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The Expansion of the Inverse of the Mutual Distance Between Two Planets Raised to Any Real Integer Using Taylor Theorem and Elliptic Expansions Part I: Theoretical Results

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We introduce in this part the method to obtain the literal expansion of the mutual distance between two planets of the solar system raised to any negative real integer.

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1. Introduction

The subject of the expansion of planetary disturbing function is extensive. It plays the basic role in the establishment of analytical planetary theories. Ancient astronomers: Pierce [1], Le Verrier [2], Boquet [3], Newcomb [16] expanded Δ^{-1} to degrees 6, 7, 8, 6 respectively, w.r.t. eccentricity, and mutual inclinations and nodes. We proceed two steps forward:

- We refer the inclinations and the nodes to a common fixed plane, say the ecliptic, which involve more complex computations, but lead to convenient results.
- By the present analysis we can calculate Δ^{-3}, Δ^{-5}, Δ^{-7}, ... by putting $s = 3, 5, 7, \ldots$ in the development, which is essential, for the construction of planetary theories of order > 1 w.r.t. planetary masses.

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2. Methods and results

We can make the calculation of $\cos(f)$ and $\sin(f)$ to any order of e, by using the following formulas, for given s_m, i_m and j_m:

$$\cos(f) \approx -e + \frac{2.(1-e^2)}{e}.\sum_{s=1}^{s_m} J_s\,(se)\cos(sM) =$$

$$-e + \frac{2(1-e^2)}{e}\sum_{s=1}^{s_m}\sum_{i=0}^{i_m}\frac{(-1)^i(\frac{se}{2})^{s+2i}}{i!\ (s+i)!}\cos(sM)\,.$$

$$\sin(f) \approx 2\sqrt{1-e^2}\sum_{s=1}^{s_m}\frac{1}{s}\frac{d}{de}J_{\ s}\,(se)\sin(sM) \approx$$

$$\left[-2\sum_{j=0}^{j_m}\frac{(2j)!\ e^{2j}}{2^{2j}\ (j!)^2\,(2j-1)}\right]\cdot\sum_{s=1}^{s_m}\frac{1}{s}\left[\frac{s}{2}\int_{i=0}^{i_m}\frac{(-1)^i\,(s+2i)\left(\frac{se}{2}\right)^{s+2i-1}}{i!\ (s+i)!}\right]\sin(sM)\,.$$

$$\sin f \approx -\sum_{j=0}^{j_m}\int_{s=1}^{s_m}\int_{i=0}^{i_m}\frac{(2j)!\ e^{2j}}{2^{2j}\ (j!)^2\,(2j-1)}\frac{(-1)^i.\,(s+2i)\left(\frac{se}{2}\right)^{s+2i-1}}{i!\ (s+i)!}\sin(sM)\,.$$

We have the following results concerning the calculation of $\cos(f)$ and $\sin(f)$ to e^6 obtained by taking $i_m = j_m = s_m = 6$.

$$\cos(f) = \cos(M) + e[\cos(2M) - 1] + 9e^2\,\frac{\cos(3M)\ -\cos(M)}{8}+$$

$$4e^3\,\frac{\cos(4M)\ -\cos(2M)}{3} + 25e^4\,\frac{25\cos(5M) - 27\cos(3M) + 2\cos(M)}{384}+$$

$$3e^5\,\frac{27\cos(6M) - 32\cos(4M) + 5\cos(2M)}{40}+$$

$$49e^6\,\frac{[2401\cos(7M) - 3125\cos(5M) + 729\cos(3M) - 5\cos(M)]}{46080}\,.$$

$$\sin(f) = \sin(M) + e\sin(2M) + e^2\,\frac{9\sin(3M) - 7\sin(M)}{8}+$$

$$e^3\,\frac{8\sin(4M) - 7\sin(2M)}{6}+$$

$$e^4\,\frac{625\sin(5M) - 621\sin(3M) + 34\sin(M)}{384}+$$

$$e^5\,\frac{243\sin(6M) - 272\sin(4M) + 40\sin(2M)}{120}+$$

$$e^6\,\frac{117649\sin(7M) - 146875\sin(5M) + 33129\sin(3M) - 1355\sin(M)}{46080}\,.$$

We have also the expansion series of r/a:

$$\frac{r}{a} \approx 1 + \frac{e^2}{2} - 2e\sum_{s=1}^{s_m}\frac{1}{s^2}\frac{d}{de}J_s\,(se)\cos(sM) \approx$$

$$1 - e\,\cos(M) + e^2\,\frac{1-\cos(2M)}{2} + e^3\,\frac{3\cos(M)-\cos(3M)}{8} +$$
$$e^4\,\frac{\cos(2M)-\cos(4M)}{3} - e^5\,\frac{5[25\cos(5M) + -27\cos(3M) + 2\cos(M)]}{384} +$$
$$-e^6\,\frac{[27\cos(6M) - 32\cos(4M) + 5\cos(2M)]}{80}$$

Then we deduce the expansion of

$$\varepsilon \equiv \frac{r}{a} - 1 \approx \frac{e^2}{2} - 2e\sum_{s=1}^{s_m}\frac{1}{s^2}\frac{d}{de}J_s(se)\cos(sM) \approx$$

$$e\cos(M) + e^2\,\frac{1-\cos(2M)}{2} + e^3\,\frac{3[\cos(M)-\cos(3M)]}{8} + e^4\,\frac{[\cos(2M)-\cos(4M)]}{3} +$$
$$-e^5\,\frac{5[25\cos(5M) - 27\cos(3M) + 2\cos(M)]}{384} +$$
$$-e^6\,\frac{27\cos(6M) - 32\cos(4M) + 5\cos(2M)]}{80}\,.$$

Hence we obtain the expansion of $\varepsilon' \equiv \frac{r'}{a'} - 1$:

$$\varepsilon' \approx -e'\cos(M') + e'^2\,\frac{1-\cos(2M')}{2} + e'^3\,\frac{3[\cos(M')-\cos(3M')]}{8} +$$
$$+e'^4\,\frac{\cos(2M')-\cos(4M')}{3} - e'^5\,\frac{5\,[25\cos(5M') - 27\cos(3M') + 2\cos(M')]}{384}$$
$$-e'^6\,\frac{27\cos(6.M') - 32\cos(4.M') + 5\cos(2M')}{80}$$

Then we deduce

$$\cos(\omega + f) = \cos(\omega)\cos(f) - \sin(\omega)\sin(f)$$

and

$$\sin(\omega + f) = \sin(\omega)\cos(f) + \cos(\omega)\sin(f)\,.$$

After calculation we obtain:

$$\cos(\omega + f) = \cos(M+\omega) + e\,[\cos(2M+\omega) - \cos(\omega)] +$$
$$e^2\,\frac{9\cos(3M+\omega) - 8\cos(M+\omega) - \cos(M-\omega)}{8} +$$
$$e^3\,\frac{16\cos(4M+\omega) - 15\cos(2M+\omega) - \cos(2M-\omega)}{12} +$$
$$e^4\,\frac{625\cos(5M+\omega) - 648\cos(3M+\omega) - 27\cos(3M-\omega) + 42\cos(M+\omega) + 8\cos(M-\omega)}{384} +$$

$$+\frac{e^5}{240}\left[\begin{array}{c} 486\cos(6M+\omega) - 560\cos(4M+\omega) - 16\cos(4M-\omega) + \\ 85\cos(2M+\omega) + 5\cos(2M-\omega) \end{array}\right] +$$

$$\frac{e^6}{46080}\left[\begin{array}{c}117649\cos(7M+\omega)-150000\cos(5M+\omega)-3125\cos(5M-\omega)+\\ 34425\cos(3M+\omega)+1296\cos(3M-\omega)-800\cos(M+\omega)+555\cos(M-\omega)\end{array}\right].$$

