

Research Article

Amit K. Thawait*, Lakshman Sondhi, Shubhashis Sanyal, and Shubhankar Bhowmick

Stress and Deformation Analysis of Clamped Functionally graded Rotating Disks with Variable Thickness

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Abstract: The present study reports the linear elastic analysis of variable thickness functionally graded rotating disks. Disk material is graded radially by varying the volume fraction ratios of the constituent components. Three types of distribution laws, namely power law, exponential law and Mori–Tanaka scheme are considered on a concave thickness profile rotating disk, and the resulting deformation and stresses are evaluated for clamped-free boundary condition. The investigation is carried out using element based grading of material properties on the discretized elements. The effect of grading on deformation and stresses is investigated for each type of material distribution law. Further, a comparison is made between different types of distributions. The results obtained show that in a rotating disk, the deformation and stress fields can be controlled by the distribution law and grading parameter n of the volume fraction ratio.

Keywords: Functionally graded material (FGM), linear elastic analysis, rotating disk of variable thickness, element based material gradation

1 Introduction

Components made of functionally graded materials (FGMs) are widely used in space vehicles, aircrafts, nuclear power plants and many other engineering applications. FGMs are special composites with continuous spatial variations of physical and mechanical properties. A

rotating disk, made up of such an FGM is widely used in the field of aerospace, mechanical and marine industry for machines and machine elements like gas turbines, gears and flywheels. In rotating disks, the centrifugal load produces deformation and stresses, thus limiting the application range, and as a result, these need to be constrained by varying the material property and thickness of the disk. Disks made up of functionally graded materials and of variable thickness have significant stress reduction over the disks made up of homogeneous material and of uniform thickness. Therefore, a higher limit speed and higher pressure is permissible for FGM disks.

A few researchers have reported their work on the analysis of FGM disks, plates, shells, beams and bars using the analytical and finite element method. Gupta *et al.* [1] reported creep stress and strain rates for a rotating thin annulus of varying density based on Seth's transition theory. Based on Tresca's yield criterion under plane stress assumption, Eraslan [2] presented analytical solutions for stresses in elastic–plastic regime of rotating parabolic disks. Using von Mises yield criterion, J2–deformation theory and nonlinear isotropic hardening, Eraslan *et al.* [3] conducted the parametric elastic–plastic analysis of rotating annular disks of variable thickness under pressurized boundary condition. Ramanjaneyulu *et al.* [4] have studied the effect of taper stiffeners and material properties on critical speeds of very thin spinning disc. Analysis was carried out for different numbers of stiffeners and variable geometry of disk under clamped-free boundary condition. Stump *et al.* [5] explained the topology optimization framework for material distribution of an FG rotating disk under mechanical stress constraints. The scheme of functionally graded material distribution is based on the material model derived using the Hashin-Strikhman upper and lower bounds.

Bayat *et al.* [6] reported the analysis of variable thickness FGM rotating disk with power law property distribution and the disk is subjected to both mechanical and thermal loads. Afsar *et al.* [7] analyzed a rotating FGM circular disk subjected to thermo-mechanical load using finite element method. The disk has exponentially varying

*Corresponding Author: Amit K. Thawait: Institute of Technology, Guru Ghasidas Vishwavidyalaya, Bilaspur, 495009, Chhattisgarh, India; Email: amkthawait@gmail.com; Tel.: +91 8103180085

Lakshman Sondhi: Shri Shankaracharya Technical Campus (SSGI), Bhilai, 490020, Chhattisgarh, India

Shubhashis Sanyal, Shubhankar Bhowmick: National Institute of Technology, Raipur, 492010, Chhattisgarh, India

material properties in radial direction. The disk is subjected to a thermal load along with centrifugal load due to non-uniform temperature distribution. Zenkour *et al.* [8] reported numerical solution for varying thickness annular disk rotating under clamped–clamped condition. Callioglu *et al.* [9] studied the internally pressurized rotating FG annular disk under different temperature distributions and in another work, [10] reported the analytical thermo-elastic solution for FG disk. Bayat *et al.* [11] studied the displacement and stresses in a rotating FG disk of varying thickness using the semi-analytical method. Radially varying one-dimensional FGM is taken and material properties vary according to the power law and Mori–Tanaka scheme. In another work, Callioglu *et al.* [12] reported the effect of power law material property distribution on stress and displacement in thin FGM disks. For two different thickness profiles of annular disks, Zenkour *et al.* [13] presented the exact analytical and numerical solutions based on Runge–Kutta method under free–free boundary conditions.

