Derivation of the equations of motion of the compound manipulator

Łódź 2010

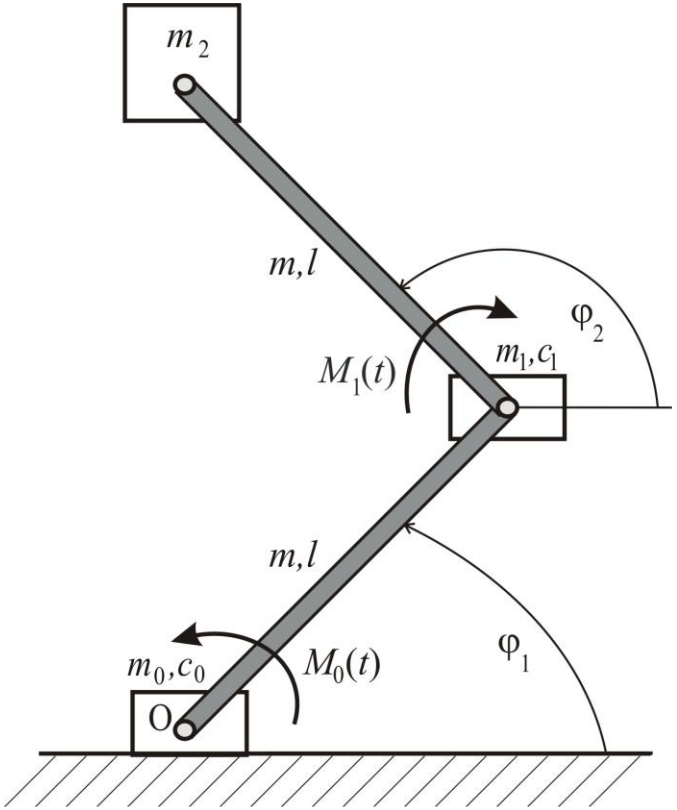

Fig. 1

Analyzed system

φ_1, φ_2 – angular coordinates of manipulator's arms,

m_0, m_1 – masses of driving systems,

m_2 – main mass,

m, l – mass and length of manipulator's arms pulatora,

c_0, c_1 – viscous damping coefficients,

$M_0(t)$, $M_1(t)$ – driving moments.

During the analysis of moving parts nodal masses m_1 and m_2 were treated as concentrated in the nodes at the ends of arms

LAGRANGE'S EQUATIONS

General form:

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\varphi}_i}\right) - \frac{\partial U}{\partial \varphi_i} + \frac{\partial V}{\partial \varphi_i} + \frac{\partial D}{\partial \dot{\varphi}_i} = Q_i(t) \qquad i = 1,2 \qquad (1)$$