Then we have:

$$\sin(\omega+f)=\sin(M+\omega)+e\left[\sin(2M+\omega)-\sin(\omega)\right]+$$
$$e^2\,\frac{9\sin(3M+\omega)-8\sin(M+\omega)+\sin(M-\omega)}{8}+$$
$$e^3\,\frac{16\sin(4M+\omega)-15\sin(2M+\omega)+\sin(2M-\omega}{12}+$$
$$e^4\,\frac{625\sin(5M+\omega)-648\sin(3M+\omega)+27\sin(3M-\omega)+42\sin(M+\omega)-8\sin(M-\omega)}{384}+$$
$$e^5\,\frac{486\sin(6M+\omega)-560\sin(4M+\omega)+16\sin(4M-\omega)+85\sin(2M+\omega)-5\sin(2M-\omega)}{240}+$$
$$e^6\left[\frac{117649\sin(7M+\omega)-150000\sin(5M+\omega)+3125\sin(5M-\omega)}{46080}+\right.$$
$$\left.\frac{+34425\sin(3M+\omega)-1296\sin(3M-\omega)-800\sin(M+\omega)-555\sin(M-\omega)}{46080}\right].$$

We have, on the other hand,

$$\frac{x}{r}=\cos(\Omega)\cos(\omega+f)-\sin(\Omega)\sin(\omega+f)\cos(i)$$
$$\frac{y}{r}=\sin(\Omega)\cos(\omega+f)+\cos(\Omega)\sin(\omega+f)\cos(i)$$
$$\frac{z}{r}=\sin(\omega+f)\sin i$$

and

$$\frac{x_1}{r_1}=\cos(\Omega_1)\cos(\omega_1+f_1)-\sin(\Omega_1)\sin(\omega_1+f)\cos(i)$$
$$\frac{y_1}{r_1}=\sin(\Omega_1)\cos(\omega_1+f_1)+\cos(\Omega_1)\sin(\omega_1+f_1)\cos(i)$$
$$\frac{z_1}{r_1}=\sin(\omega_1+f_1)\sin i$$

which gives:

$$\cos(\psi)=\frac{xx_1+yy_1+yy_1}{rr_1}.$$

We have also:

$$\theta=\varpi+f=\Omega+\omega+f,\quad \theta_1=\varpi_1+f_1=\Omega_1+\omega_1+f_1\,,$$
$$\theta-\theta_1=(\Omega-\Omega_1)+\left[(\omega+f)-(\omega_1+f_1)\right]$$
$$\cos(\theta-\theta_1)=\cos[\Omega+\omega+f]\cos[\Omega_1+\omega_1+f_1]+\sin[\Omega+\omega+f]\sin[\Omega_1+\omega_1+f_1]=$$
$$=[\cos\Omega\cos(\omega+f)-\sin(\Omega)\sin(\omega+f)][\cos(\Omega_1)\cos(\omega_1+f_1)]-\sin(\Omega_1)\sin(\omega_1+f_1)]+$$
$$[\sin(\Omega\cos(\omega+f)+\cos(\Omega)\sin(\omega+f)][\sin(\Omega_1)\cos(\omega_1+f_1)]+\cos(\Omega_1)\sin(\omega_1+f_1)]$$

$$= \cos(\Omega - \Omega_1)[\cos(\omega + f)\cos(\omega_1 + f_1) + \sin(\omega + f)\sin(\omega_1 + f_1)] +$$
$$- \sin(\Omega - \Omega_1)[\sin(\omega + f)\cos(\omega_1 + f_1) - \cos(\omega + f)\sin(\omega_1 + f_1)]$$
$$\sin(I/2) = \gamma, \quad \cos(I) = 1 - 2\sin^2(I/2) = 1 - 2\gamma^2\,,$$
$$\sin(I) = 2\gamma\sqrt{1-\gamma^2} \approx 2\left(1 - \frac{\gamma^2}{2} - \frac{\gamma^4}{8} - \frac{\gamma^6}{16} - \frac{5\gamma^8}{128}\right) = 2 - \gamma^3 - \frac{\gamma^5}{4} - \frac{\gamma^7}{8} - \frac{5\gamma^9}{64}$$
$$\approx 2 - \gamma^3 - \frac{\gamma^5}{4} - \frac{\gamma^7}{8}$$

Let us use the longitudes $\lambda = M + \varpi$ and $\lambda_1 = M_1 + \varpi_1$, by replacing, in the expressions of $\cos(\psi)$ and $\cos(\theta - \theta_1)$, ω and M, ω_1 and M_1 respectively by:

$$\omega = \varpi - \Omega, \quad M = \lambda - \varpi, \quad \omega_1 = \varpi_1 - \Omega_1, \quad M_1 = \lambda_1 - \varpi_1\,.$$

The expression of $\cos(\theta - \theta_1)$ can be obtained from the expression of $\cos(\psi)$ by setting $=' = 0$. Then we can deduce the expression of

$$\Psi = \cos(\psi) - \cos(\theta - \theta_1)\,.$$

We obtain:

$$\Psi = -2e\gamma\gamma'\cos(2\lambda - \varpi + \lambda' -') + e\gamma'^2\cos(2\lambda - \varpi + \lambda' - 2')$$
$$+e\gamma^2\cos[2\lambda - \varpi + \lambda' - 2] + 2e\gamma\gamma'\cos(2\lambda - \varpi - \lambda' +')$$
$$-e(\gamma^2 + \gamma'2)\cos(2\lambda - \varpi - \lambda') + 2e\gamma\gamma'\cos(\lambda + \varpi' - 2\lambda' + ' -)$$
$$-e'(\gamma^2 + \gamma'^2)\cos(\lambda + \varpi' - 2\lambda') + 2e'\gamma\gamma'\cos(\lambda + \varpi' - ' -)$$
$$-e'\gamma'^2\cos(\lambda + \varpi' - 2') - e'\gamma^2\cos(\lambda + \varpi' - 2)$$
$$-2e'\gamma\gamma'\cos(\lambda - \varpi' + 2\lambda' - ' -)$$
$$+e'\gamma'^2\cos(\lambda - \varpi' + 2\lambda' - 2') + e'\gamma^2\cos(\lambda - \varpi' + 2\lambda' - 2)$$
$$-2e'\gamma\gamma'\cos(\lambda - \varpi' + ' -) + e'(\gamma^2 + \gamma'^2)\cos(\lambda - \varpi')$$
$$-2\gamma\gamma'\cos(\lambda + \lambda' - ' -)$$
$$+\gamma'^2\cos(\lambda + \lambda' - 2') + \gamma^2\cos(\lambda + \lambda' - 2)$$
$$+2\gamma\gamma'\cos(\lambda - \lambda' + ' -) - (\gamma^2 + \gamma'^2)\cos(\lambda - \lambda')$$
$$+e\gamma^2\cos(\varpi - \lambda') - e\gamma^2\cos(\varpi + \lambda' - 2)$$
$$+2e\gamma\gamma'\cos(\varpi + \lambda' - ') - 2e\gamma\gamma'\cos(\varpi - \lambda' + ' -)$$
$$-e\gamma'^2\cos(\varpi + \lambda' - 2') + e\gamma'^2\cos(\varpi - \lambda')$$

We note that Ψ is of second order in γ, γ', e and e'. The terms of the second order are independent of e and e'.

On the other hand, we have

$$\cos\theta = \cos(\Omega + \omega + f) = \cos(\varpi + f) = \cos(\varpi)\cos(f) - \sin(\varpi)\sin(f)\,,$$

$$\sin\theta = \sin(\Omega+\omega+f) = \sin(\varpi+f) = \sin(\varpi)\cos(f)+\cos(\varpi)\sin(f)\,.$$

Then we can deduce

$$e^{i\theta} = \cos(\theta + i\sin(\theta)\,,$$

$$e^{ij\theta} = (\cos(\theta)+i\sin(\theta))^j = \cos(j\theta)+i\sin(j\theta),$$

so we obtain $\cos(j\theta)$ and $\sin(j\theta)$ as series of ϖ, λ and M. We notice that it is possible to calculate f as a function of M. For this aim, we use the differential relation $dM = n\,dt$, with $n = \sqrt{\frac{\mu}{a^3}}$, $M = E - e\sin(E)$,

$$dM = (1-e\cos(E))\,dE$$

$$df = \frac{\sqrt{\mu a(1-e^2)}}{r^2} \quad dt = \frac{\sqrt{\mu a(1-e^2)}}{a(1-e\cos(E)^2)}\,dt,$$

with $r = a(1-e\cos(E))$,

$$df = \sqrt{\frac{\mu}{a^3}}\frac{\sqrt{1-e^2}}{(1-e\cos(E)} \quad dt = \frac{\sqrt{1-e^2}dM}{(1-e\cos E)^2} = \sqrt{1-e^2}\left(\frac{dE}{dM}\right)^2 dM\,.$$

