i=1}^{3} b_i \Delta_i coshm_i z$$

$$\widetilde{\phi_\theta} = \frac{1}{\Delta}\int_{i=4}^{5} d_i \Delta_i sinhm_i z$$

$$\widetilde{t_{zz}} = \frac{1}{\Delta}\int_{i=1}^{3} L_i \Delta_i coshm_i z + \frac{1}{\Delta}\int_{i=4}^{5} M_i \Delta_i coshm_i z$$

$$\widetilde{t_{zr}} = \frac{1}{\Delta}\int_{i=1}^{3} N_i \Delta_i sinhm_i z + \frac{1}{\Delta}\int_{i=4}^{5} S_i \Delta_i sinhm_i z$$

$$\widetilde{m_{z\theta}} = \frac{1}{\Delta}\int_{i=4}^{5} T_i \Delta_i coshm_i z$$

where:

$$\Delta^{**} = \begin{vmatrix} U_1^{**} & U_2^{**} & U_3^{**} & 0 & 0 \\ V_1^{**} & V_2^{**} & V_3^{**} & Y_4^{**} & Y_5^{**} \\ W_1^{**} & W_2^{**} & W_3^{**} & Z_4^{**} & Z_5^{**} \\ 0 & 0 & 0 & O_4^{**} & O_5^{**} \\ P_1^{**} & P_2^{**} & P_3^{**} & 0 & 0 \end{vmatrix}$$

and $\Delta_i^{**}(i = 1, 2, 3, 4, 5)$ are obtained from $\Delta^{**}$ by replacing i$^{th}$ column of $\Delta^{**}$ with $\mid Q, \ 0, \ 0, \ 0, \ R|^{tr}$,

also:

$$U_1^{**} = b_i m_i sinhm_i d, \quad V_i^{**} = L_i^{**} coshm_i d \quad W_i^{**} = N_i^{**} sinhm_i d,$$

$$P_i^{**} = K_i^{**} coshm_i d \ i = 1,\ 2,\ 3$$

$$O_i^{**} = T_i^{**} coshm_i d \quad Y_i^{**} = M_i^{**} coshm_i d \quad Z_i^{**} = S_i^{**} sinhm_i d \quad i = 4,\ 5$$

$$L_i^{**} = m_i^2 - \frac{\lambda \xi^2}{\rho c_1^2} - b_i - d_i,\ i = 1,\ 2,\ 3$$

$$M_i^{**} = \xi \left( \frac{\lambda}{\rho c_1^2} - 1 \right) m_i,\ i = 4,\ 5$$

$$N_i^{**} = -\left( \frac{2\mu}{\rho c_1^2} + p \right) \xi m_i,\ i = 1,\ 2,\ 3$$

$$S_i^{**} = \frac{(\mu + K) m_i^2 + \mu \xi^2}{\rho c_1^2} - p d_i \quad i = 4,\ 5$$

$$T_i^{**} = \frac{\gamma {\omega^*}^2}{\rho c_1^4} d_i m_i,\ i = 4,\ 5$$

$$b_i = \left[ \left( m_i^2 - \xi^2 - s^2 \right) B^* s - D^* \left( m_i^2 - \xi^2 \right) + \left( m_i^2 - \xi^2 \right)^2 \right] / \mathcal{H}$$

$$d_i = [-\left( m_i^2 - \xi^2 \right)^2 - A^* \left( m_i^2 - \xi^2 \right) \left( m_i^2 - \xi^2 - s^2 \right)] / \mathcal{H}$$

$$\mathcal{H} = B^* s - (D^* + A^*) \left( m_i^2 - \xi^2 \right)$$

## 6. Inversion of transforms

We have to obtain the transformed displacements, microrotation, volume fraction field, temperature distribution, concentration and stresses in the physical domain, so, we invert the transforms in the resulting expressions (38)-(44). All these expressions are functions of the form $\tilde{f}(\xi, z, s)$. Therefore, we get the function $f(r, z, t)$ by using the inversion of the Hankel and Laplace transforms are defined by:

$$\tilde{f}(\xi, z, s) = \int_0^\infty \xi \overline{f}(\xi, z, s) J_n(\xi r) d\xi \tag{45}$$

$$f(r, z, t) = \frac{1}{2\iota\pi} \int_{c-\iota\infty}^{c+\iota\infty} \overline{f}(r, z, s) e^{-st} ds \tag{46}$$

where $c$ is an arbitrary constant greater than all real parts of the singularities of $\overline{f}(r, z, t)$.

