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Polarization of the Longitudinal Pochhammer–Chree Waves

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The exact solutions of the linear Pochhammer – Chree equation for propagating harmonic waves in a cylindrical rod, are analyzed. Spectral analysis of the matrix dispersion equation for longitudinal axially symmetric modes is performed. Analytical expressions for displacement fields are obtained. Variation of wave polarization on the free surface due to variation of Poisson's ratio and circular frequency is analyzed. It is observed that at the phase speed coinciding with the bulk shear wave speed all the components of the displacement field vanish, meaning that no longitudinal axisymmetric Pochhammer – Chree wave can propagate at this phase speed.

Keywords: Pochhammer–Chree waves, polarization, dispersion, spectral analysis.

1. Introduction

The equation for propagating harmonic waves in a cylindrical rod, now known as the Pochhammer – Chree equation, was for the first time derived in [1 – 3]. However, the corresponding solutions binding the phase or group speed with frequency remained unexplored until mid of the last century, when the first branches of the dispersion curves were obtained numerically in [4 – 22]. In [4 – 20] longitudinal axially symmetric modes were explored, and in [21, 22] flexural and torsional modes were also considered. According to [16] the axially symmetric longitudinal modes are denoted by $L(0, m)$, where m is the mode number.

In [4 – 6] by asymptotic methods were obtained analytical formulas for both short-wave ($c_{1,lim}$) and long-wave ($c_{2,lim}$) limits for the phase speed for the lowest (fundamental) branch of the longitudinal axially symmetric modes. Following [6] (see also [15]), the short-wave limit speed ($c_{1,lim}$) at $\omega \to \infty$:

$$c_{1,lim} = c_R \tag{1}$$

coincides with Rayleigh wave speed (c_R), while the long-wave limit speed $c_{2,lim}$ yields [15]:

$$c_{2,lim} = \sqrt{\frac{E}{\rho}} \tag{2}$$

where E is Young's modulus, and ρ is the material density. In [6, 15] the long-wave limit $c_{2,lim}$ was named as the "rod" wave speed.

Dispersion curves related to higher axially symmetric modes were studied in [4–20]. In [8] the first several roots of the dispersion equation were (numerically) obtained and it was revealed that some of the roots were complex relating to attenuating modes. Beside dispersion curves, variation of the displacement magnitudes along radius of the rod for the first three $L(0,m)$ modes at fixed Poisson's ratio $\nu = 0.3317$ was analyzed in [19].

One of the interesting peculiarities of propagating $L(0,m)$, $m > 1$ modes at $\gamma \to 0$, where γ is the wave number ($\gamma = 2\pi/\lambda$, λ is the wavelength), corresponds to the zero slope of the dimensionless frequency Ω [15]:

$$\lim_{\gamma \to 0} \frac{\partial \Omega}{\partial \gamma} = 0 \tag{3}$$

In (3) $\Omega = \omega R/c_2$ with ω being circular frequency, R is radius of the rod cross section, and c_2 speed of the bulk shear wave. Actually, condition (3) means presence of the horizontal asymptote in the dispersion relation $\omega(c)$ at the phase speed $c \to \infty$ for higher longitudinal axially symmetric modes. Resemblance with the dispersion curves at $c \to \infty$ for higher modes of Lamb waves can be observed, see [23].

Extensions of the Pochhammer – Chree waves to helical waves (longitudinal axially symmetric modes) that relate to non-integer coefficients at the angle coordinate in the corresponding potentials, were analyzed in [24–26].

2. Principle equations

Equation of motion for an isotropic medium at absence of body forces can be represented in a form

$$c_1^2 \nabla \mathrm{div} \mathbf{u} - c_2^2 \mathrm{rot}\,\mathrm{rot}\,\mathbf{u} = \partial_{tt}^2 \mathbf{u} \tag{4}$$

where $\mathbf{u}$ is the displacement field, c_1, c_2 are speeds of bulk longitudinal and shear waves respectively, and:

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad c_2 = \sqrt{\frac{\mu}{\rho}} \tag{5}$$

In (5) λ, μ are Lame's constants, and ρ is a material density.

The Helmholtz representation for the displacement field $\mathbf{u}$ yields:

$$\mathbf{u} = \nabla \Phi + \mathrm{rot} \boldsymbol{\Psi} \tag{6}$$

where Φ and $\boldsymbol{\Psi}$ are scalar and vector potentials respectively.

