

## Research Article

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# Finite element model updating using Lagrange interpolation

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**Abstract:** In this paper, an iterative finite element model updating method in structural dynamics is proposed. This uses information matrices and element connectivity matrices to reconstruct the corrected model by reproducing the frequency response at measured degrees of freedom. Indicators have been proposed to quantify the mismodelling errors based on a development in Lagrange matrix interpolation. When applied on simulated truss structures, the model gives satisfactory results by detecting and quantifying the defaults of the initial model.

**Keywords:** model updating, Lagrange interpolation, information matrices, frequency domain

## 1 Introduction

Finite element model updating is a technique that improves the correlation between the measured data and the theoretical prediction. A significant number of methods exist. These methods can be classified into two categories, namely direct methods and iterative methods. Direct methods are also called basic reference methods, and they reproduce the experimental measurement so that it can be used later; these methods do not localize the modelling errors. Iterative methods are also called parametric identification by updating; these methods are more interesting because not only they correct the model, but also they express these corrections by parameters.

The majority of direct methods are developed at the end of the 1970s and the beginning of the 1980s. Thanks to their advantage, these methods are always used and have witnessed major developments. The first developed methods used Lagrange multiplier constraints in the minimization of cost functions [1–3]. Other methods called

error matrix methods [4, 5] are used to estimate directly the errors on the mass and stiffness matrices, and inverse techniques are also exploited in this field [6–9]. Iterative methods are usually cost function methods [10–14] or minimal variation methods [15–17]. In addition to the preceding methods, other specific methods exist. Some methods use genetic algorithms to find global minima or maximum [18, 19], while other methods use perturbed boundary conditions [20–23].

Incomplete experimental measurements in terms of frequency band and a number of degrees of freedom show that updating cannot often be applied to all the parameters of the system. The choice of measurements and the parameters to be corrected as well as the model reduction or expansion method is of primary importance for the success of the updating procedure. Several types of measurements can be used (natural frequencies, Eigen-modes, frequency response functions,...). They have different sensitivities on the updating parameters.

In this paper, a direct updating method using frequency response measurements refined by an iterative system is proposed, which allows detecting and localizing modelling errors.

## 2 Updating method

\*\*\*For a linear damped system, the dynamic characteristics are described by a set of second order differential equations in time domain

$$M_a \ddot{y}_a(t) + D_a \dot{y}_a(t) + K_a y_a(t) = f(t) \quad (1)$$

where  $M_a$ ,  $D_a$  and  $K_a$  are respectively the mass, damping and stiffness real symmetric  $n$  by  $n$  matrices, the response functions  $y_a(t)$ ,  $\dot{y}_a(t)$ ,  $\ddot{y}_a(t)$  are respectively the displacement, velocity and acceleration vectors,  $f(t)$  is the excitation force vector. Index “a” express analytical initial model.

This system can be analysed for its harmonic response properties by assuming that  $y_a(t) = Y_a e^{st}$  and  $f(t) = F e^{st}$  with  $s = i\omega$ . This lead to

$$(s^2 M_a + s D_a + K_a) Y_a = F \quad (2)$$

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Where  $Z_a(s) = (s^2 M_a + s D_a + K_a)$  is the damping stiffness matrix. The frequency response function matrix is written as  $H_a(s) = Z_a^{-1}(s)$

The updating process consists in determining the new updated matrices  $M_u, D_u$  and  $K_u$ , which reproduces the frequency response measurements  $Y_x$ , according to the harmonic response

$$(s^2 M_u + s D_u + K_u) Y_x = F \quad (3)$$

For this purpose consider the relation between the measured and analytical responses

$$Y_a = P(s) Y_x \quad (4)$$

Where  $P$  is an unknown information matrix between the analytical responses and the measured ones. When the initial model is well modeled the two sets of response function are very close and the information matrix is sensibly equal to identity matrix.

Index “x” expresses the measurements, and index “u” expresses the updated components.

