Mechanics and Mechanical Engineering
Vol. 22, No. 4 (2018) 959–965

https://doi.org/10.2478/mme-2018-0075

Optimization of the Inverted Pendulum Controller with Friction Compensation by Means of the New Method of Lyapunov Exponents Estimation

Marek Balcerzak
Division of Dynamics, Lodz University of Technology, ul. Stefanowskiego 1/15, 90-924 Lodz, Poland
e-mail: marek.balcerzak@dokt.p.lodz.pl

Received ((15 December 2017)
Revised (16 June 2018)
Accepted (25 September 2018)

This text covers optimization of an inverted pendulum control system with friction compensator. The control system is tuned with respect to a performance index based on the novel method of the Largest Lyapunov Exponent estimation. The detailed description of the method is provided. Model of the control object is presented. A simple controller is proposed. Two control systems are compared: the one with compensator and the one without. Parameters of both controllers are optimized with respect to the novel criterion by means of the Differential Evolution method. Results of numerical simulations are presented. It is shown that the new criterion can be successfully applied to both: typical linear regulators and controllers with compensators.

Keywords: Inverted pendulum, control, friction compensator, parameters optimization, differential evolution.

1. Introduction

Lyapunov Exponents (LE) are one of the commonly used tools for the analysis of non-linear dynamical systems. These exponents indicate the exponential convergence or divergence of trajectories that start close to each other. The existence of such numbers has been proved by Oseledec theorem [1], but the first numerical study of the system's behavior using Lyapunov exponents had been done by Henon and Heiles [2].

Recently, a simple and effective method of estimation of the Largest Lyapunov Exponent (LLE) from the perturbation vector and its derivative dot product has been presented. It is based on simple computations involving only basic mathematical operations such as summing, subtracting, multiplying, dividing. The LLE can be extracted from information known before each integration step. The method can be used in different aspects of the nonlinear systems control. The applications presented so far include: continuous systems [3], synchronization phenomena detection [4], time series in control systems [5–7].

In this paper, the control performance index [5] based on the LLE estimation algorithm [3] is used to optimize a control system with friction compensator. Derivation of the novel control performance index is presented, its properties are explained. Features of the new index are checked on an exemplary control system – the inverted pendulum. Equations of the control object are presented. Two controllers are proposed: the linear one with a friction compensator and the one without a compensator. Optimization of both controllers parameters with respect to the new criterion is performed by means of the Differential Evolution method [8]. Finally, results of simulations are presented and conclusions are drawn.

2. Description of the Control Object

The control object analyzed in this paper is an inverted pendulum (Fig. 1). The inverted pendulum is a kind of pendulum in which the axis of rotation is fixed to a cart. The cart is able to move along the horizontal axis x in a controlled way. The fundamental problem of the inverted pendulum is to find such a control of the cart that keeps the pendulum's bar in the vicinity of the upright vertical position $\alpha(t) = 0$ even if external disturbances appear.

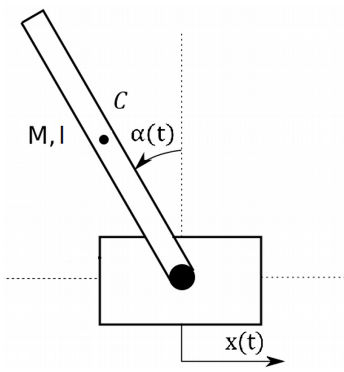

Figure 1 Sketch of the considered control object – the inverted pendulum

It has been assumed that the pendulum's drive is velocity-controlled. It means that the control signal $u(t)$ supplied to the drive is equal to the desired velocity of the cart. If the drive is stiff enough, then the motion of the pendulum's bar does not influence position of the cart $x(t)$. Providing that the drive can be approximated by a linear differential equation of the first order, the dependence between acceleration of the cart and the control signal is as follows (4):

$$\ddot{x}(t) = a[u(t) - \dot{x}(t)] \tag{1}$$

where *u(t)* is the control signal and a is a drive constant, which can be determined in the identification process.

The equation of motion of the inverted pendulum can be easily derived using Lagrange approach [9]. Assume that the pendulum's bar is uniform, its mass center

C is in the middle of its length and it is loaded by a friction torque $\tau * ml^2/3$. Then, the following equation of motion (2) is obtained:

$$\ddot{\alpha}(t) = \frac{3g}{2l}\sin(\alpha(t)) + \frac{3\ddot{x}(t)}{2l}\cos(\alpha(t)) - \tau(\dot{\alpha}(t)) \tag{2}$$

where l is the length of the bar. Substituting the equation (1) into (2) removes acceleration of the cart from the equation of motion of the bar and indicates direct influence of the control on acceleration of the bar (3):

$$\ddot{\alpha}(t) = \frac{3g}{2l}\sin(\alpha(t)) + \frac{3a}{2l}(u(t) - \dot{x}(t))\cos(\alpha(t)) - \tau(\dot{\alpha}(t)) \tag{3}$$

Equations (1) and (3) constitute a complete mathematical description of the inverted pendulum.