We have also

$$dM = \frac{nr^2\,df}{\sqrt{\mu a(1-e^2)}} = \left(1-e^2\right)^{3/2}\frac{df}{\left(1+e\cos(f)\right)^2}\,.$$

We develop the last member as a series before integrating term by term, the constant of integration being equal to zero:

$$M = f - 2e\sin(f) + \frac{3e^2\sin(2f)}{4} - \frac{e^3\sin(3f)}{3} + e^4\left[\frac{5\sin(4f)}{32}+\frac{\sin(4f)}{8}\right] +$$
$$-e^5\left[\frac{3\sin(5f)}{40}+\frac{\sin(3f)}{8}\right] + e^6\left[\frac{7\sin(6f)}{192}+\frac{3\sin(4f)}{32}+\frac{3\sin(2f)}{64}\right]+\ldots$$

We can deduce: $f = M + e\Phi\,(f)$, with

$$\Phi(f) \equiv \frac{f-M}{e} = 2\sin(f) + \frac{3e\sin(2f)}{4} + \frac{e^2\sin(3f)}{3} - e^3\left[\frac{5\sin(4f)}{32}+\frac{\sin(4f)}{8}\right]$$
$$+e^4\left[\frac{3\sin(5f)}{40}+\frac{\sin(3f)}{8}\right] - e^5\left[\frac{7\sin(6f)}{192}+\frac{3\sin(4f)}{32}+\frac{3\sin(2f)}{64}\right]$$
$$+e^6\left[\frac{\sin(7f)}{56}+\frac{\sin(5f)}{16}+\frac{\sin(3f)}{16}\right]+\ldots$$

So we can write.

$$f = M + \sum_{j=1}^{\infty}\frac{d^{j-1}}{dM^{j-1}}\left[\Phi(M)\right]^j$$

$$f = M + 2e\sin(M) + \frac{5e^2\sin(2M)}{4} + e^3\left[\frac{13\sin(3M)}{12}-\frac{\sin(M)}{4}\right]+$$

$$+e^4\left[\frac{103\sin(4M)}{96}-\frac{11\sin(2M)}{24}\right]+e^5\left[\frac{1097\sin(5M)}{960}-\frac{43\sin(3M)}{64}+\frac{5\sin(Mt)}{96}\right]$$

$$+e^6\left[\frac{1223\sin(6M)}{960}-\frac{451\sin(4M)}{480}+\frac{17\sin(2M)}{192}\right]$$

We notice that $(f-M)$ is of order 1 at least with respect to e. We have:

$$j(f-M)=2je\sin(M)+\frac{5je^2\sin(2M)}{4}+je^3[\frac{13\sin(3M)}{12}-\frac{\sin(M)}{4}]$$

$$+e^4\left[\frac{103\sin(4M)}{96}-\frac{11\sin(2M)}{24}\right]+je^5\left[\frac{1097\sin(5M)}{960}-\frac{43\sin(3M)}{64}+\frac{5\sin(M)}{96}\right]$$

$$+je^6\left[\frac{1223\sin(6M)}{960}-\frac{451\sin(4M)}{480}+\frac{17\sin(2M)}{192}\right]$$

$$\cos\left((j(f-M)\right)=$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i[j(f-M)]^{2i}}{(2i)!}\approx 1-\frac{[j(f-M)]^2}{2}+\frac{[j(f-M)]^4}{24}+\frac{[j(f-M)]^6}{720}$$

to order 6.

$$\approx 1-\frac{[j(f-M)]^2}{2}$$

to order 3.

$$\sin\left((j(f-M)\right)=$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i[j(f-M)]^{2i+1}}{(2i+1)!}\approx 1-j(f-M)]-\frac{[j(f-M)]^3}{6}+\frac{[j(f-M)]^5}{120}$$

to order 6.

$$\approx[1-j(f-M)]-\frac{[j(f-M)]^3}{6}$$

to order 3.

We obtain to order 3:

$$\cos\left(j(f-M)\right)\approx 1+e^2j^2[\cos(2M)-1]+\frac{5}{4}e^3j^2[\cos(3M)-\cos(M)]$$

$$\sin[j(f-M)]\approx 2ej\sin(M)+\frac{5}{4}e^2j\sin(2M)+$$

$$+\frac{1}{12}e^3j\left[(4j^2+13\sin(3M)-3(4j^2+1\sin(M)\right]$$

On the other hand, we can write, since: $\theta=\omega+\Omega+f$ and $M=\lambda-\varpi$:

$$\cos(j\theta)\equiv\cos[j(\omega+\Omega+f)]=\cos[j(\varpi+f)]=\cos[j\varpi+jM+j(f-M)]=$$

$$=\cos[j\lambda+j(f-M)]$$

$$= \cos(j\lambda)\cos[j(f-M)] - \sin(j\lambda)\sin[j(f-M)] =$$

$$\sin(j\theta) \equiv \sin[j(\omega+\Omega+f)] = \sin[j(\varpi+f)] = \sin[j\varpi + jM + j(f-M)] =$$

$$\sin[j\lambda + j(f-M)] = \sin(j\lambda)\cos[j(f-M)] + \cos(j\lambda)\sin[j(f-M)],$$

We find, after calculation:

$$\cos(j\theta) \approx \cos(j\lambda) + e\,[j\cos[(j+1)\lambda-\varpi] - \cos[(j-1)\lambda+\varpi]]$$

$$+\frac{e^2 j}{8}\{(4j+5)\cos[(j+2)\lambda-2\varpi] + (4j-5)\cos[(j-2)\lambda+2\varpi] - 8j\cos(j\lambda)\}$$

$$+\frac{e^3 j}{24}\{(4j^2+15j+13)\cos[(j+3)\lambda-3\varpi] - 3(4j^2+5j+1)\cos[(j+1)\lambda-\varpi]$$

$$+3(4j^2-5j+1)\cos[(j-1)\lambda+\varpi] - (4j^2-15j+13)\cos[(j-1)\lambda+\varpi]\}$$

$$\sin(j\theta) \approx \sin(j\lambda) + ej\{\sin[(j+1)\lambda-\varpi] - \sin[(j-1)\lambda+\varpi]\}$$

$$+\frac{e^2 j}{8}\{(4j+5)\sin[(j+2)\lambda-2.\varpi] + (4j-5)\sin[(j-2)\lambda+2.\varpi] - 8j\sin(j\lambda)\}$$

$$+\frac{e^3 j}{24}\{(4j^2+15j+13)\sin[(j+3)\lambda-3\varpi] - 3(4j^2+5j+1)\sin[(j+1)\lambda-\varpi]$$

$$+3(4j^2-5j+1)\sin[(j-1)\lambda+\varpi] - (4j^2-15j+13)\sin[(j-1)\lambda+\varpi]\}$$

So we can calculate:

$$\cos[j(\theta-\theta')] = \cos(j\theta)\cos(j\theta') + \sin(j\theta)\sin(j\theta')$$