In cylindrical coordinates representation (6) for the physical components of the displacement field, becomes:

$$\begin{aligned} u_r &= \tfrac{\partial \Phi}{\partial r} + \tfrac{1}{r}\tfrac{\partial \Psi_z}{\partial \theta} - \tfrac{\partial \Psi_\theta}{\partial z} \\ u_\theta &= \tfrac{1}{r}\tfrac{\partial \Phi}{\partial \theta} + \tfrac{\partial \Psi_r}{\partial z} - \tfrac{\partial \Psi_z}{\partial r} \\ u_z &= \tfrac{\partial \Phi}{\partial z} + \tfrac{1}{r}\tfrac{\partial}{\partial r}\left(r\Psi_\theta\right) - \tfrac{1}{r}\tfrac{\partial \Psi_r}{\partial \theta} \end{aligned} \tag{7}$$

In (7) it is assumed that coordinate z directs along central axis of the rod. It is assumed that the displacement field is axially symmetric, that yields:

$$u_\theta = 0 \tag{8}$$

Substituting (6) into equation of motion (4) yields:

$$c_1^2 \Delta \Phi = \ddot{\Phi} \qquad c_2^2 \Delta \mathbf{\Psi} = \ddot{\mathbf{\Psi}} \tag{9}$$

For a harmonic wave propagating along axis z, potentials (9) can be represented in a form:

$$\Phi = \Phi_0(x')e^{i\gamma(z-ct)} \qquad \mathbf{\Psi} = \mathbf{\Psi}_0(\mathbf{x}')e^{i\gamma(z-ct)} \tag{10}$$

where, as before, γ is the wave number related to the phase speed c and circular frequency ω by equation:

$$\gamma = \frac{\omega}{c} \tag{11}$$

In (10) x' is the (vector) coordinate in the cross section of a rod ($\mathbf{x}' = \mathbf{x} - (\mathbf{n}\cdot\mathbf{x})\mathbf{n}$), $\mathbf{n}$ is the wave vector; and $z = \mathbf{n}\cdot\mathbf{x}$.

Substituting representations (10) into Eqs. (9), yields the Helmholtz equations for the potentials:

$$\Delta\Phi_0 + \left(\frac{c^2}{c_1^2} - 1\right)\gamma^2\Phi_0 = 0 \qquad \Delta\mathbf{\Psi}_0 + \left(\frac{c^2}{c_2^2} - 1\right)\gamma^2\mathbf{\Psi}_0 = 0 \tag{12}$$

Axial symmetry of Φ_0 ensures [13, 14]:

$$\frac{\partial \Phi_0}{\partial \theta} = 0 \tag{13}$$

Equations (12), (13) result in Bessel's equation:

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\Phi_0(r) + \left(\frac{c^2}{c_1^2} - 1\right)\gamma^2\Phi_0(r) = 0 \tag{14}$$

where c is the phase speed. The solution of Eq. (14) can be represented in terms of the corresponding Bessel functions:

$$\Phi_0(r) = C_1 J_0(q_1 r) + C_2 Y_0(q_1 r) \tag{15}$$

where C_k, $k = 1, 2$ are the unknown complex coefficients, and:

$$q_1^2 = \left(\frac{c^2}{c_1^2} - 1\right)\gamma^2 \tag{16}$$

Axial symmetry of potential $\mathbf{\Psi}_0$ is satisfied by the following equations [13, 14]:

$$\frac{\partial \Psi_r}{\partial \theta} = \frac{\partial \Psi_\theta}{\partial \theta} = \frac{\partial \Psi_z}{\partial \theta} = 0 \tag{17}$$

Equations (12), (17) yield Bessel equations (for physical components):

$$\begin{array}{l} \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\Psi_r(r) + \left(\left(\frac{c^2}{c_2^2} - 1\right)\gamma^2 - \frac{1}{r^2}\right)\Psi_r(r) = 0 \\ \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\Psi_\theta(r) + \left(\left(\frac{c^2}{c_2^2} - 1\right)\gamma^2 - \frac{1}{r^2}\right)\Psi_\theta(r) = 0 \\ \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\Psi_z(r) + \left(\frac{c^2}{c_2^2} - 1\right)\gamma^2\Psi_z(r) = 0 \end{array} \tag{18}$$

The solutions of Eqs. (18) are:

$$\begin{array}{l} \Psi_\theta(r) = C_3 J_1(q_2 r) + C_4 Y_1(q_2 r) \\ \Psi_r(r) = C_5 J_1(q_2 r) + C_6 Y_1(q_2 r) \\ \Psi_z(r) = C_7 J_0(q_2 r) + C_8 Y_0(q_2 r) \end{array} \tag{19}$$