## 7. Numerical results and discussions

Following Eringen [46], the values of micropolar parameters is taken as:

$$\lambda = 9.4 \times 10^{10} Nm^{-2},\ \mu = 4.0 \times 10^{10} Nm^{-2},\ K = 1.0 \times 10^{10} Nm^{-2},$$

$$\rho = 1.74 \times 10^3 Kgm^{-3},\ j = 0.2 \times 10^{-19} m^2,\ \gamma = 0.779 \times 10^{-9} N.$$

Following Dhaliwal and Singh [47], the values of thermal parameters are given by:

$$C^* = 1.04 \times 10^3 JKg^{-1}K^{-1},\ K_1^* = 1.7 \times 10^6 Jm^{-1}s^{-1}K^{-1},\ \alpha_t = 2.33 \times 10^{-5} K^{-1},$$

$$\tau_t = 0.1s \times 10^{-13} sec,\ \tau_q = 0.2s \times 10^{-13} sec,\ \tau_0 = 6.131 \times 10^{-13} sec,$$
$$T_0 = 0.298 \times 10^3 K,\ m = 1.13849 \times 10^{10} N/m^2 K,$$

The diffusion parameters are given by:

$$\alpha_{t1} = 2.33 \times 10^{-5} K^{-1},\ \alpha_{t2} = 2.48 \times 10^{-5} K^{-1},\ \alpha_{c1} = 2.65 \times 10^{-4} m^3 Kg^{-1},$$
$$\alpha_{c2} = 2.83 \times 10^{-4} m^3 Kg^{-1},\ a_0 = 2.9 \times 10^4 m^2 s^{-2} K^{-1},\ b_0 = 3.2 \times 10^5 Kg^{-1} m^5 s^{-2},$$
$$D = 0.85 \times 10^{-8} Kgm^{-3} s.$$

The values of void parameters are given as:

$$\alpha_1 = 3.688 \times 10^{-9} N, \quad b = 1.138494 \times 10^{10} N/m^2,\ \xi_1 = 1.1475 \times 10^{10} N/m^2,$$
$$\chi = 1.1753 \times 10^{-19} m^2,\ \omega_0 = 0.0787 \times 10^{-1} N \times sec/m^2.$$

In the Figures 1-6, we have determined the variations of normal stress, shear stress, couple shear stress, volume fraction field, temperature distribution and concentration with distance $r$ in case of dual phase lag micropolar thermoelastic porous with diffusion (MTPD), dual phase lag micropolar thermodiffusion (MTD), dual phase lag microplar thermoelastic porous (MTP) and micropolar thermoelastic porous with Lord Shulman theory (MTPL)

In all these figures, MTPD, MTD, MTP and MTPL corresponding to solid line (——), small dash line with centered symbol (-∗-∗-∗-∗-), small dash line (———) and dash line (– - – -) respectively.

Figure 1 displays that $t_{zz}$ starts increase for MTD and MTPL for $1 \leq r \leq 1.5$, decreases for $1.5 \leq r \leq 3$ and then its values become stationary for $3 \leq r \leq 4$. The values of $t_{zz}$ for MTPD and MTP initially decay for $1 \leq r \leq 1.7$ and then increase for $1.7 \leq r \leq 4$. The value for MTPD is small in comparison to other cases for $1.2 \leq r \leq 3.6$. The maximum magnitude of $t_{zz}$ is seen for MTPL and then the magnitude will decay and decays gradually away from the source. The values are coinciding for MTPD, MTD and MTPL away from the source. For the range $1 \leq r \leq 3$, the behavior of MTD and MTPL is opposite to MTPD and MTP except for $3.6 \leq r \leq 4$.

Figure 2 exhibits that $t_{zr}$ decreases for $1 \leq r \leq 4$ in the cases of MTPD and MTPL. For MTP, its value decreases for $1 \leq r \leq 2.3$ increases for $2.3 \leq r \leq 4$. For MTD, its value decreases for $1 \leq r \leq 1.3$, increases for $1.3 \leq r \leq 3.3$ and again decreases $3.3 \leq r \leq 4$. The behavior of MTPD and MTPL is similar for all the value of $r$. The value is higher in the case of MTP and smaller value is noticed for MTD in comparison to the other cases near the application of the source. Away from the source, the values for MTP and MTD are similar.