Sharma *et al.* [14] reported the finite element method (FEM) based two-dimensional thermo-elastic results for a thin circular rotating disk subjected to a thermal load. In another work, Sharma *et al.* [15] studied stresses, displacements and strains in a thin circular (FGM) disk using FEM under thermo-mechanical loading. Ali *et al.* [16] reported their study on the elastic analysis of two sigmoid FGM rotating disks, wherein metal–ceramic–metal disk is analyzed for both uniform and variable thickness disks, and the effect of grading index on the displacement and stresses are reported. In the case of a three-layered (perfectly bonded) composite rotating elastic stresses are reported by Peng *et al.* [17] for two cases: firstly, considering FGM with power-law gradient; and in second, FGM with radial non-homogeneity by reducing the problem into a Fredholm integral equation. Using VMP method, Jahromi *et al.* [18] presented the elasto-plastic stresses and effect of metal–ceramic grading patterns and relative densities and elastic moduli on the stresses in a rotating FG disk. Maruthi *et al.* [19] reported a finite element model to predict over-speed and burst-margin limits of aero-engine turbine disc rotating in the range of 10,000 rpm to 22,000 rpm at different temperatures. In a recent work, Nejad *et al.* [20] proposed the analytical solution of an exponentially varying FG disk under internal and external pressure boundary conditions. Shariyat *et al.* [21] reported 3D bending and stress analyses of rotating annular disks having functional grading in two directions using Hermite elements.

Jabbari *et al.* [22] developed an analytical solution for computing plastic stresses and critical angular speed in FG solid and annular disks assuming power law material property distribution. Tutuncu *et al.* [23] analyzed

the FG rotating disks of varying thickness under non-uniform temperature difference using the complementary functions method. Nejad *et al.* [24] obtained an exact solution for the elastic stresses in FG rotating disk assuming exponential material property variation under internal and external pressure boundary conditions. In a recent work, Zafarmand *et al.* [25] presented a 2D elastic analysis of annular and solid FG rotating disks of non-uniform thickness based on graded finite element method (GFEM). Rosyid *et al.* [26] reported the solution of rotating, nonhomogeneous disk of arbitrary thickness for three types of grading law namely power law, sigmoid and exponential distribution law. For similar material distribution, Bhandari *et al.* [27] investigated the behavior of functionally graded plates and reported parametric studies for varying volume fraction distributions and boundary conditions. Assuming nonlinear elasticity, Zafarmand *et al.* [28] investigated the FG single-walled carbon nanotubes reinforced thick rotating disks of varying thickness, wherein the nonlinear governing equations are derived and solved using nonlinear graded finite element method (NGFEM). Based on Galerkin's error minimization principle, Sondhi *et al.* [29] investigated the stress and deformation state of FG rotating annular disks of constant thickness. Thawait *et al.* [30, 31] studied the FGM disks having exponential material property variation subjected to clamped-free boundary condition and reported the effect of the thickness variation on stress and deformation for different geometries of constant mass.

In the present work, rotating disks of parabolic concave thickness profile and of different grading distribution functions are analyzed. The distribution functions of material grading along the radial direction considered in this study are power law function, exponential function and Mori–Tanaka scheme. These distributions are implemented in the FEM using element based material grading. A finite element formulation for the problem is reported, which is based on the principle of stationary total potential. Disks are subjected to centrifugal body load and have clamped-free boundary condition. The work aims to investigate the effect of grading parameter “ n ” on the deformation and stresses for different material gradation law.

2 Problem Formulation

In this section, geometric equations as well as different types of material property distributions are presented and the governing equations for the rotating disk are derived.

2.1 Material modeling

Three types of material models, namely Mori–Tanaka scheme [11], power law distribution [9] and exponential distribution [7] are considered in the present analysis. The effective Young’s modulus $E(r)$ and density $\rho(r)$ of a disk having inner radius a and outer radius b can be obtained as:

2.1.1 Mori–Tanaka scheme

For the FG disk, the effective bulk modulus $B(r)$ and shear modulus $G(r)$ based on Mori–Tanaka scheme [11] are given by:

$$B(r) = (B_b - B_a) / V_b \left(1 + (1 - V_b) \frac{3(B_b - B_a)}{3B_a + 4G_a} \right) + B_a \quad (1)$$

$$G(r) = (G_b - G_a) / V_b \left(1 + (1 - V_b) \frac{(G_b - G_a)}{G_a + f_1} \right) + G_a \quad (2)$$

$$f_1 = \frac{G_a (9B_a + 8G_a)}{6(B_a + 2G_a)} \quad (3)$$

Here, V is the volume fraction of the phase material, subscripts “ a ” and “ b ” correspond to the inner and outer materials, respectively. The inner and outer material volume fractions are related by:

$$V_a + V_b = 1 \quad (4)$$

Here, V_b is expressed as:

$$V_b = \left(\frac{r - a}{b - a} \right)^n \quad (5)$$

where n ($n \geq 0$) is the volume fraction exponent. The elastic modulus E can be found as:

$$E(r) = \frac{9B(r)G(r)}{3B(r) + G(r)} \quad (6)$$

Density, ρ is calculated using the rule of mixtures as:

$$\rho(r) = (\rho_b - \rho_a) \left(\frac{r - a}{b - a} \right)^n + \rho_a \quad (7)$$

2.1.2 Power law

$$E(r) = E_b \left(\frac{r}{b} \right)^n \quad (8)$$

$$\rho(r) = \rho_b \left(\frac{r}{b} \right)^n \quad (9)$$

where E_b is modulus of elasticity and ρ_b is density at the outer radius.