Let the state vector of the system (1), (3) be defined as (4):

$$x(t) = [x_1, x_2, x_3, x_4]^T = [\alpha, \dot{\alpha}, x, \dot{x}]^T \tag{4}$$

In this paper a linear control of the pendulum has been assumed. Therefore, the control function $u(t)$, in the case without compensator, can be described in the form (5):

$$u(t) = [k_1, k_2, k_3, k_4]^T \cdot x(t) = k \cdot x(t) \tag{5}$$

where $\mathbf{k} = [k_1, k_2, k_3, k_4]^T$ is the vector of controller parameters and $k_1, \ldots, k_4$ are constants to be determined.

The controller with compensator should operate in such a way that the influence of friction τ on the motion of the bar is removed or, at least, minimized. Consider a compensated control $u^*(t)$ in the form (6):

$$u^*(t) = u(t) + \frac{2l\tau(\dot{\alpha}(t))}{3a\cos(\alpha(t))} \tag{6}$$

Obviously, the friction function value τ in (6) is the one obtained from a model. Inserting the new control $u^*(t)$ to Eq. (6) in the place of $u(t)$ in (3) leads to the new equation of motion of the bar with friction compensation (7):

$$\ddot{\alpha}(t) = \frac{3g}{2l}\sin(\alpha(t)) + \frac{3a}{2l}(u(t) - \dot{x}(t))\cos(\alpha(t)) \tag{7}$$

The equation (7) states that, as long as the friction model used in (6) is accurate enough and delays in the system are negligible, application of the compensator (6) can minimize influence of friction torque on motion of the pendulum's bar. In this paper an ideal case is analyzed: the friction model in (6) is assumed to be exact and no delays are taken into account.

3. Controller Optimization

This section is devoted to optimization of the controllers (5), (6), i.e. selection of such values of parameters $k_1, \ldots, k_4$ that minimize value of the performance index based on the new LLE estimation method [3]. According to [5], the performance index is obtained by calculating the mean value λ_e of coefficients λ^* (8) calculated in each integration step.

$$\lambda^* = \frac{\frac{dx}{dt} \cdot x}{|x|^2} \tag{8}$$

For fixed initial conditions of the systems (1,3,5) or (1,3,6), λ_e can be treated as a function of regulator constants: $\lambda_e = \lambda_e(k_1, \ldots, k_4)$. Values of this function can be obtained by numerical simulation of the systems (1,3,5), or (1,3,6) and direct calculation of λ_e. Therefore, optimization of the controller can be regarded as minimization of the function $\lambda_e(k_1, \ldots, k_4)$. This task has been solved by means of the Differential Evolution (DE) method [8]. DE is a heuristic method which does not require differentiability or even continuity of the optimized function. Therefore, it suits very well to the presented task, because properties of the function $\lambda_e(k_1, \ldots, k_4)$ are not known. However, it is not guaranteed that the result of DE method is strictly the optimal one. Please note that different values of optimal parameters $k_1, \ldots, k_4$ may be obtained for the controller with compensation and for the one without compensation.

Assume that for small angular velocities of the bar, the following cubic friction model is valid (9):

$$\tau(\dot{\alpha}) = c_1\dot{\alpha} + c_2(\dot{\alpha})^3 \tag{9}$$

Under such conditions, the whole inverted pendulum control system can be described in the state space as follows (10):

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{3g}{2l}\sin(x_1) + \frac{3a}{2l}[u(t) - x_4]\cos(x_1) - c_1x_2 - c_2(x_2)^3 \\ x_4 \\ a[u(t) - x_4] \end{bmatrix} \tag{10}$$

where the control $u(t)$ in (10) may be taken either from the formula (5) in the case of control without compensation, or from the formula (6) in the case of control with compensation.