Figure 3 shows that the values of $m_{z\theta}$ increase in the start for MTPD, MTP and MTPL for the range $1 \leq r \leq 2.5$, decrease for $2.5 \leq r \leq 4$ and tend to zero away from the source. Its value for MTD increases rapidly for $1 \leq r \leq 2.7$ and then decays to zero for $2.7 \leq r \leq 4$. The maximum magnitude is obtained for $2 \leq r \leq 3.3$ in the case of MTD and beyond this, the magnitude decays gradually and tends to zero. Minimum magnitude is also seen in the case of MTD near the application of the source. The values are coincides for MTPD, MTD and MTPL away from the source.

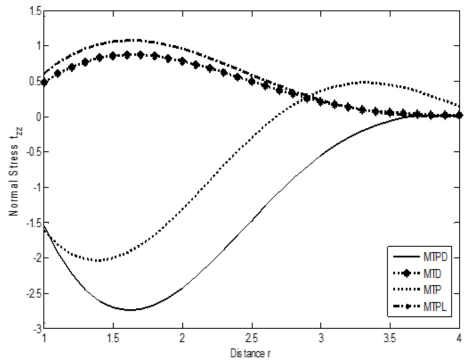


**Figure 1** Variations of normal stress $t_{zz}$

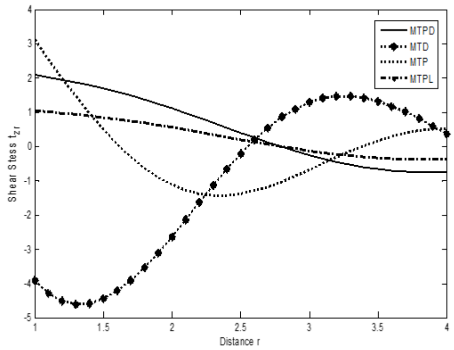


**Figure 2** Variations of shear stress $t_{zr}$

Figure 4 shows that the values of $\phi^*$ initially decrease for $1 \le r \le 1.7$ and then decrease for $1.7 \le r \le 4$ for MTPD and MTPL. For MTP, $\phi^*$ decreases for $1 \le r \le 1.5$, increases for $1.5 \le r \le 3.3$ and then decreases for $3.3 \le r \le 4$. The value is maximum for MTP and minimum for MTPL over the whole range. The variation for MTPD, MTP and MTPL are similar except away from the source.
Figure 5 displays that the values of $T$ initially increase for $1 \le r \le 1.8$ and then decrease gradually for $1.8 \le r \le 4$ for MTPD, MTD and MTPL. The value of $T$ for MTP initially increases for $1 \le r \le 1.5$, decreases for $1.5 \le r \le 3.3$ and again increases for $3.3 \le r \le 4$. $T$ has maximum value for $1 \le r \le 2.2$ and minimum value for $2.6 \le r \le 3.8$ for MTP. For the range $3.3 \le r \le 4$, the values are similar for MTPD, MTD and MTPL. The variations of MTPD, MTD and MTPL are similar over the entire range. Away from the sources, all the quantities have similar behavior.

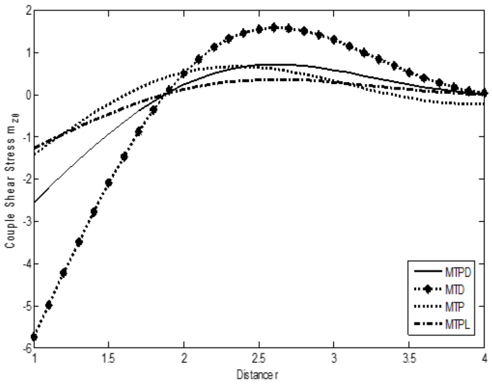


**Figure 3** Variations of couple shear stress $m_{z\theta}$

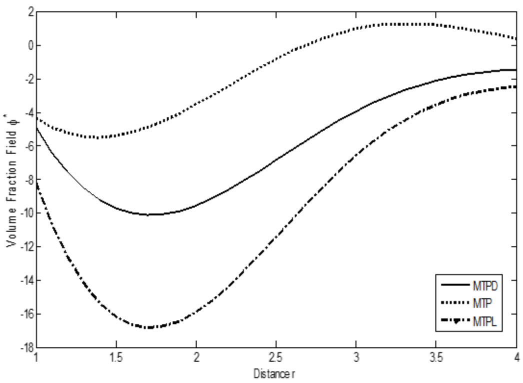


**Figure 4** Variations of Volume Fraction Field $\phi^*$

Figure 6 represents that the values of $C$ initially decrease for $1 \leq r \leq 2.7$ and increase for $2.7 \leq r \leq 4$ for MTPD and MTPL. For MTD, its value decreases for $1 \leq r \leq 1.8$ and increases for $1.8 \leq r \leq 4$. It is seen that the values for MTPD and MTPL decays sharply as compared to that of MTD. The values for MTPD and MTPL are higher for $1 \leq r \leq 2.4,\ 3.4 \leq r \leq 4$ and smaller for $2.4 \leq r \leq 3.4$ as compared to those MTD.