2.1.3 Exponential law

$$E(r) = E_0 e^{\beta r} \quad (10)$$

$$\rho(r) = \rho_0 e^{\gamma r} \quad (11)$$

$$E_0 = E_a e^{-\beta a} \quad (12)$$

$$\rho_0 = \rho_a e^{-\gamma a} \quad (13)$$

$$\gamma = \frac{1}{a - b} \ln \left(\frac{\rho_a}{\rho_b} \right) \quad (14)$$

$$\beta = \frac{1}{a - b} \ln \left(\frac{E_a}{E_b} \right) \quad (15)$$

E_i and E_o are modulus of elasticity, and ρ_i and ρ_o are density at the inner and outer radius, respectively.

In FEM, the functional grading is popularly carried out by assigning the average material properties over a given geometry followed by adhering to the geometries, thus resulting into layered functional grading of material properties. The downside of this approach is that it yields singular field variable values at the boundaries of the glued geometries, means a jump of the values of the material properties can be observed at the element boundaries. To get better results, instead of assigning average material properties to each element, material properties are varied inside the element boundaries, using the same shape functions used to interpolate the displacement fields [25].

$$\phi^e = \sum_{i=1}^8 \phi_i N_i \quad (16)$$

where ϕ^e is element material property, ϕ_i is material property at node i and N_i is the shape function.

2.2 Geometric modeling

Figure 1 shows the axisymmetric cross section and geometric parameters of an annular disk. The governing equation of radially varying thickness disk is assumed as:

$$h(r) = h_0 \left[1 - q \left(\frac{r - a}{b - a} \right)^k \right] \quad (17)$$

where $h(r)$ and h_0 are half of the thickness at radius r and root of the disk, respectively; a and b are inner and outer radius; k and q are the constants that control the thickness profiles of the disk. For uniform thickness disk, $q = 0$, and for variable thickness, $q > 0$; $k < 1$ for concave thickness profile.

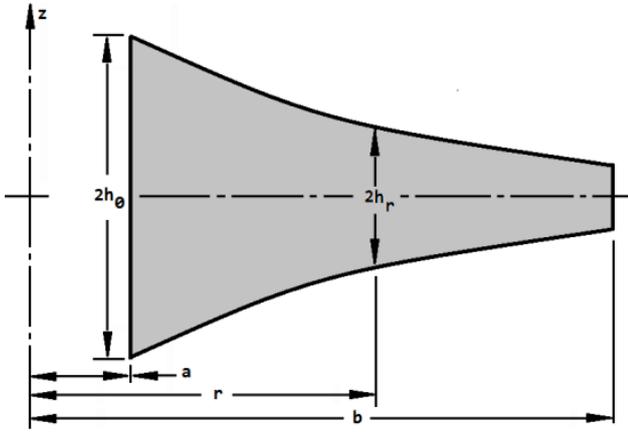


Figure 1: Cross section and geometric parameters of the rotating disk

2.3 Finite element modeling

The rotating disk, being thin, is modeled as a plane stress axisymmetric problem. Using quadratic quadrilateral element, the element displacement vector $\{u\}$ can be obtained as [32]:

$$\{u\} = [N] \{\delta\}^e \tag{18}$$

where $\{u\}$ is element displacement vector, $[N]$ is matrix of quadratic quadrilateral shape functions and $\{\delta\}^e$ is nodal displacement vector.

$$[N] = [N_1 \quad N_2 \quad \dots \quad N_8]$$

$$\{\delta\}^e = \{u_1 \quad u_2 \quad \dots \quad u_8\}^T$$

The strain components are related to elemental displacement components as:

$$\{\varepsilon\} = \{\varepsilon_r \quad \varepsilon_\theta\}^T = [B] \{\delta\}^e \tag{19}$$

where ε_r and ε_θ are radial and circumferential strain, respectively. $[B]$ is strain–displacement relationship matrix, which contains derivatives of shape functions. For a quadratic quadrilateral element, it is calculated as:

$$[B] = [B_1] \times [B_2] \times [B_3] \tag{20}$$

$$[B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{21}$$

$$[B_2] = \begin{bmatrix} \frac{I_{22}}{|J|} & \frac{-J_{12}}{|J|} & 0 \\ \frac{-J_{21}}{|J|} & \frac{I_{11}}{|J|} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

where J is the Jacobian matrix, used to transform the global coordinates into natural coordinates. It is given as:

$$[J] = \begin{bmatrix} \sum_{i=1}^8 \frac{\partial N_i}{\partial \xi} r_i & \sum_{i=1}^8 \frac{\partial N_i}{\partial \xi} z_i \\ \sum_{i=1}^8 \frac{\partial N_i}{\partial \eta} r_i & \sum_{i=1}^8 \frac{\partial N_i}{\partial \eta} z_i \end{bmatrix} \tag{23}$$

$$[B_3] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \dots & \frac{\partial N_8}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \dots & \frac{\partial N_8}{\partial \eta} \\ \frac{N_1}{r} & \frac{N_2}{r} & \dots & \frac{N_8}{r} \end{bmatrix} \tag{24}$$