U – kinetic energy,

V – potential energy,

D – function of damping,

$Q_i(t)$ – external forces,

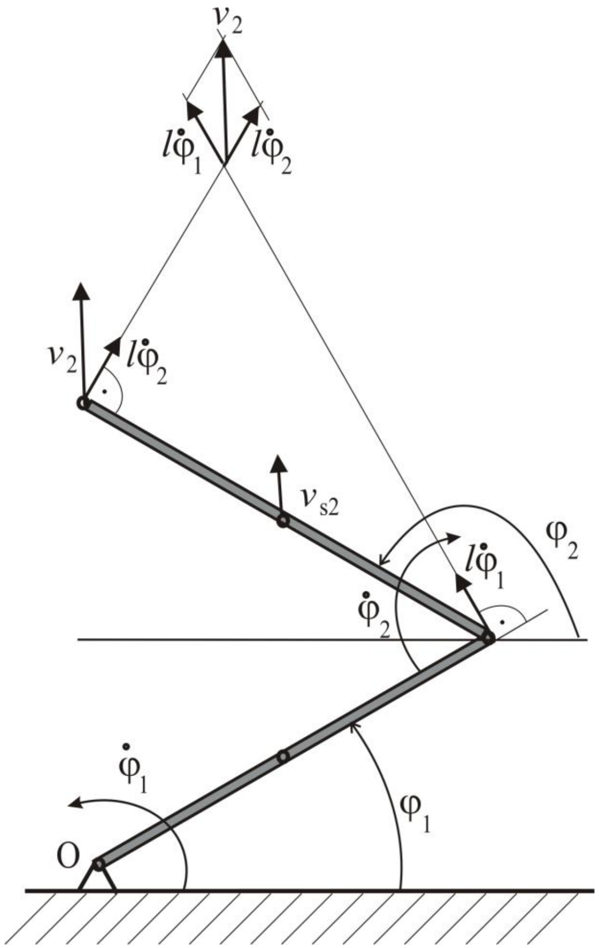

Fig. 2

1) kinetic energy

$$U = \frac{1}{2}\left(\frac{1}{3}ml^2\dot{\varphi}_1^2 + m_1 l^2\dot{\varphi}_1^2 + \frac{1}{12}ml^2\dot{\varphi}_2^2 + mv_{S2}^2 + m_2 v_2^2\right), \qquad (2)$$

where:

$$v_{S2}^2 = (l\dot{\varphi}_1)^2 + (0.5l\dot{\varphi}_2)^2 + l^2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2 - \varphi_1), \qquad (2a)$$

$$v_2^2 = (l\dot{\varphi}_1)^2 + (l\dot{\varphi}_2)^2 + 2l^2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2 - \varphi_1). \qquad (2a)$$

Substituting equations (2a) and (2b) into the equation (2) we obtain

$$U=\frac{1}{2}\left(\frac{1}{3}m+m_1\right)l^2\dot{\varphi}_1^2+\frac{1}{2}\frac{1}{12}ml^2\dot{\varphi}_2^2+\frac{1}{2}ml^2\dot{\varphi}_1^2+\frac{1}{2}m\frac{1}{4}l^2\dot{\varphi}_2^2+\frac{1}{2}ml^2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2-\varphi_1)+$$
$$\frac{1}{2}m_2l^2\dot{\varphi}_1^2+\frac{1}{2}m_2l^2\dot{\varphi}_2^2+m_2l^2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2-\varphi_1). \qquad (3)$$

Finally we have

$$U=\frac{1}{2}\left(\frac{4}{3}m+m_1+m_2\right)l^2\dot{\varphi}_1^2+\frac{1}{2}\left(\frac{1}{3}m+m_2\right)l^2\dot{\varphi}_2^2+\left(\frac{1}{3}m+m_2\right)l^2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2-\varphi_1). \qquad (4)$$

2) potential energy

$$V = mg\frac{l}{2}\sin\varphi_1 + m_1 gl\sin\varphi_1 + mg\left(l\sin\varphi_1 + \frac{l}{2}\sin\varphi_2\right) + m_2 g\left(l\sin\varphi_1 + l\sin\varphi_2\right), \tag{5}$$

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$$V = \left(\frac{4}{3}m + m_1 + m_2\right)gl\sin\varphi_1 + \left(\frac{1}{3}m + m_2\right)gl\sin\varphi_2. \tag{6}$$