The following parameters values have been used for simulations: $g = 9.81$, $a = 19.72688$, $c_1 = 0.21075$, $c_2 = -0.11161$. These numbers have been obtained from identification of a real inverted pendulum. Please note that $c_2 < 0$. Therefore, the friction model (9) can be applied in limited range of velocities only. It has been assumed that the system starts with the following initial conditions: $x_1(0) = 0.3$, $x_2(0) = 1.0$, $x_3(0) = x_4(0) = 0.0$. The boundaries for optimized parameters have been selected as follows: $[2.0, 100.0]$ for k_1, $[0.0, 10.0]$ for k_2, $[-5.0, 0.0]$ for k_3 and $[-5.0, 0.0]$ for k_4. Such boundaries approximate the region in the parameters space for which the control system (10) is stable. Integration of the system (10) has been performed by means of Runge-Kutta method implemented in the SciPy package for Python programming language. The maximum integration step has been set to 10^{-3}. The implementation of Differential Evolution method provided by the SciPy package has been applied to find the best parameters $\mathbf{k}$.

Throughout the optimization process, the system (10) has been simulated for 6630 different controller parameter vectors $\mathbf{k}$ in the case with compensator and for 6245 different values of $\mathbf{k}$ in the case without compensator. In each trial the index λ_e has been calculated.

The graph depicting changes of the angle $\alpha(t)$ during stabilization of the pendulum with the optimal compensated controller (dashed line) and with the optimal non-compensated one (solid line) is presented in the Fig. 2. Zoom of the Fig. 2 in the neighbourhood of the second local maximum of $\alpha(t)$ is shown in the Fig. 3.

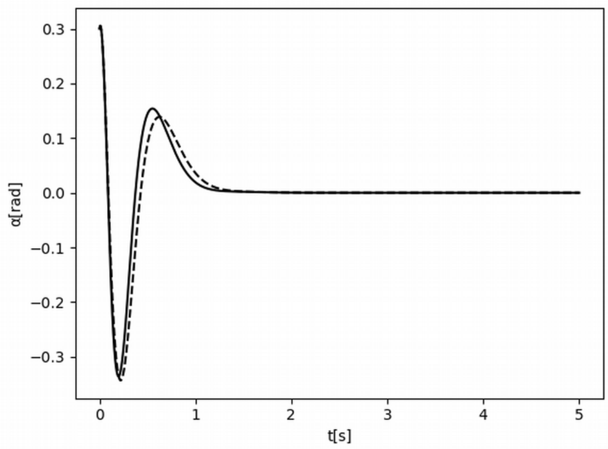

Figure 2 Stabilization of the pendulum with two types of controllers: the optimized linear controller with compensator (dashed line) and the optimized linear controller without compensator (solid line)

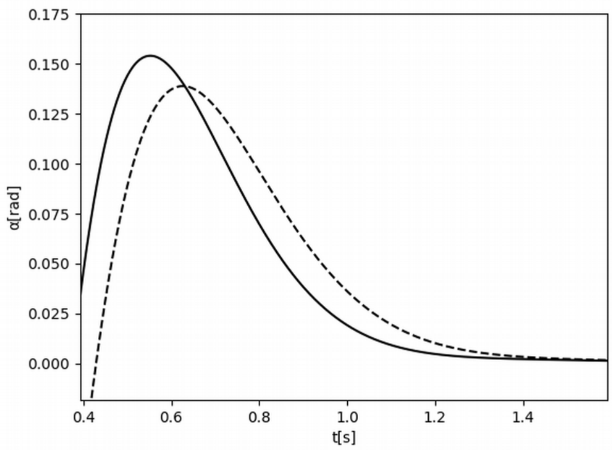

Figure 3 Zoom of the Fig. 2 in the neighborhood of the second local maximum

4. Conclusions

This paper presents application of a new criterion of control performance assessment to optimize a controller with friction compensation. The control performance index based on the new method of the largest Lyapunov exponent estimation has been presented. An exemplary control system – the inverted pendulum – has been described. Equations of the control object have been presented. A simple linear controller has been proposed. A friction compensator has been introduced. The controller with compensator and the one without compensator have been optimized with respect to the new control performance index. The Differential Evolution method has been applied in the optimization process.

Results show that the performance of the optimized controller is similar for the case with compensator and for the case without compensator. Application of the compensated controller results of slightly larger overshoot in the first minimum of the controlled variable, a bit smaller overshoot in the first maximum and longer regulation time. However, these differences are minor. This results from the fact that friction in the presented control system is relatively small. Nevertheless, the main conclusion of this work is that the presented method can be successfully applied to optimize control systems with compensators.

Acknowledgements

This study has been supported by Polish Ministry of Science and Higher Education under the program Diamond Grant, Project no. D/2013 019743.

This study has been supported by Polish National Centre of Science (NCN) under the program MAESTRO: *Multi-scale modelling of hysteretic and synchronous effects in dry friction process*, Project no. 2012/06/A/ST8/00356.

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