Introducing expression Eq. (4) in Eq. (2) and equalizing with Eq. (3) we have

$$F = Z_a(s) P(s) Y_x = Z_u(s) Y_x \quad (5)$$

$$P(s) = Z_a^{-1}(s) Z_u(s) \quad (6)$$

For  $s = 0$ , we have

$$P(0) = Z_a^{-1}(0) Z_u(0) \quad (7)$$

$$P(0) = K_a^{-1} K_u \quad (8)$$

We deduce the updated stiffness matrix

$$K_u = K_a P(0) \quad (9)$$

The first derivative with respect to variable “s” of the information matrix for  $s = 0$  is

$$\dot{P}(0) = \dot{Z}_a^{-1}(0) Z_u(0) + Z_a^{-1}(0) \dot{Z}_u(0) \quad (10)$$

$$\dot{P}(0) = \left( -Z_a^{-1}(0) \dot{Z}_a(0) Z_a^{-1}(0) \right) Z_u(0) + Z_a^{-1}(0) \dot{Z}_u(0) \quad (11)$$

$$\dot{P}(0) = \left( -K_a^{-1} D_a K_a^{-1} \right) K_u + K_a^{-1} D_u \quad (12)$$

From what, we can deduce the updated damping matrix

$$D_u = K_a \left( \dot{P}(0) + \left( K_a^{-1} D_a K_a^{-1} \right) K_u \right) \quad (13)$$

Introducing Eq. (9) in this equation we can finally write the updated damping matrix as a function of the information matrix and its first derivative as follow

$$D_u = K_a \dot{P}(0) + D_a P(0) \quad (14)$$

The expression of the updated mass matrix is derived from the second derivative according to variable “s” of the information matrix  $P$

$$\ddot{P}(0) = \ddot{Z}_a^{-1}(0) Z_u(0) + 2 \dot{Z}_a^{-1}(0) \dot{Z}_u(0) + Z_a^{-1}(0) \ddot{Z}_u(0) \quad (15)$$

After derivation and development the above equation becomes

$$\begin{aligned} \ddot{P}(0) = & -2 K_a^{-1} \left( M_a + D_a K_a^{-1} D_a \right) K_a^{-1} K_u \\ & + 2 \left( -K_a^{-1} D_a K_a^{-1} \right) D_u + 2 K_a^{-1} M_u \end{aligned} \quad (16)$$

From what we can deduce the updated mass matrix

$$\begin{aligned} M_u = & K_a \left[ \frac{1}{2} \ddot{P}(0) + K_a^{-1} \left( M_a + D_a K_a^{-1} D_a \right) K_a^{-1} K_u \right. \\ & \left. + \left( K_a^{-1} D_a K_a^{-1} \right) D_u \right] \end{aligned} \quad (17)$$

This can be expressed in terms of  $P$  and her first and second derivatives as follow

$$M_u = \frac{1}{2} K_a \ddot{P}(0) + D_a \dot{P}(0) + M_a P(0) \quad (18)$$

Finally we have the updated unknown matrices

$$K_u = K_a P(0)$$

$$D_u = K_a \dot{P}(0) + D_a P(0)$$

$$M_u = \frac{1}{2} K_a \ddot{P}(0) + D_a \dot{P}(0) + M_a P(0)$$

The updating method is performed when the information matrix  $P$  and its derivatives  $\dot{P}$  and  $\ddot{P}$  are known. It is well indicated that these matrices are to be determined from the measurement components.

### 3 Expression for the information matrices

The principal idea in this section is to interpolate the information matrices for each s-value using Eq. (4). Suppose that “m” measurements are performed for a set of complex

frequencies  $\{s_0, s_1, s_2, \dots, s_{m-1}, s_m\}$ , we have then “ $m$ ” equation of the form

$$Y_a(s_i) = P(s_i)Y_x(s_i) \quad (19)$$

From what, we can estimate the information matrix for the complex frequency  $s_i$

$$P(s_i) = \left( Y_a(s_i) Y_x^T(s_i) \right) \left( Y_x(s_i) Y_x^T(s_i) \right)^{-1} \quad (20)$$

To avoid ill-conditioned matrices the part  $\left( Y_x(s_i) Y_x^T(s_i) \right)^{-1}$  in Eq. (20) is approximated by  $\left( \tilde{Y}_x \cdot \tilde{Y}_x^T \right)^{-1}$ , where  $\tilde{Y}_x$  is a modulated matrix of all measured responses. This matrix is constructed so that it has maximum rank and minimal condition number. From the matrix of all measured responses the response vector which do not rise the rank of the matrix is eliminated, and in the same way the response vector which perturbs the condition number of the matrix is also eliminated.