From Hooke’s law, components of stresses in radial and circumferential direction are related to components of total strain as:

$$\{\sigma\} = \{\sigma_r \quad \sigma_\theta\}^T = [D(r)] \{\varepsilon\} \tag{25}$$

where $D(r)$ is stress strain relationship matrix, which is given as:

$$D(r) = \frac{E(r)}{(1-\nu)^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \tag{26}$$

Upon rotation, the disk experiences a body force, which under constrained boundary results in deformation and stores internal strain energy [32]. Also, there exists a work potential due to body force resulting from centrifugal action. The element level strain energy U^e and work potential V^e are given as:

$$U^e = \int_V \pi r h_r \{\delta\}^e T [B]^T [D(r)] [B] \{\delta\}^e dr \tag{27}$$

$$V^e = -2 \int_V \pi r h_r \{\delta\}^e T [N]^T \{q_v\} dr \tag{28}$$

where $\{q_v\}$ is body force vector. For a disk rotating at ω rad/sec, $\{q_v\}$ for each element is given by:

$$\{q_v\} = \begin{bmatrix} \rho(r) \omega^2 r \\ 0 \end{bmatrix} \tag{29}$$

The total potential of the element is obtained as:

$$\pi_p^e = U^e + V^e = \frac{1}{2} \{\delta\}^e T [K]^e \{\delta\}^e - \{\delta\}^e T \{f\}^e \tag{30}$$

Here, element stiffness matrix, $[K]^e$ and element load vector, $\{f\}^e$ are:

$$[K]^e = 2 \int_V \pi r h_r [B]^T [D(r)] [B] dr \tag{31}$$

$$\{f\}^e = 2 \int_V \pi r h_r [N]^T \{q_v\} dr \tag{32}$$

By transforming the global coordinates into natural coordinates,

$$[K]^e = 2\pi \int_{-1}^1 \int_{-1}^1 [B]^T [D(r)] [B] r |J| d\xi d\eta \quad (33)$$

$$\{f\}^e = 2\pi \int_{-1}^1 \int_{-1}^1 [N]^T \{q_v\} r |J| d\xi d\eta \quad (34)$$

The element matrices are then assembled to yield the global stiffness matrix and global load vector, respectively. Total potential energy of the disk is given by:

$$\pi_p = \sum \pi_p^e = \frac{1}{2} \{\delta\}^T [K] \{\delta\} - \{\delta\}^T \{F\} \quad (35)$$

where

$$[K] = \sum_{n=1}^N [K]^e = \text{Global Stiffness matrix}$$

$$\{F\} = \sum_{n=1}^N \{f\}^e = \text{Global load vector}$$

N = number of elements.

Using the Principle of Stationary Total Potential (PSTP), the total potential is set to be stationary with respect to small variation in the nodal degree of freedom, that is,

$$\frac{\partial \pi_p}{\partial \{\delta\}^T} = 0 \quad (36)$$

From above, the system of simultaneous equations is obtained as follows:

$$[K] \{\delta\} = \{F\} \quad (37)$$

In this section, geometric equations as well as different types of material property distributions are presented and the governing equations for the rotating disk are derived.

3 Results and Discussion

3.1 Validation

A uniform thickness rotating disk ($n = 0$ in ceramic–metal FGM disk modeled by Mori–Tanaka scheme) and a concave thickness profile disk ($n = 0$ in metal–ceramic FGM disk modeled by Mori–Tanaka scheme) [11] is reconsidered and analyzed again and results are presented in Figure 2. The results obtained are in good agreement with the pre-established results of reference.

3.2 Numerical results and discussion

In this section, the rotating annular disks of parabolic concave thickness profile and made of aluminum and zirconia ceramic as per different FG distribution functions are analyzed and the effects of grading parameter n on stresses and deformation states are investigated. The material properties of aluminum and zirconia are given in Table 1 [11]. The disks have geometric parameters as $k = 0.5$, inner diameter = 0.2 m, outer diameter = 1 m, $q = 0.96$ and h_0 is 0.075 m. Disks are assumed to be rotating at unit angular velocity and have clamped-free boundary condition. Figure 3 to Figure 8 show the distributions of E and ρ for different distribution laws. Grading index $n = 0$ indicates that the disk is made of outer material completely, that is, the disk is homogeneous in composition. For ceramic–metal

Table 1: Aluminum and zirconia properties [11]

Material	E (GPa)	ρ (kg/m ³)	G (GPa)	ν
Aluminum	70	2700	26.9231	0.3
Zirconia	151	5700	58.0769	0.3