$$= \frac{1}{24}\{e^3 j(4j^2+15j+13)\cos[(j+3)\lambda-3\varpi-j\lambda']$$

$$+3e^2e'j^2(4j+5)\cos[(j+2)\lambda-(j+1)\lambda'+\varpi'-2\varpi]$$

$$-3e^2e'j^2(4j+5)\cos[(j+2)\lambda-(j-1)\lambda'-\varpi'-2\varpi]$$

$$+3e^2 j(4j+5)\cos[(j+2)\lambda-j\lambda'-2\varpi]$$

$$+3ee'^2 j^2(4j+5)\cos[(j+1)\lambda-(j+2)\lambda'+2\varpi'-\varpi]$$

$$+24ee'j^2\cos[(j+1)(\lambda-\lambda')+\varpi'-\varpi]$$

$$-24ee'j^2\cos[(j+1)\lambda-(j-1)\lambda'-\varpi'-\varpi]$$

$$+\ 3ee'^2 j^2(4j-5)\cos[(j+1)\lambda-(j-2)\lambda'-2\varpi'-\varpi]$$

$$+3ej[-4(2e'^2+e^2)j^2 - e^2(5j+1)+8]\cos[(j+1)\lambda-j\lambda'-\varpi]$$

$$+\ e'^3 j(4j^2+15j+13)]\cos[j\lambda-(j+3)\lambda'+3\varpi']$$

$$+3e'^2 j(4j+5)\cos[j\lambda-(j+2)\lambda'+2\varpi']$$

$$-3e'j[4(2e^2+e'^2)j^2 + e'^2\,(5j+1)-8]\cos[j\lambda-(j+1)\lambda'+\varpi']$$

$$+3e'j[4(2e^2+e'^2)j^2 - e'^2\,(5j-1)-8]\cos[j\lambda-(j-1)\lambda'-\varpi']$$

$$+3e'^2 j(4j-5)\cos[j\lambda-(j-2)\lambda'-2\varpi']$$

$$-e'^3 j(4j^2-15j+13))\cos[j\lambda-(j-3)\lambda'-3\varpi']$$
$$-24[(e^2+e'^2)j^2-1]\cos[j(\lambda-\lambda')]$$
$$-3ee'^2 j^2(4j+5)\cos[(j-1)\lambda-(j+2)\lambda'+2\varpi'+\varpi]$$
$$-24ee'j^2\cos[(j-1)\lambda-(j+1)\lambda'+\varpi'+\varpi]$$
$$+24ee'j^2\cos[(j-1)\lambda-(j-1)\lambda'-\varpi'+\varpi]$$
$$-3ee'^2 j^2(4j-5)\cos[(j-1)\lambda-(j-2)\lambda'-2\varpi'+\varpi]$$
$$+3ej[(e^2+2e'^2)j^2-e^2(5j-1)-8]\cos[(j-1)\lambda-j\lambda'+\varpi]$$
$$+3e^2e'j^2(4j-5)\cos[(j-2)\lambda-(j+1)\lambda'+2\varpi+\varpi']$$
$$-3e^2e'j^2(4j-5)\cos[(j-2)\lambda-(j-1)\lambda'+2\varpi-\varpi']$$
$$+3e^2j(4j-5)\cos[(j-2)\lambda-j\lambda'+2\varpi]$$
$$-e^3j(4j^2-5j+13))\cos[(j-3)\lambda-j\lambda'+3\varpi]\}$$

In the next step, we calculate $\left(\frac{r}{a}\,\frac{r}{a'}\,\Psi\right)$. We obtain to order 3:

$$\left(\frac{r}{a}\frac{r'}{a'}\Psi\right)=\frac{rr'\Psi}{aa'}=-e\cos(2\lambda+\lambda'-\varpi--')+(e'^2/2)\cos(2\lambda+\lambda'-\varpi-2\Omega')$$
$$+(e^2/2)\cos(2\lambda+\lambda'-\varpi-2\Omega+e'\cos(2\lambda-\lambda'-\varpi-\Omega+\Omega')$$
$$-[(e^2+'^2)/2]\cos(2\lambda-\lambda'-\varpi)+e''\cos(\lambda-2\lambda'+\varpi'-\Omega+\Omega')$$
$$-[e'(^2+'^2)/2]\cos(\lambda-2\lambda'+\varpi')+3e''\cos(\lambda+\varpi'-\Omega-\Omega')$$
$$-3(e'^2/2)\cos(\lambda+\varpi'-2\Omega')-3[e'^2/2]\cos(\lambda+\varpi'-2\Omega)$$
$$-e''\cos(\lambda+2\lambda'-\varpi'-\Omega-\Omega')$$
$$+[e''^2/2]\cos(\lambda+2\lambda'-\varpi'-2\Omega')+[e'^2/2]\cos(\lambda+2\lambda'-\varpi'-2\Omega)$$
$$-3e''\cos(\lambda-\varpi'-\Omega+\Omega')$$
$$+3[e'(^2+'^2)/2]\cos(\lambda-\varpi')-2'\cos(\lambda+\lambda'-\Omega-\Omega')$$
$$+'^2\cos(\lambda+\lambda'-2\Omega')+^2\cos(\lambda+\lambda'-2\Omega)+2'\cos(\lambda-\lambda'-\Omega+\Omega')$$
$$-(^2+'^2)\cos(\lambda-\lambda')+[3e(^2+'^2)/2]\cos(\lambda'-\varpi)$$
$$-[e^2/2]\cos(\lambda'+\varpi-2\Omega)$$
$$+3e'\cos(\lambda'+\varpi-\Omega-\Omega')-3e'\cos(\lambda'-\varpi+\Omega-\Omega')$$
$$-[3e'^2/2]\cos(\lambda'+\varpi-2\Omega')\,.$$

We note that $\left(\frac{r}{a}\frac{r'}{a'}\Psi\right)$ is of the second order in $'$ and $'$ at least. Consequently:

$$\left(\frac{r}{a}\frac{r'}{a'}\Psi\right)^2$$

is of order 4 at least, and to order 3 we have:

$$\left(\frac{r}{a}\frac{r'}{a'}\Psi\right)^2 \approx 0\,.$$

We have:

$$\Delta^{-s} = \left(r^2 + r'^2 - 2rr'\cos\psi\right)^{-s/2}$$

$$\equiv \left[r^2 + r'^2 - 2rr'\ \Psi + \cos(\theta - \theta')\right]^{-s/2}$$

$$\equiv \left\{\left[r^2 + r'^2 - 2rr'\cos\left(\theta - \theta'\right)\right] - 2rr'\Psi\right\}^{-s/2}$$

$$\equiv \left[\Delta_0^2 - 2rr'\Psi\right]^{-s/2} = \Delta_0^{-s}\left[1 - \frac{2rr'\Psi}{\Delta_0^2}\right]^{-s/2}$$

$$= \Delta_0^{-s}\sum_{k=0}^{\infty}\frac{(rr'\Psi)^k}{k!\Delta_0^{2k}}\prod_{l=0}^{k-1}(s+2l) = \sum_{k=0}^{\infty}\prod_{l=0}^{k-1}(s+2l)\frac{(rr'\Psi)^k}{k!}\Delta_0^{-(2k+s)}\,,$$

We use the formula:

$$(1-u)^{-\frac{s}{2}} = \int_{k=0}^{\infty}\frac{u^k}{2^k k!}\int_{l=0}^{k-1}(s+2l),$$

with

$$u = \frac{2rr'\Psi}{\Delta_0^2}, \quad \left(\frac{u}{2}\right)^k = (rr'\Psi)^k\,,$$

where

$$\prod_{l=0}^{-1}(s+2l) = 1\,,$$

so we have:

$$(1-u)^{-s/2} = 1 + \frac{su}{2(1!)} + \frac{s\,(s+2)}{(2!)}\left(\frac{u}{2}\right)^2 + \ldots\,,$$

with

$$\Delta_0^2 = \left[r^2 + r'^2 - 2rr'\cos\left(\theta - \theta'\right)\right]\,,$$

$$\Delta_0^{-(2k+s)} = \left[r^2 + r'^2 - 2rr'\cos\left(\theta - \theta'\right)\right]^{-(2k+s)/2}\,,$$

$$\Psi = \cos\psi - \cos\left(\theta - \theta'\right)$$

and

$$(\theta - \theta') = (\varpi + f) - (\varpi' + f') = (\varpi - \varpi') + (f - f')$$

whence after some algebra:

$$\Delta^{-s} = \left(\frac{1}{\Delta}\right)^s = \sum_{k=0}^{\infty}\prod_{l=0}^{k-1}(s+2l)\frac{(rr'\Psi)^k}{k!}\left(\frac{1}{\Delta_0}\right)^{(2k+s)}\,,$$

with

$$\prod_{l=0}^{-1}(s+2l)=1\,,$$

where

$$\frac{1}{\Delta_0}=\left[r^2+r'^2-2rr'\cos(\theta-\theta')\right]^{-1/2}$$

We remark that Ψ is of order 2 at least with respect to γ and γ'. Furthermore the Ψ terms of order 2 depend on the inclinations not on the eccentricities, so $(rr'\Psi)$ is of order 2 w.r.t. γ and γ'.