The information matrix  $P$  is expressed in terms of  $P(s_i)$  using Lagrange polynomial interpolation applied to  $n$  by  $n$  matrices. The formulation is then

$$P(s) = \sum_{i=1}^m L_i(s) P(s_i) \quad (21)$$

with

$$L_i(s) = \prod_{\substack{j=1 \\ j \neq i}}^m \frac{s - s_j}{s_i - s_j} \quad (22)$$

This leads to final polynomial form of

$$P(s) = s^{m-1} A_{m-1} + s^{m-2} A_{m-2} + \dots + s^2 A_2 + s A_1 + A_0 \quad (23)$$

From what, we can deduce

$$\begin{cases} P(0) = A_0 \\ \dot{P}(0) = A_1 \\ \ddot{P}(0) = 2A_2 \end{cases} \quad (24)$$

This discretization is used instead of the full information matrix which uses simultaneously all the response functions to avoid ill-conditioned problems and bad pseudo-inverse. In fact, in the interpolation process, information matrix for any complex frequency  $s_i$  which poses a bad pseudo inverse will be eliminated and thus the polynomial form of Eq. (23) is cleaned from all ill-conditioned information matrices.

Finally, the updated mass, damping and stiffness matrices are

$$K_u = K_a A_0 \quad (25)$$

$$D_u = K_a A_1 + D_a A_0 \quad (26)$$

$$M_u = K_a A_2 + D_a A_1 + M_a A_0 \quad (27)$$

The updating method is refined by an iterative system which computes and replaces the analytical matrices by the updated matrices at each iteration until  $P(s)$  is equal to identity ( $A_0 = I, A_1 = 0, A_2 = 0$ ).

## 4 Error localization

It is important to localize modelling errors in the initial matrices. This step consists then in the splitting of the updated matrices to compare with the corresponding splitting of the initial matrices. It is well established that the initial global matrices are assembled by a summation of the corresponding mass, damping and stiffness elementary matrices using connectivity elementary matrices. For example the global mass matrix is written

$$M_a = \sum_{i=1}^N C e_i^T M e_i C e_i \quad (28)$$

Where  $C e_i$  are the connectivity elementary  $ne$  by  $n$  matrices,  $M e_i$  are the mass elementary  $ne$  by  $ne$  matrices,  $N$  is the total number of elements in the structure, and  $ne$  is the number of degrees of freedom of each element, knowing that  $n$  is the total number of degrees of freedom of the structure.

The inverse of the assemblage principle is used to split the global matrices to  $N$  elementary sub-matrices  $Mu_i, Du_i$  and  $Ku_i$   $ne$  by  $ne$  respectively for the updated mass, damping and stiffness matrices, and  $Ma_i, Da_i$  and  $Ka_i$  for the initial corresponding matrices. This is summarized by the following equations

$$K u_i = C e_i K_u C e_i^T \quad (29)$$

and  $K a_i = C e_i K_a C e_i^T$

$$D u_i = C e_i D_u C e_i^T \quad (30)$$

and  $C a_i = C e_i D_a C e_i^T$

$$M u_i = C e_i M_u C e_i^T \quad (31)$$

and  $M a_i = C e_i M_a C e_i^T$

It is important to indicate that the splitting initial matrices are different from the elementary matrices; these are simply used to quantify the initial state. Finally for each element of the structure the comparison between the initial

and the updated situation can be performed by quantifying the differences

$$\Delta K_i = Ku_i - Ka_i \quad (32)$$

$$\Delta D_i = Du_i - Da_i \quad (33)$$

$$\Delta M_i = Mu_i - Ma_i \quad (34)$$

To have a scalar quantification of the severity of modelling errors various matrix norms can be used. In our case we use the following form

$$Ek_i = \frac{\left( \sum_j \sum_k^{ne} \Delta K_i(j, k)^2 \right)^{1/2}}{ne^2} \quad (35)$$