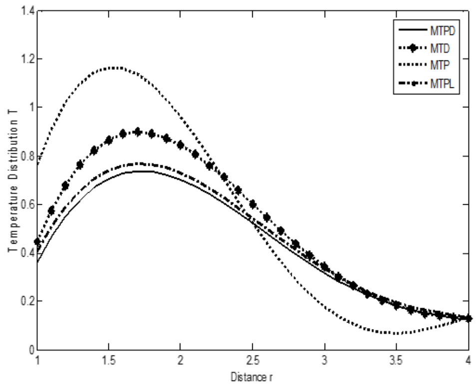


**Figure 5** Variations for temperature distribution $T$

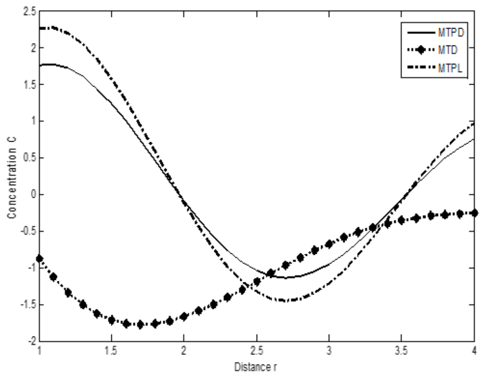


**Figure 6** Variations of concentration $C$

## 8. Conclusions

we have investigated a two dimensional axisymmetric problem of micropolar porous circular plate with thermal and chemical potential sources in the context of the theory of dual phase lag generalized thermoelastic diffusion. The potential functions are used to solve the problem. The Laplace and Hankel transforms techniques are used to obtain the expressions of displacements, microrotation, volume fraction field, temperature distribution, concentration and stresses in the transformed domain. The inversion of transforms is applied to obtain the results in the physical domain. We have presented the effect of porosity, LS theory and phase lag on normal stress, shear stress, couple shear stress, volume fraction field, temperature distribution and concentration numerically and graphically. For LS theory and porosity effect, the behavior are similar for normal stress and the similarity also obtained for MTPD and MTP but the behavior of LS theory and porosity effect is opposite to MTPD and MTP. Couple shear stress and volume fraction field have similar behavior for MTPD, MTP and MTPL. For the cases MTPD, MTP and MTPL, the variation of couple shear stress initially slowly increased and then decreased. Due to porosity effect, its value sharply increased and then decreased. The behavior of concentration is similar for MTPD and MTPL. The value of concentration initially decreased and then increased. It is also observed that all the quantities have similar behavior away from the sources. The ideas presented in the article are useful for material science and devices for a variety of thermomechanical applications.

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## Appendix I

$$a_i = [\{\left(m_i^2-\xi^2\right)^3(1-D^*) + (s\left(m_i^2-\xi^2\right)^2(B^*+D^*s) - \left(m_i^2-\xi^2\right)s^3B^*)\}R_1$$
$$+\{\left(m_i^2-\xi^2\right)^2(\epsilon\,(A^*+D^*) + Q^*\,(D^*-1) + S^*\,(A^*+1))$$
$$-\left(m_i^2-\xi^2\right)\{B^*s(\epsilon+Q^*) + s^2(D^*Q^*+A^*S^*)\} + s^3B^*Q^*\}]R_2/\mathcal{H}$$
$$i = 1,\ 2,\ 3,\ 4$$

$$b_i = [\left(m_i^2-\xi^2\right)^2\{\epsilon\,(1-D^*)\,\overline{\beta}_1 - (\epsilon+S^*)\,\overline{\beta}_2 + \epsilon D^*p_0 + p_0S^*\}$$
$$+\left(m_i^2-\xi^2\right)\epsilon sB^*\left(\overline{\beta}_1-p_0\right) + s^2\left(\epsilon D^*\overline{\beta}_1 + S^*\overline{\beta}_2\right) - \epsilon s^3B^*\overline{\beta}_1]R_2/\mathcal{H}$$
$$i = 1,\ 2,\ 3,\ 4$$