Hence to order 3, $k=0$ and 1:

$$\left(\frac{1}{\Delta}\right)^s=\sum_{k=0}^{1}\frac{(rr'\Psi)^k}{k!}\left(\frac{1}{\Delta_0}\right)^{(2k+s)}\prod_{l=0}^{k-1}(s+2l)\,,$$

$$\left(\frac{1}{\Delta}\right)^s=\frac{1}{\Delta_0}^s+(rr'\Psi)\left(\frac{1}{\Delta_0}\right)^{(2+s)}s=\frac{1}{\Delta_0{}^s}\left[1+s\,(rr'\Psi)\left(\frac{1}{\Delta_0}\right)^2\right]$$

Therefore, we have:

$$\left(\frac{1}{\Delta_0}\right)^{2k+s}=\left[r^2+r'^2-2rr'\cos(\theta-\theta')\right]^{-(2k+s)/2}\,.$$

Let

$$\frac{1}{\rho_0}=\left[a^2+a'^2-2aa'\cos(\theta-\theta')\right]^{-1/2}$$

$$=a'^{-1}\left[1+\alpha^2-2\alpha\cos(\theta-\theta')\right]^{-1/2}\,,\quad \alpha=\frac{a}{a'}\,,$$

so we have:

$$\left(\frac{1}{\rho_0}\right)^{2k+s}=a'^{-(2k+s)}\left[1+\alpha^2-2\alpha\; cos\,(\theta-\theta')\right]^{-(k+s/2)}$$

Applying Taylor's series expansion in ρ_0, we may write

$$\frac{1}{\Delta_0}=\sum_{i=0}^{\infty}\frac{1}{i!}\left[(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}\right]^i\left(\frac{1}{\rho_0}\right)$$

We have also:

$$\left(\frac{1}{\Delta_0}\right)^{(2k+s)}=\sum_{i=0}^{\infty}\frac{1}{i!}\left[(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}\right]^i\left(\frac{1}{\rho_0}\right)^{(2k+s)}$$

Then we deduce:

$$\left(\frac{1}{\Delta}\right)^s=$$

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(rr'\Psi)^k}{k!}\frac{\prod_{l=0}^{k-1}(2.l+s)}{i!}\left\{\left[(r-a)\frac{\partial}{\partial a}+(r'-a')\frac{\partial}{\partial a'}\right]^i\left(\frac{1}{\rho_0}\right)^{(2k+s)}\right\}$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{(rr'\Psi)^k}{k!i!} \frac{\prod_{l=0}^{k-1}(2l+s)}{i!} \left[(r-a)\frac{\partial}{\partial a} + (r'-a')\frac{\partial}{\partial a'}\right]^i$$

$$\left[a^2 + a^{'2} - 2aa'\cos(\theta-\theta')\right]^{-(2k+s)/2}$$

But, we have:

$$\left(\frac{1}{\rho_0}\right)^v = \left[a^2 + a^{'2} - 2aa'\cos(\theta-\theta')\right]^{-v/2} = a^{'-v}\left[1+\alpha^2 - 2\alpha\cos(\theta-\theta')\right]^{-v/2} =$$

$$= \frac{a^{'-v}}{2}\sum_{j=-\infty}^{\infty} b_{v/2}^{(j)}\cos[j(\theta-\theta')] = \sum_{j=-\infty}^{\infty} B_{v/2}^{(j)}\cos[j(\theta-\theta')]\,,$$

with

$$B_{v/2}^{(j)} = \frac{a^{'-v}}{2} b_{v/2}^{(j)}\,,$$

where $b_{v/2}^{(j}$ is a function of $\alpha = a/a'$, for all j. We have in particular

$$B_{(2k+s)/2}^{(j)} = \frac{a^{'-(2k+s)}}{2} b_{(2k+s)/2}^{(j)}(\alpha)$$

for all j with:

$$\left(\frac{1}{\rho_0}\right)^{2k+s} = a'^{-(2k+s)}\left[1+\alpha^2 - 2\alpha\cos(\theta-\theta')\right]^{-(2k+s)/2}$$

$$= \frac{a'^{-(2k+s)}}{2}\sum_{j=-\infty}^{\infty} b_{(2k+s)/2}^{(j)}(\alpha)\cos[j(\theta-\theta')]$$

Then we deduce:

$$\left(\frac{1}{\Delta}\right)^s = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty} \frac{(rr'\Psi)^k}{k!i!}\sum_{l=0}^{k-1}(2l+s)\cdot$$

$$\cos[j(\theta-\theta')]\left[(r-a)\frac{\partial}{\partial a} + (r'-a')\frac{\partial}{\partial a'}\right]^i B_{(2k+s)/2}^{(j)}$$

or

$$\left(\frac{1}{\Delta}\right)^s = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty}\prod_{l=0}^{k-1}(2l+s)\frac{(rr'\Psi)^k}{k!i!}\cos[j(\theta-\theta')]\cdot$$

$$\cdot\left[(r-a)\frac{\partial}{\partial a} + (r'-a')\frac{\partial}{\partial a'}\right]^i \left[\frac{a'-(2k+s)}{2} b_{\frac{2k+s}{2}}^{(j)}(\alpha)\right]$$

But we can write, for any function $c(a,a')$:

$$\left[(r-a)\frac{\partial}{\partial a} + (r'-a')\frac{\partial}{\partial a'}\right]^i c(a,a')$$

$$= \sum_{p=0}^{i} C_i^p (r-a)^p (r'-a')^{i-p} \frac{\partial^p}{\partial a^p}\left[\frac{\partial^{i-p}}{\partial {a'}^{i-p}} c\,(a,a')\right],$$

where C_i^p= i!/p!/(i-p)!. So we have:

$$\left(\frac{1}{\Delta}\right)^s = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty} \frac{\prod_{l=0}^{k-1}(2l+s)\,(rr'\Psi)^k}{k!i!}\cos\left[j\,(\theta-\theta')\right].$$

$$\sum_{p=0}^{i} C_i^p (r-a)^p (r'-a')^{i-p} \frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial {a'}^{i-p}}\left[\frac{{a'}^{-(2k+s)}}{2} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right],$$

$$\left(\frac{1}{\Delta}\right)^s = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty} \frac{\prod_{l=0}^{k-1}(2l+s)\,(rr'\Psi)^k}{k!i!}\cos\left[j(\theta-\theta')\right]\sum_{p=0}^{i} C_i^p \varepsilon^p {\varepsilon'}^{i-p} a^p {a'}^{i-p}$$

$$\frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial {a'}^{i-p}}\left[\frac{{a'}^{-2k+s)}}{2} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right]$$

with $\varepsilon = r/a - 1$, $\varepsilon' = r'/a' - 1$.

Finally, we obtain:

$$\left(\frac{1}{\Delta}\right)^s = \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty}\int_{i=0}^{\infty}\int_{p=0}^{i} a^p {a'}^{i-p} \frac{\prod_{l=0}^{k-1}(2l+s)\varepsilon^p {\varepsilon'}^{i-p}\,(rr'\Psi)^k \cos\left[j(\theta-\theta')\right]}{k!p!(i-p)!}.$$

$$\frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial {a'}^{i-p}}\left[\frac{{a'}^{-(2k+s)}}{2} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right],$$

where

$$\frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial {a'}^{i-p}}\left[\frac{{a'}^{-(2k+s)}}{2} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right] = \frac{\partial^{i-p}}{\partial {a'}^{i-p}}\left[\frac{{a'}^{-2k+p+s)}}{2}\frac{d^p}{d\alpha^p} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right]$$

$$= \sum_{l=0}^{i-p} C_{i-p}^l \prod_{m=1}^{l}(2k+s+p+m)\left[\frac{{a'}^{-(2k+p+s+l)}}{2}\right]\frac{\partial^{i-p-l}}{\partial {a'}^{i-p-l}}\left[\frac{d^p}{d\alpha^p} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right].$$

We remark that:

$$\frac{\partial}{\partial a'}\left[\frac{d^p}{d\alpha^p} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right] = -\frac{\alpha}{a'}\left[\frac{d^{p+1}}{d\alpha^{p+1}} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right]$$

$$\frac{\partial^2}{\partial {a'}^2}\left[\frac{d^p}{d\alpha^p} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right] = \frac{1}{{a'}^2}\left[2\alpha\frac{d^{p+1}}{d\alpha^{p+1}} b^{(j)}_{\frac{2k+s}{2}}(\alpha) + \alpha^2\frac{d^{p+1}}{d\alpha^{p+1}} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right],$$

$$\frac{\partial^3}{\partial {a'}^3}\left[\frac{d^p}{d\alpha^p} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right] =$$

$$-\frac{1}{{a'}^3}\left[6\alpha\frac{d^{p+1}}{d\alpha^{p+1}} b^{(j)}_{\frac{2k+s}{2}}(\alpha) + 6\alpha^2\frac{d^{p+2}}{d\alpha^{p+2}} b^{(j)}_{\frac{2k+s}{2}}(\alpha) + \alpha^3\frac{d^{p+3}}{d\alpha^{p+3}} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right],\ \ldots$$