Where  $Ek_i$  is the stiffness error in the  $i^{th}$  element. The damping errors  $Ed_i$  and the mass errors  $Em_i$  are also calculated with similar forms.

$$Ed_i = \frac{\left( \sum_j \sum_k^{ne} \Delta D_i(j, k)^2 \right)^{1/2}}{ne^2} \quad (36)$$

$$Em_i = \frac{\left( \sum_j \sum_k^{ne} \Delta M_i(j, k)^2 \right)^{1/2}}{ne^2} \quad (37)$$

## 5 Application

To evaluate the effectiveness of the proposed method we consider the truss structure Figure 1 which is discretized into 16 finite elements and 24 degrees of freedom. This later is considered made of material with Young's modulus  $E=2.1 \times 10^{11}$  Pa and mass density  $\rho=7800$  kg/m<sup>3</sup>. To simulate the real structure, modelling errors are introduced in the sixth element increasing the Young's modulus to  $E_6=2.3 \times 10^{11}$  Pa and in the twelfth element reducing the mass density to  $\rho_{12}=6630$  kg/m<sup>3</sup>. The damping is assumed to be proportional to the mass and stiffness. A vertical time-harmonic transverse force is applied on the junction of beams 13, 14 and 16.

To simulate real measurements, 5% random noise is added to the simulated response functions of the structure.

Figure 2 shows that the introduced errors are localized and quantified. In fact the mass error modelling of 15% is detected in the twelfth element, and the stiffness modelling error of 10% is also detected in the sixth element.

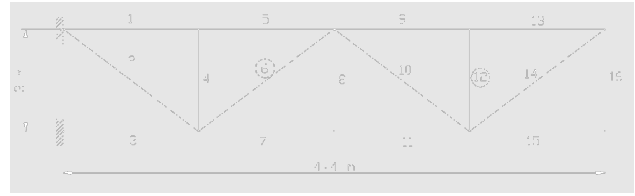


Figure 1: Simulated truss structure

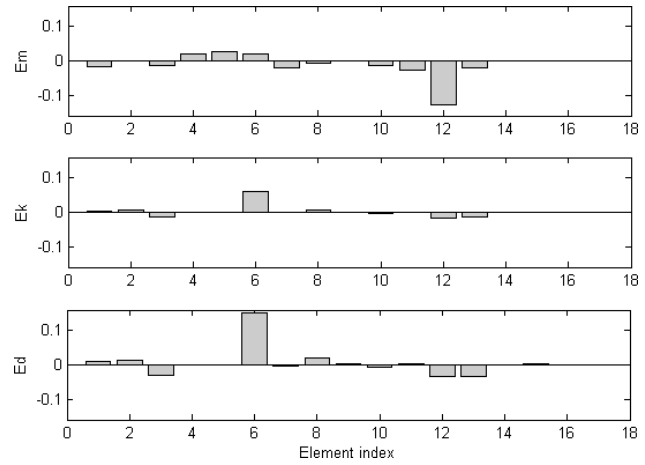


Figure 2: Damage localization

## 6 Conclusion

A global iterative finite element model updating method is proposed using information matrices and Lagrange polynomial interpolation. An iterative procedure is performed to refine the results. To detect the modelling errors or any damage in the structure, the global updated matrices are decomposed by a splitting procedure based on the inverse of the assemblage technique. This uses the connectivity elementary matrices. To avoid bad pseudo-inverse of a full information matrix, this last is parametrized by information matrices at each complex frequency, this allows us to eliminate any ill-conditioned matrix.

This global iterative technique is tested on simulated structures shows interesting results for finite element model updating of damped systems.