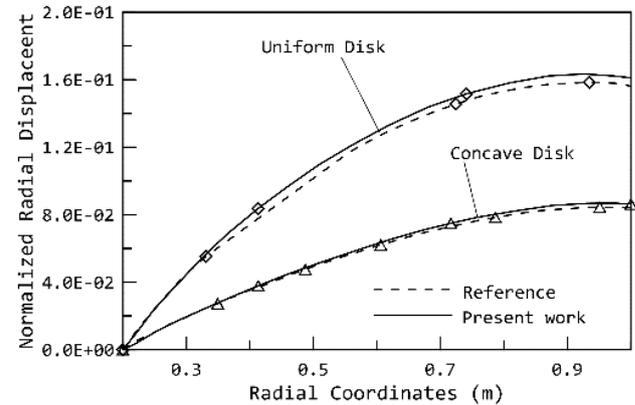


Figure 2: Comparison of the results of current work with reference

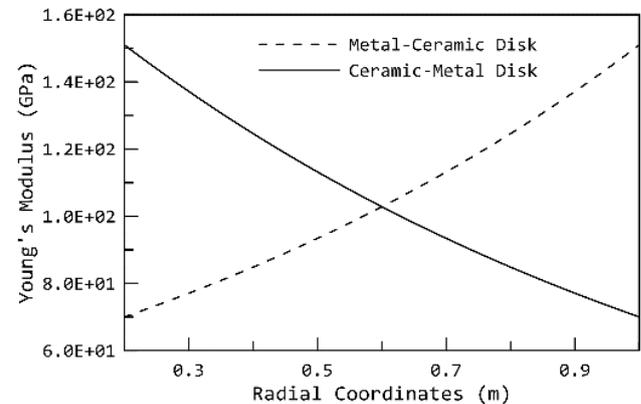


Figure 3: Distribution of E for exponential FG disks

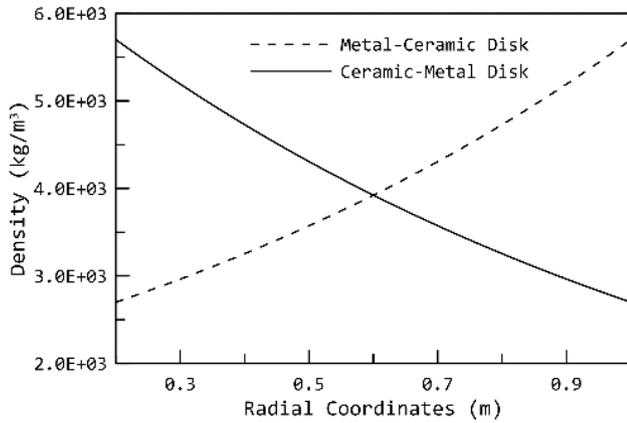


Figure 4: Distribution of density for exponential FG disks

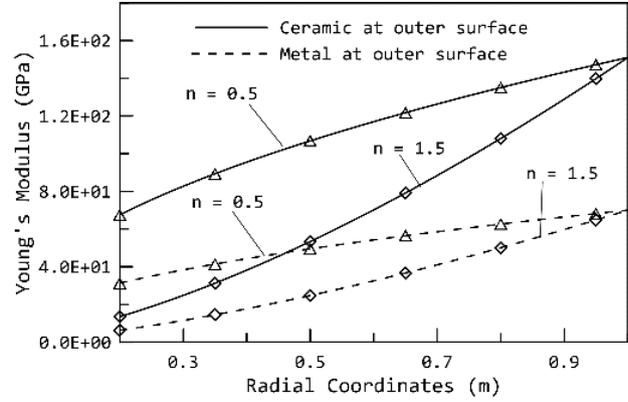


Figure 7: Distribution of E for power law FG disks

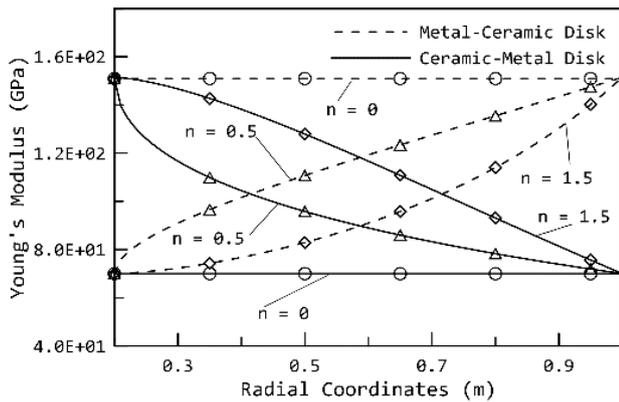


Figure 5: Distribution of E for Mori-Tanaka disks

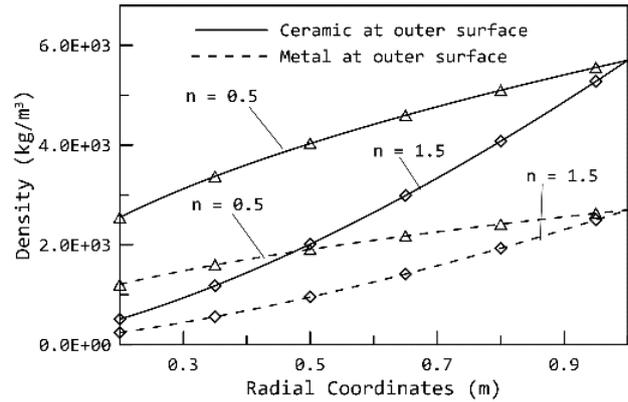


Figure 8: Distribution of density for power law FG disks

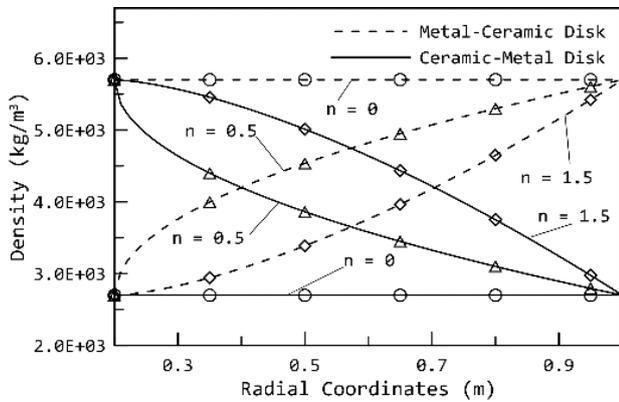


Figure 6: Distribution of density for Mori-Tanaka disks

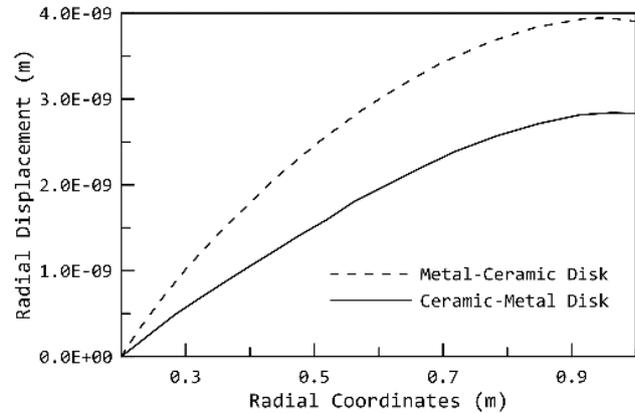


Figure 9: Displacements for exponential FG disks

FGM, $n = 0$ indicates homogeneous metallic (aluminum) disk, while for metal-ceramic FGM, it indicates homogeneous ceramic (zirconia) disk. For values of n other than 0, volume fraction varies with the radius according to different equations and gives different types of FGMs.

Figure 9 to Figure 20 report the distributions of displacement and stresses for all the three material distribution functions. It can be observed that the ceramic-metal