The expression of $\left(\frac{1}{\Delta}\right)^s$ can be approximated by the truncated series:

$$g(s, i_m, j_m, k_m) =$$

$$\sum_{i=0}^{i_m}\sum_{k=0}^{k_m}\sum_{j=-j_m}^{j_m}\sum_{p=0}^{i} a^p a'^{i-p} \frac{\prod_{l=0}^{k-1}(2l+s)\varepsilon^p \varepsilon'^{i-p}(rr'\Psi)^k}{k!p!(i-p)!} \cos\left[j(\theta-\theta')\right] ,$$

$$\frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial a'^{i-p}}\left[\frac{a'^{-(2k+s)}}{2} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right] .$$

In particular, we have for $s = 1$, since $\prod_{l=0}^{k-1}(2l+1) = \frac{(2k)!}{(k!)2^k}$:

$$\frac{1}{\Delta} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{p=0}^{i} a^p a'^{i-p} \frac{(2k)!\varepsilon^p \varepsilon'^{i-p}\left(rr'\Psi/2\right)^k}{(k!)^2 p!(i-p)!} \cos\left[j(\theta-\theta')\right] .$$

$$\frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial a'^{i-p}}\left[\frac{a'^{-(2k+1)}}{2} b^{(j)}_{k+\frac{1}{2}}(\alpha)\right] ,$$

$$\frac{1}{\Delta} \cong g(1, i_m, j_m, k_m) =$$

$$\sum_{j=-j_m}^{j_m}\sum_{k=0}^{k_m}\sum_{i=0}^{i_m}\sum_{p=0}^{i} a^p a'^{i-p} \frac{(2k)!\varepsilon^p \varepsilon'^{i-p}(rr'\Psi/2)^k \cos\left[j(\theta-\theta')\right]}{k!p!(i-p)!}$$

$$\frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial a'^{i-p}}\left[\frac{a'^{-(2k+1)}}{2} b^{(j)}_{k+\frac{1}{2}}(\alpha)\right]$$

$$= \sum_{j=-j_m}^{j_m}\sum_{k=0}^{k_m}\sum_{i=0}^{i_m}\sum_{p=0}^{i}$$

$$\frac{(2k)!a^{p+k}a'^{i-p+k}}{(k!)^2 p!(i-p)!}\left(\frac{rr'\Psi}{2aa'}\right)^k \varepsilon^p\varepsilon'^{i-p}\frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial a'^{i-p}}\left[\frac{a'^{-(2k+1)}}{2} b^{(j)}_{k+\frac{1}{2}}(\alpha)\right] \cos\left[j(\theta-\theta')\right]$$

where:

$$\frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial} a'^{i-p}\left[\frac{a'^{-(2k+1)}}{2} b^{(j)}_{k+\frac{1}{2}}(\alpha)\right] = \frac{\partial^{i-p}}{\partial a'^{i-p}}\frac{a'^{-(2k+p+1)}}{2}\frac{d^p}{d\alpha^p}\left[b^{(j)}_{k+\frac{1}{2}}(\alpha)\right] = \dots$$

Consequently, to have an approximation of $\left(\frac{1}{\Delta}\right)^s$ to order six, we must proceed as follows.

1. We calculate, to order 6, the expression of the product

$$r_c[k,j] = \left(rr'\Psi\right)^k \cos\left[j(\theta-\theta')\right] .$$

2. We calculate:

$$pr_j(s,i,,k,p) = \frac{\prod_{l=0}^{k-1}(2l+s)}{p!k!(i-p)!} a^p {a'}^{i-p} \varepsilon^p {\varepsilon'}^{i-p} \left[\frac{\partial^p}{\partial a^p} \frac{\partial^{i-p}}{\partial {a'}^{i-p}} B^{(j)}_{k+\frac{s}{2}} \right]$$

$$= \alpha^p \frac{\prod_{l=0}^{k-1}(2l+s)}{p!k!(i-p)!} \varepsilon^p {\varepsilon'}^{i-p} {a'}^i \left[\frac{\partial^{i-p}}{\partial {a'}^{i-p}} \left[{a'}^{-p} \frac{d^p}{d\alpha^p} \right] B^{(j)}_{k+\frac{s}{2}} \right]$$

with $\varepsilon = \frac{r}{a} - 1$ and $\varepsilon' = \frac{r'}{a'} - 1$. We remark that $r_c[k,j]$ is of order $(2k)$ because $\cos[j(\theta - \theta')]$ contains a constant term (independent of parameters e, e', and γ nad γ'). The product $\varepsilon^p \, \varepsilon'^{i-p}$ is of order i because ε and ε' are of order one with respect to e and e' respectively.

Consequently the product $pc_j[s,i,k,p] = r_c[k,j] pr_j(s,i,k,p)$ is of order $(2k + i)$.

To have the terms of order 2, we must take $0 \leq 2k + i \leq 2$, with $0 \leq p \leq i$. So the only terms of order ≤ 2 are

$$pc_j(s,0,0,0), \quad pc_j(s,0,1,0), \quad pc_j(s1,0,0), \quad pc_j(s,1,0,1), \quad pc_j(s,2,0,0),$$
$$pc_j(s,2,0,1) \quad \text{and} \quad pc_j(s,2,0,2)\,.$$

3. We calculate

$$g(s,i_m,j_m,k_m) = \sum_{i=0}^{i_m} \sum_{k=0}^{k_m} \sum_{j=-j_m}^{j_m} \sum_{p=0}^{i} pc_j[s,i,k,p] = \sum_{j=-j_m}^{j_m} g_j(s,i_m,\ k_m),$$

with

$$g_j(s,i_m,\ k_m) = \sum_{i=0}^{i_m} \sum_{k=0}^{k_m} \sum_{p=0}^{i} pc_j[s,i,k,p]$$
$$= \sum_{i=0}^{i_m} \sum_{k=0}^{k_m} \sum_{p=0}^{i}$$
$${a'}^i . \alpha^p \frac{\prod_{l=0}^{k-1}(2l+s)}{p!k!(i-p)!} \varepsilon^p {\varepsilon'}^{i-p} \left[\frac{\partial^p}{\partial a^p} \frac{\partial^{i-p}}{\partial {a'}^{i-p}} B^{(j)}_{k+\frac{s}{2}} \right] (rr'\Psi)^k \cos[j(\theta - \theta')]$$

where

$$pc_j[s,i,,k,p] = pr_j[s,i,k,p] \, r_c[k,j]$$
$$= {a'}^i \alpha^p \frac{\prod_{l=0}^{k-1}(2l+s)}{p!k!(i-p!} \varepsilon^p {\varepsilon'}^{i-p} \left[\frac{\partial^p}{\partial a^p} \frac{\partial^{i-p}}{\partial {a'}^{i-p}} B^{(j)}_{k+\frac{s}{2}} \right] (rr'\Psi)^k \cos[j(\theta - \theta')]$$

It is sufficient to take $k_m = 3$, to obtain the series to order 6, since $r_c[k,j] = 0$, for $k > 3$.

[We will not use the expansion of Laplace coefficients $B(s,j)$ with respect to $\alpha = a/a'$. These coefficients will appear with their derivatives in the coefficients of the cosine terms as well as e and γ. When we require numerical results we should assign the per limit of power of alpha, according to the number of decimals we desire.]