## References

- [1] Baruch, M.: Optimisation procedure to correct stiffness and flexibility matrices using vibration data. *AIAA J.*, 16(11) , 208–1210, 1978.
- [2] Berman, A., Nagy, E.J.: Improvement of large analytical model using test data. *AIAA J.*, 21(8) , 1169–2273, 1983.
- [3] Wei, F.S.: Analytical dynamic model improvement using vibra-

- tion test data. *AIAA J.*, 28, 175–177, 1990.
- [4] Gysin, H.P.: Critical application of the error matrix method for localisation of finite element modelling inaccuracies. Proceedings of the 4th International Modal Analysis Conference, 1339–1351, 1986.
- [5] He, J., Ewins, D.J.: Analytical stiffness matrix correction using measured vibration modes. *Modal Anal.* 1(3), 9–14, 1986.
- [6] Starek, L., Ewins, D.J.: A symmetric inverse vibration problem with over damped modes. *J. Sound Vib.*, 181(5), 893–903, 1995.
- [7] Starek, L., Inman, D.J.: On the inverse vibration problem with rigid body modes. *ASME J. Appl. Mech.*, 58(4), 1101–1104, 1991.
- [8] Starek, L., Inman, D.J., Kress, A.: A symmetric inverse vibration problem. *ASME J. Vib. Acoust.*, 114(4), 564–568, 1992.
- [9] Starek, L., Musil, M., Inman, D.J.: Updating of non-conservative systems using inverse methods. Proceedings of the ASME Design Engineering Technical Conference, 41–58, 1997.
- [10] Link, M., Zhang, L.: Experience with different procedures for updating structural parameters of analytical models using test data. Proceedings of the 10th International Modal Analysis Conference, 730–738, 1992.
- [11] Link, M., Rohrmann, R.G., Pietrzko, S.: Experience with automated procedures for adjusting the finite element model of a complex highway bridge to experimental modal data. Proceedings of the 14th International Modal Analysis Conference, 218–225, 1996.
- [12] Natke, H.G.: On regularisation methods applied to the error localisation of mathematical models. Proceedings of the 9th International Modal Analysis Conference, 70–73, 1991.
- [13] Visser, W.J., Imregun, M.: A technique to update finite element models using frequency response data. Proceedings of the 9th International Modal Analysis Conference, 462–468, 1991.
- [14] Imregun, M., Visser, W.J., Ewins, D.J.: Finite element model updating using frequency response function data – I. theory and initial investigation. *Mech. Syst. Signal Pr.*, 9(2), 187–202, 1995.
- [15] Reix, C., Gerard, A., Tombini, C.: Updating the damping matrix using frequency response data. Proceedings of the ASME Design Engineering Technical Conferences, 803–808, 1995.
- [16] Lindholm, B.E., West, R.L.: Finite element model reconciliation with dynamic response data using a statistics-based parameter-update approach. Proceedings of the 13th International Modal Analysis Conference, 1540–1546, 1995.
- [17] Mares, C., Friswell, M.I., Mottershead, J.E.: Model updating using robust estimation. *Mech. Syst. Signal Pr.*, 16(1), 169–183, 2002.
- [18] Larson, C.B., Zimmerman, D.C.: Structural model refinement using a genetic algorithm approach. Proceedings of the 11th International Modal Analysis Conference, 1095–1101, 1993.
- [19] Carlin, R.A., Garcia, E.: Parameter optimisation of a genetic algorithm for structural damage detection. Proceedings of the 14th International Modal Analysis Conference, 1292–1298, 1996.
- [20] Yang, M., Allemang, R., Brown, D.: Practical aspects of perturbed boundary condition (PBC) finite element model updating techniques. *SAE Transactions*, 106 (6), 2836–2844, 1997.
- [21] Barney, P., St. Pierre, M., Brown, D.: Identification of a modal model utilising a perturbed boundary condition test method. Proceedings of 10th International Modal Analysis Conference, 1221–1231, 1992.
- [22] Chen, K.N., Brown, D., Nicolas, V.T.: Perturbed boundary condition model updating. Proceedings of the 11th International Modal Analysis Conference, 661–667, 1993.
- [23] Li, S., Shelley S., Brown, D.: Perturbed boundary condition testing concepts. Proceedings of the 13th International Modal Analysis Conference, 902–911, 1995.