FG disks have less deformation and higher stresses as compared to the metal-ceramic FG disks. Displacement at inner radius and radial stress at the outer radius confirms the clamped-free boundary condition applied on the disks. In the metal-ceramic FG disks modeled by Mori-Tanaka scheme, the radial deformation increases and stress decreases with an increasing value of grading parameter n , while in the ceramic-metal FG, radial deformation de-

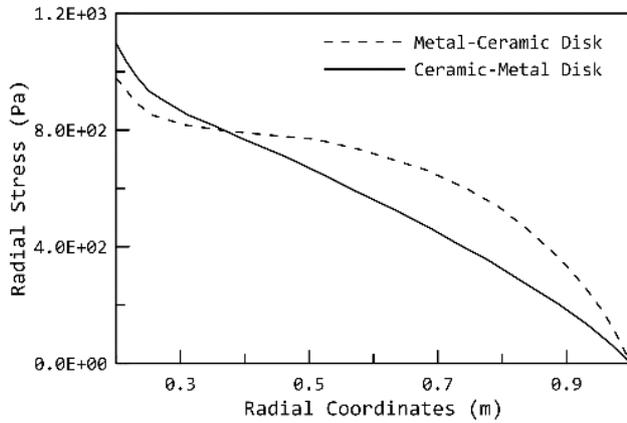


Figure 10: Radial stress for exponential FG disks

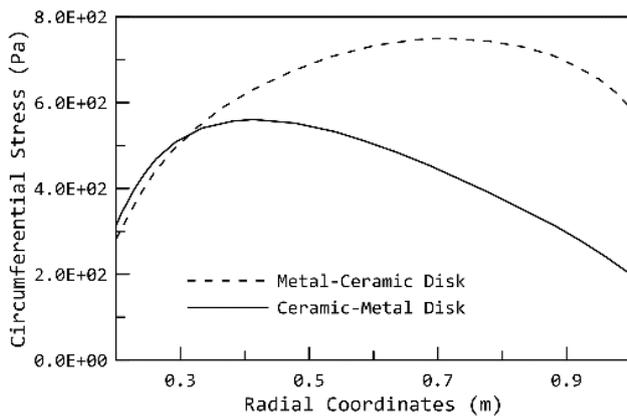


Figure 11: Circumferential stress in exponential FG disks

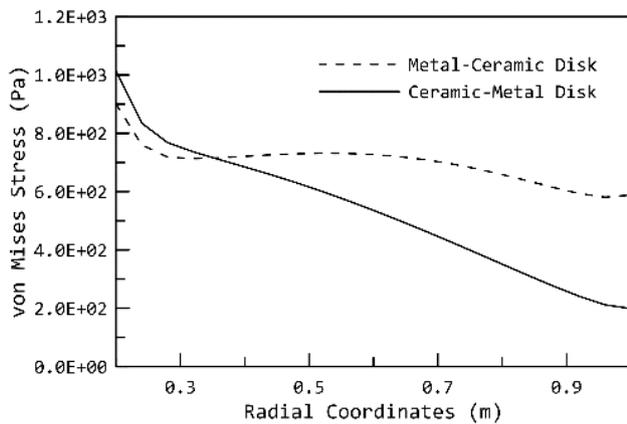


Figure 12: Equivalent stress in exponential FG disks

increases and stress increases with an increasing value of n . Increasing n means volume fraction of the outer material is decreasing and inner material is increasing. In case of metal–ceramic FG, increasing metallic content and decreasing ceramic content results in higher deformation and lesser stress, while in the case of ceramic–metal FGM,