$$d_i = [\{\left(m_i^2-\xi^2\right)^3\left(p_0-\overline{\beta}_2\right) + \left(m_i^2-\xi^2\right)^2\overline{\beta}_2s^2\}R_1$$
$$+\{\left(m_i^2-\xi^2\right)^2(-\epsilon\,(1+A^*)\,\overline{\beta}_1 + (\epsilon+Q^*)\,\overline{\beta}_2 + \epsilon p_0A^* - p_0Q^*)$$
$$+\left(m_i^2-\xi^2\right)s^2(\epsilon A^*\overline{\beta}_1 - \overline{\beta}_2Q^*)\}R_2]/\mathcal{H}$$
$$i = 1,\ 2,\ 3,\ 4$$

$$e_i = \frac{\delta^2}{p}\left(m_i^2-\xi^2-\frac{s^2}{\delta^2}\right) \qquad i = 5,\ 6$$

$$\mathcal{H} = \left(m_i^2-\xi^2\right)[\left(m_i^2-\xi^2\right)\left(D^*p_0-\overline{\beta}_2\right) - sB^*p_0\ R_1 - \{\epsilon\overline{\beta}_1\,(A^*+D^*) + Q^*\overline{\beta}_2$$
$$-p_0D^*Q^* - \overline{\beta}_2S^* - A^*S^*p_0\}R_2] + B^*s\left(\epsilon\overline{\beta}_1 + p_0Q^*\right)R_2$$
$$i = 1,\ 2,\ 3,\ 4$$

**Appendix II**

$$P_1 = [f^{11}\left(D^*-1\right) + R_1\left(f^{13}-f^{12}\right) - 2\xi^2 R_1 + \left(S^*+\epsilon D^*\right) R_2$$
$$+\overline{\beta}_2\delta_1^* R_1\left(2p_0-\overline{\beta}_2\right) + \left(\epsilon+S^*\right) A^* R_2 - p_0^2\delta_1^* R_1 D^* + R_1 D^*(f^{12}+\xi^2+s^2)]/\mathcal{G}$$

$$P_2 = [f^{11}f^{12} + 2\xi^2\left(R_1 f^{12}+f^{11}\right) + R_1\xi^4 + \epsilon\delta_1^* R_2\left(\overline{\beta}_1^2+\overline{\beta}_2^2\right) - \epsilon\overline{\beta}_1\overline{\beta}_2 R_2$$
$$+\overline{\beta}_1\overline{\beta}_2\delta_1^* R_2\left(\epsilon-S^*-\epsilon A^*\right) + \left(\overline{\beta}_1+\overline{\beta}_2\right)\left(p_0\delta_1^* S^* R_2 + \epsilon p_0\delta_1^* A^* R_2\right) + \delta_1^* R_2$$
$$\left(p_0^2\delta_1^* A^* + 2\epsilon p_0\overline{\beta}_1 D^* - \epsilon\overline{\beta}_1^2 D^*\right) - \left(f^{11}+2\xi^2 R_1\right) p_0\overline{\beta}_2\delta_1^* - S^* R_2(f^{12}+2\xi^2)$$
$$-p_0\overline{\beta}_2\delta_1^*\left(f^{11}+2\xi^2 R_1\right) + \overline{\beta}_2^2\delta_1^*\left(f^{11}+\left(2\xi^2+s^2\right)R_1\right) - \epsilon A^* R_2\left(f^{12}+2\xi^2\right)$$
$$-S^* A^* R_2\left(f^{12}+2\xi^2+s^2\right) - \epsilon R_2\left(f^{13}+\left(f^{12}+\xi^2\right)D^*\right)$$
$$+p_0^2\delta_1^*\left(D^* f^{11} + \left(f^{13}+D^*\xi^2\right)R_1\right) - f^{12}\left(\xi^2+s^2\right)R_1 D^*$$
$$-\left(f^{12}+\xi^2+s^2\right)\left(R_1 f^{13}+f^{11}D^*\right)]/\mathcal{G}$$