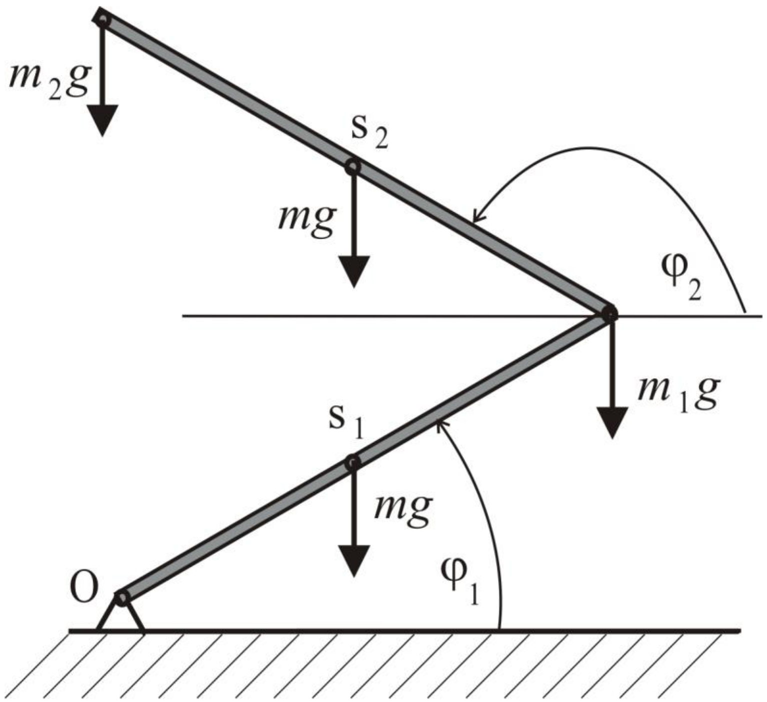


Fig. 3

3) *Damping function*

Linear viscous damping:

$$D = \frac{1}{2} c_0 \dot{\varphi}_1^2 + \frac{1}{2} c_1 (\dot{\varphi}_2 - \dot{\varphi}_1)^2. \qquad (7)$$

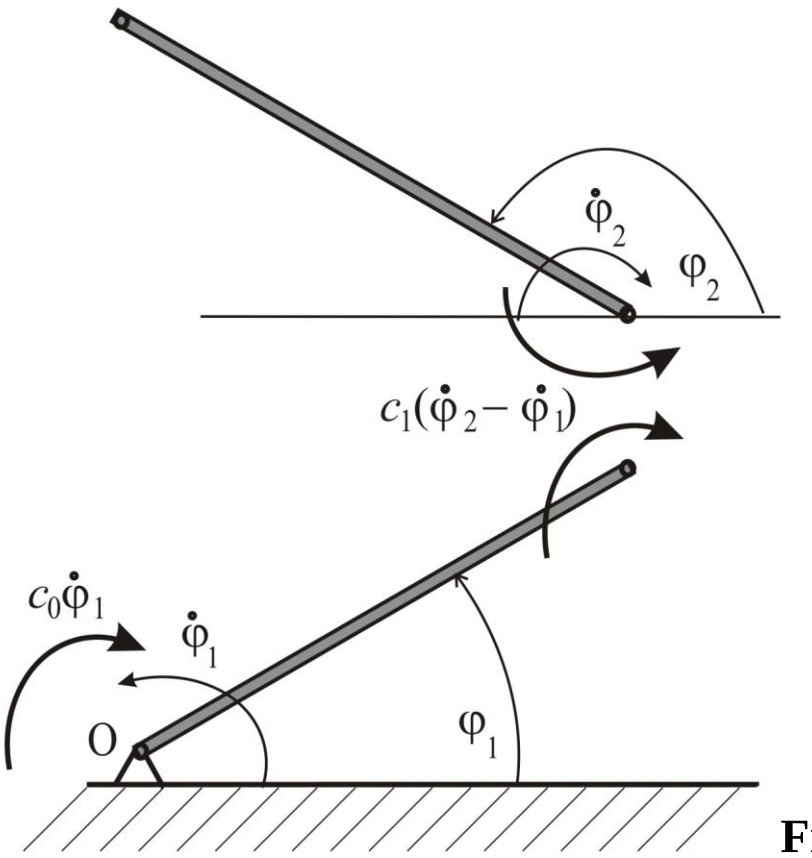
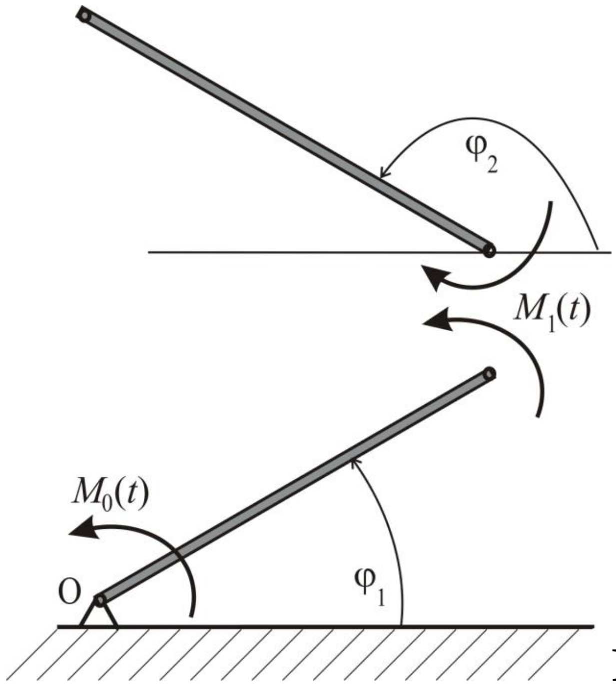