In (19) C_k, $k = 3, ..., 8$ are the unknown complex coefficients, and:

$$q_2^2 = \left(\frac{c^2}{c_2^2} - 1\right)\gamma^2 \tag{20}$$

Axial symmetry of the vector potential $\mathbf{\Psi}$ imposes another restriction [14, 16]:

$$\Psi_r = \Psi_z = 0 \tag{21}$$

Now, accounting (7), (8) (15), (19), (21), the desired vector field corresponding to the propagating longitudinal axially symmetric harmonic wave, becomes [19]:

$$\begin{array}{l} u_r = -\left[q_1\left(C_1 J_1(q_1 r) + C_2 Y_1(q_1 r)\right) + i\gamma\left(C_3 J_1(q_2 r) + C_4 Y_1(q_2 r)\right)\right] e^{i\gamma(z-ct)} \\ u_\theta = 0 \\ u_z = \left[i\gamma\left(C_1 J_0(q_1 r) + C_2 Y_0(q_1 r)\right) + q_2\left(C_3 J_0(q_2 r) + C_4 Y_0(q_2 r)\right)\right] e^{i\gamma(z-ct)} \end{array} \tag{22}$$

Since components (22) vector field should be finite at $r = 0$ and noting that at $r \to 0$ Bessel's functions Y_0, Y_1 are unbounded, the final representation flows out from (22):

$$\begin{array}{l} u_r = -\left[q_1 C_1 J_1(q_1 r) + i\gamma C_2 J_1(q_2 r)\right] e^{i\gamma(z-ct)} \\ u_\theta = 0 \\ u_z = \left[i\gamma C_1 J_0(q_1 r) + q_2 C_2 J_0(q_2 r)\right] e^{i\gamma(z-ct)} \end{array} \tag{23}$$

At deriving (23) from (22), the constant C_3 is denoted by C_2.

- *Remark 1.* Expressions (23) that at $r = 0$ the natural condition $u_r = 0$ is satisfied since $J_1(0) = 0$. At the same time $J_0(0) = 1$, so u_z at $r = 0$ takes the form:

$$u_z = \left[i\gamma C_1 + q_2 C_2\right] e^{i\gamma(z-ct)} \tag{24}$$

3. Dispersion equation

Traction free boundary conditions on a lateral cylindrical surface at $r = R$ have the form:

$$\mathbf{t}_{\boldsymbol{\nu}} \equiv \left.(\lambda(\mathrm{tr}\boldsymbol{\varepsilon})\boldsymbol{\nu} + 2\mu\boldsymbol{\varepsilon}\cdot\boldsymbol{\nu})\right|_{r=R} = 0 \tag{25}$$

where $\boldsymbol{\nu}$ is the (outward) surface normal.

Substituting the displacement representation (23) into boundary conditions (25), yields the following equations written up to exponential multiplier $e^{i\gamma(z-ct)}$):

$$\begin{array}{l} t_{rr} \equiv \lambda I_\varepsilon + 2\mu\varepsilon_{rr} \\ = -\left[\begin{array}{l} \lambda\left(q_1^2+\gamma^2\right)J_0(q_1 r)C_1 + \\ +\frac{2\mu}{r}\left[\begin{array}{l} q_1 C_1\left(q_1 r J_0(q_1 r) - J_1(q_1 r)\right) + \\ +i\gamma C_2\left(q_2 r J_0(q_2 r) - J_1(q_2 r)\right)\end{array}\right]\end{array}\right]_{r=R} = 0 \\ t_{rz} \equiv 2\mu\varepsilon_{rz} \\ = -\mu\left[\begin{array}{l} i\gamma\left[q_1 C_1 J_1(q_1 r) + i\gamma C_2 J_1(q_2 r)\right] \\ +\left[i\gamma q_1 C_1 J_1(q_1 r) + q_2^2 C_2 J_1(q_2 r)\right]\end{array}\right]_{r=R} = 0 \end{array} \tag{26}$$

Equations (26) result in the desired dispersion equation:

$$\det\mathbf{A} = 0 \tag{27}$$

where $\mathbf{A}$ is a square and generally non-symmetric 2×2 matrix with complex coefficients:

$$\begin{array}{l} A_{11} = -\left(\left(q_1^2+\gamma^2\right)\frac{c_1^2}{c_2^2} - 2\gamma^2\right)J_0(q_1 R) + \frac{2q_1}{R}J_1(q_1 R) \\ A_{12} = -\frac{2i\gamma}{R}\left(q_2 R J_0(q_2 R) - J_1(q_2 R)\right) \\ A_{21} = -2i\gamma q_1 J_1(q_1 R) \\ A_{22} = -\left(q_2^2-\gamma^2\right)J_1(q_2 R) \end{array} \tag{28}$$

At deriving (28) from (26) the following identity was used:

$$\frac{\lambda}{\mu} = \frac{c_1^2}{c_2^2} - 2 \tag{29}$$

Two-dimensional (right) eigenvectors related to vanishing eigenvalues (kernel eigenvectors) of matrix $\mathbf{A}$ define polarization of the corresponding Pochhammer – Chree waves.