$$P_3 = [-2\xi^2 f^{11}f^{12} - \left(R_1 f^{12}+f^{14}\right)\xi^4 + S^* R_2\xi^2\left(2f^{12}+\xi^2\right)$$
$$-\overline{\beta}_2^2\delta_1^*\left(f^{11}\xi^2 + \left(\xi^2+s^2\right)\left(f^{11}+\xi^2 R_1\right)\right) + \epsilon A^* R_2\xi^2\left(2f^{12}+\xi^2\right)$$
$$-2\epsilon\delta_1^* R_2\xi^2\left(\overline{\beta}_1^2+\overline{\beta}_2^2\right) + \overline{\beta}_1\overline{\beta}_2\delta_1^* R_2\ 4\epsilon\xi^2 + \left(S^*+\epsilon A^*\right)\left(2\xi^2+s^2\right)$$
$$-2p_0\delta_1^* R_2\xi^2\left(\overline{\beta}_1+\overline{\beta}_2\right)\left(S^*+\epsilon A^*\right) + 2p_0\delta_1^*\overline{\beta}_2\xi^2\left(2f^{11}+R_1\xi^2\right) - 2p_0^2\delta_1^* S^* A^* R_2$$
$$+S^* A^* R_2\ \xi^2\left(\xi^2+s^2\right) + f^{12}\left(2\xi^2+s^2\right) + \epsilon R_2\{\left(f^{12}+\xi^2\right)f^{13} + f^{12}\xi^2 D^*\}$$
$$-2\epsilon p_0\overline{\beta}_1\delta_1^* R_2\left(f^{13}+D^*\xi^2\right) + \epsilon\overline{\beta}_1^2\delta_1^* R_2\left(f^{13}+D^*\left(\xi^2+s^2\right)\right)$$
$$-p_0^2\delta_1^*\ f^{11}f^{13} + \left(D^* f^{11}+R_1 f^{13}\right)\xi^2 + f^{12}\left(\xi^2+s^2\right)\left(D^* f^{11}+R_1 f^{13}\right)$$
$$+f^{11}f^{13}\left(f^{12}+\xi^2+s^2\right)]/\mathcal{G}$$

$$P_4 = [f^{11}f^{12}\xi^4 - p_0\overline{\beta}_2\delta_1^* f^{11}\xi^4 - S^*\xi^4 R_2 f^{12} + \overline{\beta}_2^2\delta_1^* f^{11}\xi^2\left(\xi^2+s^2\right)$$
$$+p_0\delta_1^*\overline{\beta}_2 S^* R_2\xi^4 - \epsilon A^* f^{12}\xi^4 R_2 + \epsilon\delta_1^*\xi^4 R_2\left(\overline{\beta}_1^2+\overline{\beta}_2^2\right)$$
$$-\overline{\beta}_1\overline{\beta}_2\delta_1^* R_2\xi^2\{2\epsilon\xi^2 + \left(S^*+\epsilon A^*\right)\left(\xi^2+s^2\right)\} + \epsilon p_0\delta_1^* A^* R_2\xi^4\left(\overline{\beta}_1+\overline{\beta}_2\right)$$
$$+p_0\delta_1^*\xi^4\left(\overline{\beta}_1 S^* R_2 - \overline{\beta}_2 f^{11}\right) + p_0\delta_1^* R_2\xi^2\left(p_0 S^* A^*\xi^2 + 2\epsilon\overline{\beta}_1 f^{13}\right)$$
$$-S^* A^* R_2\xi^2\left(\xi^2+s^2\right)f^{12} - \epsilon f^{12}f^{13}\xi^2 R_2 + \epsilon\overline{\beta}_1^2\delta_1^* R_2 f^{13}\left(\xi^2+s^2\right)$$
$$+p_0^2\delta_1^*\xi^2 f^{11}f^{13} - f^{11}f^{12}f^{13}\left(\xi^2+s^2\right)]/\mathcal{G}$$

$$Q_1 = -\left(2\xi^2 + \frac{s^2}{\delta^2} + \frac{s^2}{\delta_1^2} + 2\delta^{*2} - \frac{p\delta^{*2}}{\delta^2}\right)$$
$$Q_2 = \frac{2\xi^2 s^2}{\delta^2} + 2\xi^2\delta^{*2} + \frac{s^2}{\delta^2}\left(\frac{s^2}{\delta_1^2} + 2\delta^{*2}\right) - \frac{p\delta^{*2}\xi^2}{\delta^2}$$

$$f^{11} = (R_1\xi^2 + R_2 Q^*), \qquad f^{12} = \left(\xi^2 + p_1\delta_1^* + \delta_2^* s + \delta_3^* s^2\right)$$

$$f^{13} = B^* s + D^*\xi^2, \qquad f^{14} = R_1 f^{12} + f^{11}, \qquad \mathcal{G} = \left(1-D^*\right)R_1$$