Fig. 4

4) external load



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$$Q_1 = M_0(t) + M_1(t), \tag{8a}$$

$$Q_2 = -M_1(t) \tag{8b}$$

DERIVATION OF THE EQUATIONS OF MOTION

Substitutions:

$$B_A = \left(\frac{4}{3}m + m_1 + m_2\right)l^2, \qquad B_B = \left(\frac{1}{3}m + m_2\right)l^2,$$

$$m_A = \frac{1}{2}m + m_2, \qquad m_B = \frac{3}{2}m + m_1 + m_2.$$

After applying the above substitutions in equations (4) and (6), kinetic and potential energies are described as follows:

$$U = \frac{1}{2}B_A\dot{\varphi}_1^2 + \frac{1}{2}B_B\dot{\varphi}_2^2 + m_A l^2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2 - \varphi_1), \tag{9}$$

$$V = m_B g l \sin\varphi_1 + m_A g l \sin\varphi_2. \tag{10}$$

Components of Lagrange's equation:

Arm 1

$$\frac{\partial U}{\partial \dot{\varphi}_1} = B_A\dot{\varphi}_1 + m_A l^2 \dot{\varphi}_2 \cos(\varphi_2 - \varphi_1),$$

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\varphi}_1}\right) = B_A\ddot{\varphi}_1 + m_A l^2 \cos(\varphi_2 - \varphi_1)\ddot{\varphi}_2 - m_A l^2 \dot{\varphi}_2 \sin(\varphi_2 - \varphi_1)(\dot{\varphi}_2 - \dot{\varphi}_1),$$

$$\frac{\partial U}{\partial \varphi_1} = m_A l^2 \dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_2 - \varphi_1), \qquad \frac{\partial V}{\partial \varphi_1} = m_B g l \cos\varphi_1, \qquad \frac{\partial D}{\partial \dot{\varphi}_1} = c_0\dot{\varphi}_1 - c_1(\dot{\varphi}_2 - \dot{\varphi}_1).$$

Arm 2

$$\frac{\partial U}{\partial \dot{\varphi}_2} = B_B\dot{\varphi}_2 + m_A l^2 \dot{\varphi}_1 \cos(\varphi_2 - \varphi_1),$$

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\varphi}_2}\right) = B_B\ddot{\varphi}_2 + m_A l^2 \cos(\varphi_2 - \varphi_1)\ddot{\varphi}_1 - m_A l^2 \dot{\varphi}_1 \sin(\varphi_2 - \varphi_1)(\dot{\varphi}_2 - \dot{\varphi}_1),$$

$$\frac{\partial U}{\partial \varphi_2} = -m_A l^2 \dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_2 - \varphi_1), \qquad \frac{\partial V}{\partial \varphi_2} = m_A g l \cos\varphi_2, \qquad \frac{\partial D}{\partial \dot{\varphi}_2} = c_1(\dot{\varphi}_2 - \dot{\varphi}_1).$$