On the other hand, $i_m = 6$, $i = 0, 1, 2, 3, \ldots, 6$. Hence the approximation of $\left(\frac{1}{\Delta}\right)^s$ to order six is $g(s, 6, 3) = \sum_{j=-j_m}^{j_m} g_j(s, 6, 3)$, with

$$g_j(s,6,3) = \sum_{i=0}^{6}\sum_{k=0}^{3}\sum_{p=0}^{i} pr_j[s,i,k,p]r_c[k,j] = \sum_{i=0}^{6}\sum_{k=0}^{3}\sum_{p=0}^{i} pc_j[s,i,k,p]\,,$$

where $pc_j[s, i, k, p] = pr_j[s, i, k, p]\, r_c[k, j]$

$$= a^p {a'}^{i-p} \frac{\prod_{l=0}^{k-1}(2l+s)\varepsilon^p {\varepsilon'}^{i-p}(rr'\Psi)^k \cos\left[j(\theta-\theta')\right]}{k!p!(i-p)!} \frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial {a'}^{i-p}} \cdot$$

$$\left[\frac{{a'}^{-(2k+s)}}{2} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right]$$

$$= \alpha^p {a'}^{i} \frac{\prod_{l=0}^{k-1}(2l+s)\varepsilon^p {\varepsilon'}^{i-p}(rr'\Psi)^k \cos\left[j(\theta-\theta')\right]}{k!p!(i-p)!} \frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial} {a'}^{i-p} \cdot$$

$$\left[\frac{{a'}^{-(2k+s)}}{2} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right]$$

$$= \alpha^p {a'}^{i} \frac{\prod_{l=0}^{k-1}(2l+s)\varepsilon^p {\varepsilon'}^{i-p}(rr'\Psi)^k \cos\left[j(\theta-\theta')\right]}{k!p!(i-p)!} \frac{\partial^{i-p}}{\partial {a'}^{i-p}} \cdot$$

$$\left[\frac{{a'}^{-(2k+p+s)}}{2} \frac{d^p}{d\alpha^p} b^{(j)}_{\frac{2k+s}{2}}(\alpha)\right]$$

3. Derivatives of $B(a, a')$ to order 6

In the following, we have the derivatives of B, with respect to a and a', calculated by using the derivatives of b as a function of α:

$$B(a, a') = \frac{{a'}^{-(2k+s)}}{2} b(\alpha)$$

$$\frac{\partial^i B}{\partial a^i} = \frac{{a'}^{-(2k+s+i)}}{2}\frac{d^i b(\alpha)}{d\alpha^i}, \quad \forall\, i = 1, 2, \ldots$$

$$\frac{\partial B}{\partial a'} = \frac{{a'}^{-(2k+s)}}{2}\left[-\frac{(2k+s)b}{a'} - \frac{a}{{a'}^2}\frac{db(\alpha)}{d\alpha}\right] = -\frac{{a'}^{-(2k+s+1)}}{2}\left[(2k+s)\,b + \alpha\frac{db(\alpha)}{d\alpha}\right]\,,$$

$$\frac{\partial}{\partial a}\left(\frac{\partial B}{\partial a'}\right) = -\frac{{a'}^{-(2k+s+2)}}{2}\left[(2k+s+1)\,\frac{db(\alpha)}{d\alpha} + \alpha\frac{d^2 b(\alpha)}{d\alpha^2}\right]\,,$$

$$\frac{\partial^2 B}{\partial {a'}^2} = \frac{{a'}^{-(2k+s+2)}}{2}\left[(2k+s)(2k+s+1)b + 2(2k+s+1)\alpha\frac{db(\alpha)}{d\alpha} + \alpha^2\frac{d^2 b(\alpha)}{d\alpha^2}\right]$$

$$\frac{\partial}{\partial a}\left(\frac{\partial^2 B}{\partial a'^2}\right) = \frac{a'^{-(2k+s+3)}}{2}\left[(2k+s+1)(2k+s+2)\frac{db(\alpha)}{d\alpha}\right.$$

$$\left.+2(2k+s+2)\alpha\frac{d^2b(\alpha)}{d\alpha^2} + \alpha^2\frac{d^3b(\alpha)}{d\alpha^3}\right]$$

$$\frac{\partial^2}{\partial a^2}\left(\frac{\partial B}{\partial a'}\right) = -\frac{a'^{-(2k+s+3)}}{2}\left[(2k+s+2)\frac{d^2b(\alpha)}{d\alpha^2} + \alpha\frac{d^3b(\alpha)}{d\alpha^3}\right]$$

$$\frac{\partial^3 B}{\partial a'^3} = -\frac{a'^{-(2k+s+3)}}{2}\left\{(2k+s)\quad(2k+s+1)(2k+s+2)b\right.$$

$$+3(2k+s+1)(2k+s+2)\alpha\frac{db(\alpha)}{d\alpha}$$

$$\left.+3(2k+s+2)\alpha^2\frac{d^2b(\alpha)}{d\alpha^2} + \alpha^3\frac{d^3b(\alpha)}{d\alpha^3}\right\}$$

$$\frac{\partial^3}{\partial a^3}\left(\frac{\partial B}{\partial a'}\right) = -\frac{a'^{-(2k+s+4)}}{2}\left[(2k+s+3)\frac{d^3b(\alpha)}{d\alpha^3} + \alpha\frac{d^4b(\alpha)}{d\alpha^4}\right]$$

$$\frac{\partial^2}{\partial a^2}\left(\frac{\partial^2 B}{\partial a'^2}\right) = \frac{a'^{-(2k+s+4)}}{2}\left[(2k+s+2)(2k+s+3)\frac{d^2b(\alpha)}{d\alpha^2}\right.$$

$$\left.+2(2k+s+3)\alpha\frac{d^3b(\alpha)}{d\alpha^3} + \alpha^2\frac{d^4b(\alpha)}{d\alpha^4}\right],$$

$$\frac{\partial}{\partial a}\left(\frac{\partial^3 B}{\partial a'^3}\right) = -\frac{a'^{-(2k+s+4)}}{2}\left\{(2k+s+1)(2k+s+2)(2k+s+3)\frac{db(\alpha)}{d\alpha}\right.$$

$$+3(2k+s+2)(2k+s+3)\alpha\frac{d^2b(\alpha)}{d\alpha^2}$$

$$\left.+3(2k+s+3)\alpha^2\frac{d^3b(\alpha)}{d\alpha^3} + \alpha^3\frac{d^4b(\alpha)}{d\alpha^4}\right\}$$

$$\frac{\partial^4 B}{\partial a'^4} = \frac{a'^{-(2k+s+4)}}{2}\left\{(2k+s)(2k+s+1)(2k+s+2)(2k+s+3)b\right.$$

$$+4(2k+s+1)(2k+s+2)(2k+s+3)\alpha\frac{db(\alpha)}{d\alpha}$$

$$+6(2k+s+2)(2k+s+3)\alpha^2\frac{d^2b(\alpha)}{d\alpha^2}$$

$$\left.+4(2k+s+3)\alpha^3\frac{d^3b(\alpha)}{d\alpha^3} + \alpha^4\frac{d^4b(\alpha)}{d\alpha^4}\right\}$$

$$\frac{\partial^5 B}{\partial a'^5} = -\frac{a'^{-(2k+s+5)}}{2}\left\{\quad(2k+s)(2k+s+1)(2k+s+2)(2k+s+3)\right.$$

$$(2k+s+4)b(\alpha)$$

$$+5(2k+s+1.(2k+s+2)(2k+s+3)(2k+s+4)\alpha\frac{db(\alpha)}{d\alpha}$$

$$+10(2k+s+2)(2k+s+3)(2k+s+4)\alpha^2\frac{d^2b(\alpha)}{d\alpha^2}$$

$$+10(2k+s+3)(2k+s+4)\alpha^3\frac{d^3b(\alpha)}{d\alpha^3}$$

$$+5(2k+s+4)\alpha^4\frac{d^4b(\alpha)}{d\alpha^4}+\alpha^5\frac{d^5b(\alpha)}{d\alpha^5}\Bigg\}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial^4 B}{\partial a'^4}\right)=\frac{a'^{-(2k+s+5)}}{2}\Bigg\{\quad(2k+s+1)(2k+s+2)(2k+s+3)$$

$$(2k+s+4)\frac{db(\alpha)}{d\alpha}+4(2k+s+2)(2k+s+3)(2k+s+4)\alpha\frac{d^2b(\alpha)}{d\alpha^2}$$

$$+6(2k+s+3)(2k+s+4)\alpha^2\frac{d^3b(\alpha)}{d\alpha^3}$$

$$+4(2k+s+4)\alpha^3\frac{d^4b(\alpha)}{d\alpha^4}+\alpha^4\frac{d^5b(\alpha}{d\alpha^5}\Bigg\}$$

$$\frac{\partial^2}{\partial a^2}\left(\frac{\partial^3 B}{\partial a'^3}\right)=-\frac{a'^{-(2k+s+5)}}{2}\Bigg\{(2k+s+2)(2k+s+3)(2k+s+4)\frac{d^2b(\alpha)}{d\alpha^2}$$