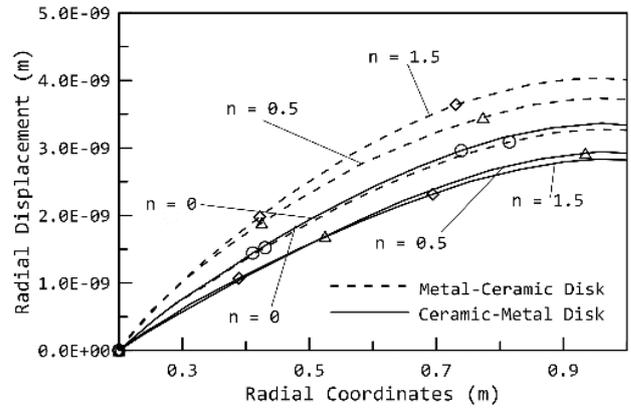


Figure 13: Displacements for Mori–Tanaka disks

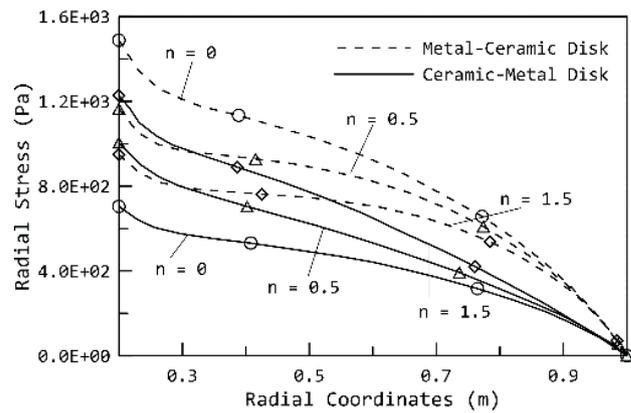


Figure 14: Radial stresses for Mori–Tanaka FGM disks

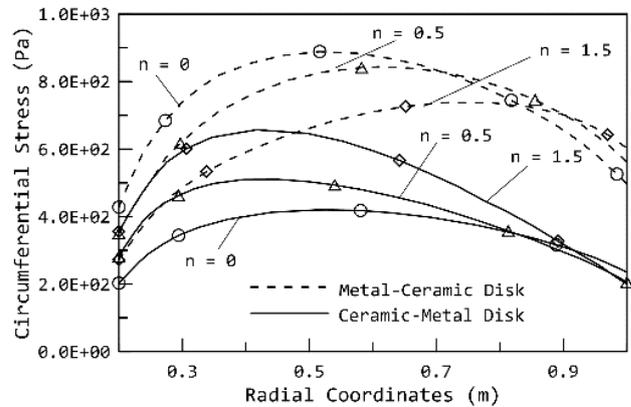


Figure 15: Circumferential stresses for Mori–Tanaka disks

increasing ceramic and decreasing metallic content results in higher stress and lesser deformation.

Metal–ceramic FG having $n = 0$ has the highest stress and $n = 1.5$ has the highest deformation, while ceramic–metal FG having $n = 0$ report the lowest stress and $n = 1.5$ report the lowest deformation among all the FG modeled by Mori–Tanaka scheme. In power law, FG disk radial deformation increases and stress decreases with an increas-