Substituting into Lagrange equation (1) above formulas, driving components (8a) and (8b), and assuming that the viscous damping coefficients are identical - c0 = c1 = c, we obtain the differential equation of motion (2nd order) of the manipulator as:

$$B_A\ddot{\varphi}_1 + m_A l^2 \cos(\varphi_2 - \varphi_1)\ddot{\varphi}_2 - m_A l^2 \sin(\varphi_2 - \varphi_1)\dot{\varphi}_2^2 + m_B g l \cos\varphi_1 + c(2\dot{\varphi}_1 - \dot{\varphi}_2) = M_0(t) + M_1(t), \tag{11a}$$

$$B_B\ddot{\varphi}_2 + m_A l^2 \cos(\varphi_2 - \varphi_1)\ddot{\varphi}_1 + m_A l^2 \sin(\varphi_2 - \varphi_1)\dot{\varphi}_1^2 + m_A g l \cos\varphi_2 + c(\dot{\varphi}_2 - \dot{\varphi}_1) = -M_1(t). \tag{11b}$$

And after another transformations:

$$\ddot{\varphi}_1 = \frac{M_0(t) + M_1(t)}{B_A} - \frac{m_A}{B_A} l^2 \cos(\varphi_2 - \varphi_1)\ddot{\varphi}_2 + \frac{m_A}{B_A} l^2 \sin(\varphi_2 - \varphi_1)\dot{\varphi}_2^2 - \frac{m_B}{B_A} g l \cos\varphi_1 - \frac{c}{B_A}(2\dot{\varphi}_1 - \dot{\varphi}_2) \tag{12a}$$

$$\ddot{\varphi}_2 = \frac{-M_1(t)}{B_B} - \frac{m_A}{B_B} l^2 \cos(\varphi_2 - \varphi_1)\ddot{\varphi}_1 - \frac{m_A}{B_B} l^2 \sin(\varphi_2 - \varphi_1)\dot{\varphi}_1^2 - \frac{m_A}{B_B} g l \cos\varphi_2 - \frac{c}{B_B}(\dot{\varphi}_2 - \dot{\varphi}_1), \tag{12b}$$

$$\ddot{\varphi}_1 = \frac{M_0(t)+M_1(t)}{B_A} - \frac{m_A}{B_A} l^2 \cos(\varphi_2-\varphi_1)\left[\frac{-M_1(t)}{B_B} - \frac{m_A}{B_B} l^2 \sin(\varphi_2-\varphi_1)\dot{\varphi}_1^2 - \frac{m_A}{B_B} gl\cos\varphi_2 - \frac{c}{B_B}(\dot{\varphi}_2-\dot{\varphi}_1)\right]+$$

$$+\frac{m_A^2}{B_A B_B} l^4 \cos^2(\varphi_2-\varphi_1)\ddot{\varphi}_1 + \frac{m_A}{B_A} l^2 \sin(\varphi_2-\varphi_1)\dot{\varphi}_2^2 - \frac{m_B}{B_A} gl\cos\varphi_1 - \frac{c}{B_A}(2\dot{\varphi}_1-\dot{\varphi}_2), \tag{13a}$$

$$\ddot{\varphi}_2 = \frac{-M_1(t)}{B_B} - \frac{m_A}{B_B} l^2 \cos(\varphi_2-\varphi_1)\left[\frac{M_0(t)+M_1(t)}{B_A} + \frac{m_A}{B_A} l^2 \sin(\varphi_2-\varphi_1)\dot{\varphi}_2^2 - \frac{m_B}{B_A} gl\cos\varphi_1 - \frac{c}{B_A}(2\dot{\varphi}_1-\dot{\varphi}_2)\right]+$$

$$+\frac{m_A^2}{B_A B_B} l^4 \cos^2(\varphi_2-\varphi_1)\ddot{\varphi}_1 - \frac{m_A}{B_B} l^2 \sin(\varphi_2-\varphi_1)\dot{\varphi}_1^2 - \frac{m_A}{B_B} gl\cos\varphi_2 - \frac{c}{B_B}(\dot{\varphi}_2-\dot{\varphi}_1), \tag{13b}$$