$$+3(2k+s+3)(2k+s+4)\alpha\frac{d^3b(\alpha)}{d\alpha^3}$$

$$+3(2k+s+4)\alpha^2\frac{d^4b(\alpha)}{d\alpha^4}+\alpha^3\frac{d^5b(\alpha)}{d\alpha^5}\Bigg\}$$

$$\frac{\partial^3}{\partial a^3}\left(\frac{\partial^2 B}{\partial a'^2}\right)=\frac{a'^{-(2k+s+5)}}{2}\Bigg\{\quad(2k+s+3)(2k+s+4)\frac{d^3b(\alpha)}{d\alpha^3}$$

$$+2(2k+s+4)\alpha\frac{d^4b(\alpha)}{d\alpha^4}+\alpha^2\frac{d^5b(\alpha)}{d\alpha^5}\Bigg\}$$

$$\frac{\partial^4}{\partial a^4}\left(\frac{\partial B}{\partial a'}\right)=-\frac{a'^{-(2k+s+5)}}{2}\left[(2k+s+4)\frac{d^4b(\alpha)}{d\alpha^4}+\alpha\frac{d^5b(\alpha)}{d\alpha^5}\right]$$

$$\frac{\partial^6 B}{\partial a'^6}=\frac{a'^{-(2k+s+6)}}{2}\Bigg\{\quad(2k+s)(2k+s+1)(2k+s+2)$$

$$(2k+s+3)(2k+s+4)(2k+s+5)b(\alpha)$$

$$+6(2k+s+1)(2k+s+2)(2k+s+3)(2k+s+4)$$

$$(2k+s+5)\alpha\frac{db(\alpha)}{d\alpha}+$$

$$+15(2k+s+2)(2k+s+3)(2k+s+4)(2k+s+5)\alpha^2\frac{d^2b(\alpha)}{d\alpha^2}$$

$$+20(2k+s+3)(2k+s+4)(2k+s+5)\alpha^3\frac{d^3b(\alpha)}{d\alpha^3}$$

$$+15(2k+s+4(2k+s+5)\alpha^4\frac{d^4b(\alpha)}{d\alpha^4}+$$

$$+6(2k+s+5)\alpha^5\frac{d^5b(\alpha)}{d\alpha^5}+\alpha^6\frac{d^6b(\alpha)}{d\alpha^6}\Bigg\}$$

$$\frac{\partial}{\partial a}\left(\frac{\partial^5 B}{\partial {a'}^5}\right)=-\frac{{a'}^{-(2k+s+6)}}{2}\Bigg\{\quad(2k+s+1)(2k+s+2)(2k+s+3)$$

$$(2k+s+4)(2k+s+5)\frac{db(\alpha)}{d\alpha}$$

$$+5(2k+s+2)(2k+s+3)(2k+s+4)(2k+s+5)\alpha\frac{d^2b(\alpha)}{d\alpha^2}$$

$$+10(2k+s+3)(2k+s+4)(2k+s+5)\alpha^2\frac{d^3b(\alpha)}{d\alpha^3}$$

$$10(2k+s+4)(2k+s+5)\alpha^3\frac{d^4b(\alpha)}{d\alpha^4}$$

$$+5(2k+s+5)\alpha^4\frac{d^5b(\alpha)}{d\alpha^5}+\alpha^5\frac{d^6b(\alpha)}{d\alpha^6}\Bigg\}$$

$$\frac{\partial^2}{\partial a^2}\left(\frac{\partial^4 B}{\partial {a'}^4}\right)=$$

$$\frac{{a'}^{-(2k+s+6)}}{2}\Bigg\{(2k+s+2)(2k+s+3)(2k+s+4)(2k+s+5)\frac{d^2b(\alpha)}{d\alpha^2}$$

$$+4\left[(2k+s+3)(2k+s+4)(2k+s+5)\right]\alpha\frac{d^3b(\alpha)}{d\alpha^3}$$

$$+\left[6(2k+s+4)(2k+s+5)\right]\alpha^2\frac{d^4b(\alpha)}{d\alpha^4}$$

$$+4(2k+s+5)\alpha^3\frac{d^5b(\alpha)}{d\alpha^5}+\alpha^4\frac{d^6b(\alpha}{d\alpha^6}\Bigg\}$$

$$\frac{\partial^3}{\partial a^3}\left(\frac{\partial^3 B}{\partial {a'}^3}\right)=-\frac{{a'}^{-(2k+s+6)}}{2}\Bigg\{(2k+s+3)(2k+s+4)(2k+s+5)\frac{d^3b(\alpha)}{d\alpha^3}$$

$$+3(2k+s+4)(2k+s+5)\alpha\frac{d^4b(\alpha)}{d\alpha^4}$$

$$+3(2k+s+5)\alpha^2\frac{d^5b(\alpha)}{d\alpha^5}+\alpha^3\frac{d^6b(\alpha)}{d\alpha^6}\Bigg\}$$

$$\frac{\partial^4}{\partial a^4}\left(\frac{\partial^2 B}{\partial {a'}^2}\right)=\frac{{a'}^{-(2k+s+6)}}{2}\Bigg\{\Bigg[(2k+s+4)(2k+s+5)\frac{d^4b(\alpha)}{d\alpha^4}$$

$$+2(2k+s+5)\quad\Bigg]\alpha\frac{d^5b(\alpha)}{d\alpha^5}+\alpha^2\frac{d^6b(\alpha)}{d\alpha^6}\Bigg\}$$

$$\frac{\partial^5}{\partial a^5}\left(\frac{\partial B}{\partial a'}\right)=-\frac{{a'}^{-(2k+s+6)}}{2}\left[(2k+s+5)\frac{d^5b(\alpha)}{d\alpha^5}+\alpha\frac{d^6b(\alpha)}{d\alpha^6}\right]$$

We use the following notations:

$$db(j,k+s/2,p,i-p) = \frac{\partial^p}{\partial a^p}\frac{\partial^{i-p}}{\partial a'^{i-p}}\left[\frac{a'^{-(2k+s)}}{2} b^{(j)}_{k+\frac{s}{2}}(\alpha)\right]$$

$$b(j,k+\frac{s}{2},\alpha) \equiv dbal(s,j,k,0) \equiv B^{(j)}_{k+\frac{s}{2}}(\alpha) = \left[\frac{a'^{-(2k+s)}}{2} b^{(j)}_{k+\frac{s}{2}}(\alpha)\right], \quad k = 0,1,\ldots,6$$

$$dbal(s,j,k,p) = \frac{\partial^p}{\partial \alpha^p} b^{(j)}_{k+\frac{s}{2}}(\alpha), \quad p = 1,2,\ldots,6; \quad k = 0,1,\ldots,6$$

in particular, we have:

$$dbal(s,j,k,0) \equiv B^{(j)}_{k+\frac{s}{2}}(\alpha) = ,$$

and

$$b\left(j,\frac{1}{2},\alpha\right) \equiv dbal(s,j,0,0) \equiv B^{(j)}_{\frac{s}{2}}(\alpha)\,.$$

4. Concluding remarks

We postulated the most general formulae, for the literal expansion of Δ^{-s}. In analytical planetary theories, s takes the values $1, 3, 5, \ldots$ according to the order in planetary masses. Brouwer and Clemence [5], Brown and Shook [4], Abu El Ata and Chapront [6], Kaula [12], Kamel [7,8], Murray [9], are responsible for low order expansions. Abu El Ata and Chapront eliminated some of the draw backs. Kaula developed methods of construction in terms of standard orbital elements. Murray extended the expansion of the disturbing function by considering $n > 2$, where n is the number of interacting planets. Kamel used the method of differential operators to carry out the expansion of Δ^{-s}, to degree 3, in eccentricity–inclination, and further in terms of Poincare‘ variables L, λ, H, K, P. Soliman [11] developed the expansion of Δ^{-s}, following the method of W.M. Smart, neglecting terms of degree > 4 in eccentricity–inclination.

In our present research paper, we applied Taylor's theory, in our derivation to acquire the formula of Δ^{-s}. In Part II the final explicit expansion, will be represented by a Poisson trigonometric series of cosines, with polynomial variables a, e, I, and harmonic variables λ, ϖ, Ω, after the implementation of a computer algebraic *MACSYMA* program.

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