4. Displacement fields

Components of the kernel eigenvectors of matrix $\mathbf{A}$, that correspond to vanishing eigenvalues, are coefficients C_1, C_2 in expressions (23). Depending on the spectral properties of matrix A, two cases can be considered.

4.1. *Matrix* $\mathbf{A}$ *is (semi) simple*

Substituting components of the kernel eigenvector that corresponds to vanishing eigenvalue into Eq. (23) yields:

$$\begin{array}{l} u_r = \frac{-\left[q_1(f\pm d)J_1(q_1 r) + i\gamma A_{21}J_1(q_2 r)\right]}{\sqrt{|A_{21}|^2+|f\pm d|^2}}e^{i\gamma(z-ct)} \\ u_z = \frac{\left[i\gamma(f\pm d)J_0(q_1 r) + q_2 A_{21}J_0(q_2 r)\right]}{\sqrt{|A_{21}|^2+|f\pm d|^2}}e^{i\gamma(z-ct)} \end{array} \tag{30}$$

where $f,\ d$ are defined by coefficients A_{ij} of matrix $\mathbf{A}$, see expressions (28). In (30) and further vanishing component u_θ is not present.

Proposition 1. For (semi) simple matrix $\mathbf{A}$ the displacement component u_z vanishes at $r = 0$ and at $c = c_2$ regardless of frequency.

Proof. For the considered case:

$$i\gamma\,(f \pm d) = -q_2 A_{21} \tag{31}$$

Equation (31) can be transformed to the equivalent equation:

$$i\gamma q_2\,(A_{11} - A_{22}) + q_2^2 A_{21} + \gamma^2 A_{12} = 0 \tag{32}$$

Substituting expressions (28) into (32) at $c = c_2$ ensures vanishing u_z at $r = 0$.

Corollary. For the considered simple matrix $\mathbf{A}$, expressions (30) are applicable for any axially symmetric mode $L(0,m)$, $m > 0$.

4.2. *Matrix* **A** *is non-semisimple (contains Jordan block)*

Substituting components of the kernel eigenvector into (23) with account of conditions of degeneracy, yields:

$$\begin{aligned} u_r &= \frac{-[q_1 f\, J_1(q_1 r) + i\gamma A_{21} J_1(q_2 r)]}{\sqrt{|A_{21}|^2 + |f|^2}} e^{i\gamma(z-ct)} \\ u_z &= \frac{[i\gamma f\, J_0(q_1 r) + q_2 A_{21} J_0(q_2 r)]}{\sqrt{|A_{21}|^2 + |f|^2}} e^{i\gamma(z-ct)} \end{aligned} \tag{33}$$

Proposition 2. For non-semisimple matrix $\mathbf{A}$ the displacement component u_z does not vanish at $r = 0$ and at $c = c_2$ regardless of frequency.

Proof. For the considered case condition of non-semisimplicity of $\mathbf{A}$ takes the form:

$$i\gamma\, f = -q_2 A_{21} \tag{34}$$

Equation (34) can be transformed to the equivalent equation:

$$i\gamma\,(A_{11} - A_{22}) + 2q_2 A_{21} = 0 \tag{35}$$

Substituting (28) into (35) at $c = c_2$ reveals that condition (34) does not hold.

Corollary. For the considered non-semisimple matrix $\mathbf{A}$, expressions (30) are applicable for any axially symmetric mode $L(0,m)$, $m > 0$.

- *Remark. 2.* Substituting phase speed $c = c_2$ into (28) reveals that at c_2 matrix $\mathbf{A}$ is simple with the following one kernel (right) eigenvector:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow \lambda = 0 \tag{36}$$

Eigenvector (36) corresponds to the following coefficients in representation (23):

$$C_1 = 0,\ C_2 = 1 \tag{37}$$

Analysis of expressions (28) and (30) for the considered case reveals that at $c = c_2$ both displacement components u_r and u_z vanish regardless of the circular frequency.

5. Conclusions

The exact solutions of the linear Pochhammer – Chree equation for propagating harmonic axisymmetric longitudinal waves $L(0, m)$, $m > 0$ in a cylindrical rod, were analyzed.

Closed form expressions for the displacement field were obtained for two cases of degeneracy of the dispersion matrix: (i) single degeneracy of a simple matrix, and (ii).double degeneracy of a non-semisimple matrix.

Spectral analysis of the matrix dispersion equation for longitudinal axially symmetric modes ($L(0, m)$, $m > 0$) of Pochhammer – Chree waves was done, revealing that no longitudinal modes can propagate at c_2 phase speed.

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