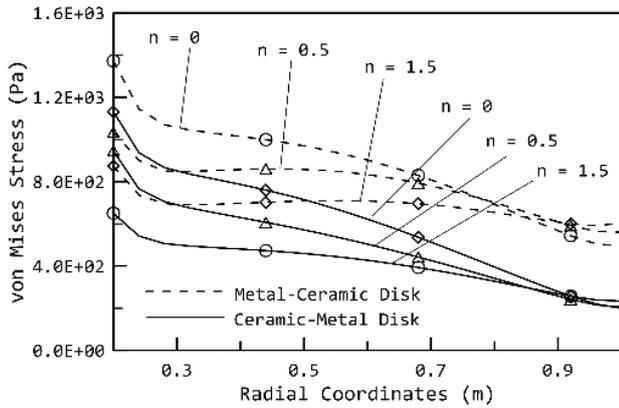


Figure 16: Equivalent stresses for Mori-Tanaka disks

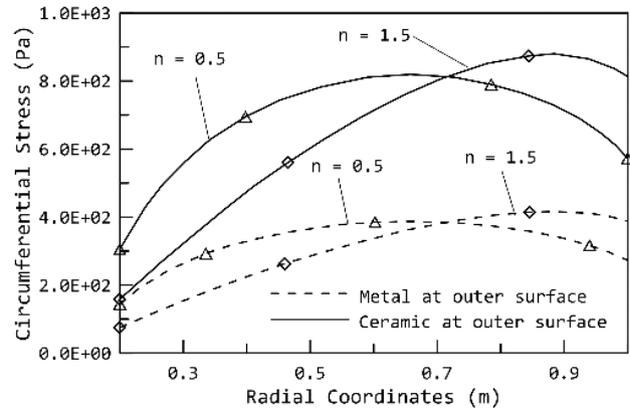


Figure 19: Circumferential stress for power law FG disks

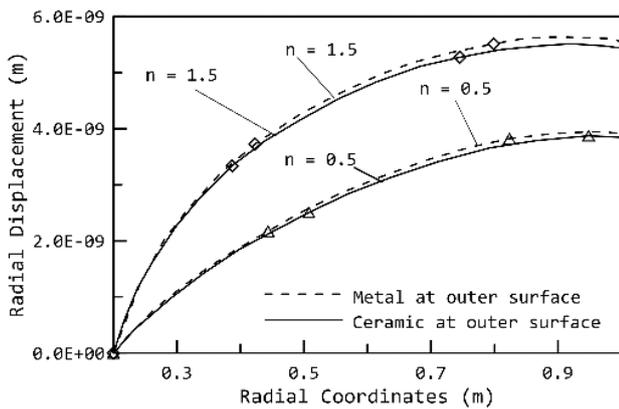


Figure 17: Displacements for power law FG disks

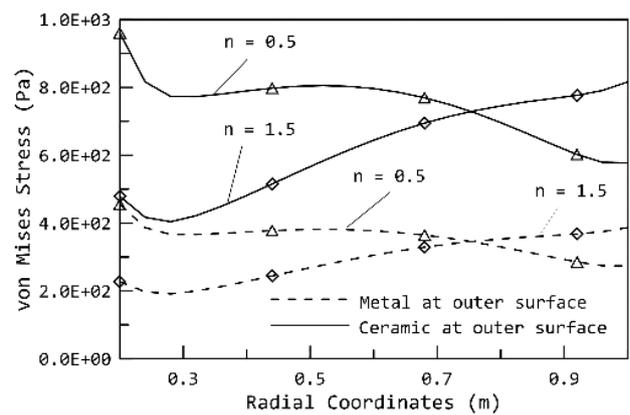


Figure 20: Equivalent stresses for power law FG disks

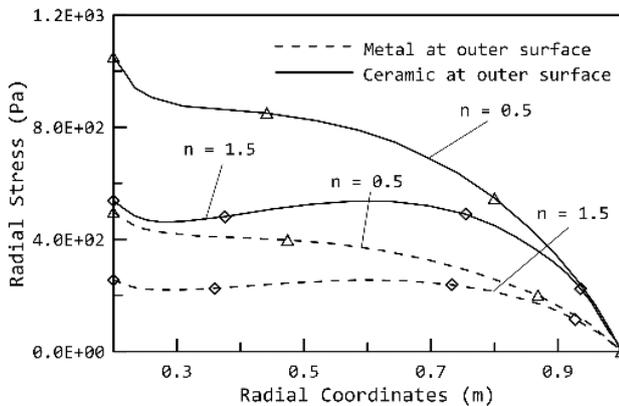


Figure 18: Radial stress for power law FG disks

ing grading parameter n . Increasing n decreases (r/b) ratio, which decreases $E(r)$, and hence, deformation increases and stress decreases. FG disk having metal at outer surface and $n = 1.5$ has maximum radial deformation and minimum radial, circumferential and von Mises or equivalent stresses, while disk having $n = 0.5$ and ceramic at outer radius has minimum radial deformation and maximum stresses. Further, it is also observed that radial stress

is higher as compared to circumferential and von Mises stress for all the cases. Therefore, for designing the rotating disks, radial stress should be taken as a limit working stress criteria. By comparing all types of distribution laws, it is observed that the power law FG disk, having metal at outer radius and $n = 1.5$, has the highest radial deformation and least radial stress, while the exponential law (ceramic-metal) disk has the lowest radial deformation and disk of full ceramic has the highest radial stress. Therefore, it is suggested that FG modeled by power law having metal at outer radius and $n = 1.5$ can be most effectively employed for rotating disk.

4 Conclusions

The present work proposes a study using element based gradation of varying material property of rotating disks and reports the stress and deformation behavior of concave thickness clamped rotating FGM disks. Different types of distribution laws having aluminum as a metal and

zirconia as a ceramic is considered, and metal–ceramic as well as ceramic–metal – both types of FGMs are analyzed. Principle of stationary total potential (PSTP) is used for finite element formulation. The results obtained are found to be in good agreement with established reports. It is observed that there is a significant reduction in stresses and deformation behavior of the FGM disks compared to the homogeneous disks. Further, it is observed that metal–ceramic FGM disk having $n = 1.5$ and modeled by power law possesses better strength than all other FGMs investigated, and therefore, is most efficient for the purpose of rotating disk under clamped-free condition.

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