We have

$$\ddot{\varphi}_1 = \left\{ \frac{M_0(t)+M_1(t)}{B_A} - \frac{m_A}{B_A} l^2 \cos(\varphi_2-\varphi_1) \left[\frac{-M_1(t)}{B_B} - \frac{m_A}{B_B} l^2 \sin(\varphi_2-\varphi_1)\dot{\varphi}_1^2 - \frac{m_A}{B_B} gl\cos\varphi_2 - \frac{c}{B_B}(\dot{\varphi}_2-\dot{\varphi}_1) \right] \right.$$
$$\left. + \frac{m_A}{B_A} l^2 \sin(\varphi_2-\varphi_1)\dot{\varphi}_2^2 - \frac{m_B}{B_A} gl\cos\varphi_1 - \frac{c}{B_A}(2\dot{\varphi}_1-\dot{\varphi}_2) \right\} \Bigg/ \left[1 - \frac{m_A^2}{B_A B_B} l^4 \cos^2(\varphi_2-\varphi_1) \right], \tag{14a}$$

$$\ddot{\varphi}_2 = \left\{ \frac{-M_1(t)}{B_B} - \frac{m_A}{B_B} l^2 \cos(\varphi_2-\varphi_1) \left[\frac{M_0(t)+M_1(t)}{B_A} + \frac{m_A}{B_A} l^2 \sin(\varphi_2-\varphi_1)\dot{\varphi}_2^2 - \frac{m_B}{B_A} gl\cos\varphi_1 - \frac{c}{B_A}(2\dot{\varphi}_1-\dot{\varphi}_2) \right] \right.$$
$$\left. - \frac{m_A}{B_B} l^2 \sin(\varphi_2-\varphi_1)\dot{\varphi}_1^2 - \frac{m_A}{B_B} gl\cos\varphi_2 - \frac{c}{B_B}(\dot{\varphi}_2-\dot{\varphi}_1) \right\} \Bigg/ \left[1 - \frac{m_A^2}{B_A B_B} l^4 \cos^2(\varphi_2-\varphi_1) \right]. \tag{14b}$$

Changing the variables

$$\varphi_1 = x_1, \qquad \dot{\varphi}_1 = x_2, \qquad \ddot{\varphi}_1 = \dot{x}_2,$$

$$\varphi_2 = y_1, \qquad \dot{\varphi}_2 = y_2, \qquad \ddot{\varphi}_2 = \dot{y}_2,$$

equations (14a) and (14b) can be written in form of four1st order differential equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \left\{ \frac{M_0(t)+M_1(t)}{B_A} - \frac{m_A}{B_A} l^2 \cos(y_1 - x_1) \left[\frac{-M_1(t)}{B_B} - \frac{m_A}{B_B} l^2 \sin(y_1 - x_1) x_2^2 - \frac{m_A}{B_B} gl \cos y_1 - \frac{c}{B_B}(y_2 - x_2) \right] \right.$$
$$\left. + \frac{m_A}{B_A} l^2 \sin(y_1 - x_1) y_2^2 - \frac{m_B}{B_A} gl \cos x_1 - \frac{c}{B_A}(2x_2 - y_2) \right\} \Big/ \left[1 - \frac{m_A^2}{B_A B_B} l^4 \cos^2(y_1 - x_1) \right],$$

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = \left\{ \frac{-M_1(t)}{B_B} - \frac{m_A}{B_B} l^2 \cos(y_1 - x_1) \left[\frac{M_0(t)+M_1(t)}{B_A} + \frac{m_A}{B_A} l^2 \sin(y_1 - x_1) y_2^2 - \frac{m_B}{B_A} gl \cos x_1 - \frac{c}{B_A}(2x_2 - y_2) \right] \right.$$
$$\left. - \frac{m_A}{B_B} l^2 \sin(y_1 - x_1) x_2^2 - \frac{m_A}{B_B} gl \cos y_1 - \frac{c}{B_B}(y_2 - x_2) \right\} \Big/ \left[1 - \frac{m_A^2}{B_A B_B} l^4 \cos^2(y_1 - x_1) \right]. \tag{15}$$