Exercise 10

RESONANCE FREQUENCIES OF TORSIONAL VIBRATIONS OF THE SHAFT

1. Aims of the experiment

- Measurement of resonance frequencies of the shaft with three discs.
- Observation of the corresponding vibration modes of the shaft.
- Comparison of the experimentally measured resonance frequencies with those calculated analytically.

2. Theoretical introduction

A model of the system under consideration is shown in Fig. 10.1. It is an example of the three-degree-of-freedom system. We consider natural torsional vibrations of the shaft fixed at one of its ends. Three identical discs having the moment of inertia B are fixed to the shaft of the torsional stiffness k_1 and k.

The differential equations of motion are as follows:

$$
\begin{aligned}
&B\ddot{\varphi}_1 + k(\varphi_1 - \varphi_2) + k_1(\varphi_1 - \varphi_0) = 0 \\
&B\ddot{\varphi}_2 + k(\varphi_2 - \varphi_3) - k(\varphi_1 - \varphi_2) = 0 \\
&B\ddot{\varphi}_3 - k(\varphi_2 - \varphi_3) = 0
\end{aligned}
\qquad (10.1)
$$

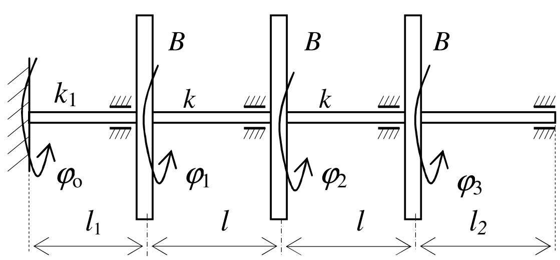


Fig. 10.1. Model of the vibrating system

In order to determine the natural frequencies of torsional vibrations of the system, the Holzer method has been applied. The particular solutions to Eqs. (10.1) have been assumed in the form of harmonic functions:

$$
\begin{aligned}
&\varphi_1 = \Phi_1 \sin \omega t; \\
&\varphi_2 = \Phi_2 \sin \omega t; \\
&\varphi_3 = \Phi_3 \sin \omega t,
\end{aligned}
\qquad (10.2)
$$

where:

ω – assumed natural frequency of torsional vibrations,

Φ_1, Φ_2, Φ_3 – amplitudes of torsional vibrations of individual discs.

Substituting the assumed solutions (Eqs. (10.2)) in Eqs. (10.1), we obtain the system of algebraic equations as follows:

$$B\Phi_1\omega^2 - k(\Phi_1 - \Phi_2) - k_1(\Phi_1 - \Phi_0) = 0; \quad (10.3)$$
$$B\Phi_2\omega^2 - k(\Phi_2 - \Phi_3) + k(\Phi_1 - \Phi_2) = 0; \quad (10.4)$$
$$B\Phi_3\omega^2 + k(\Phi_2 - \Phi_3) = 0. \quad (10.5)$$

The natural frequencies of torsional vibrations of system (10.1) can be calculated on the basis of Eqs. (10.3)÷(10.5). At the beginning, an arbitrary amplitude of the third disc has to be assumed, e.g., $\Phi_3 = 1$ [rad]. Next, we can determine the amplitude Φ_2 substituting a value of ω from the expected range of frequency:

$$\Phi_2 = \Phi_3 - \frac{B\Phi_3\omega^2}{k}. \quad (10.6)$$

Taking into consideration the determined value of Φ_2 in Eq. (10.4), we can calculate the amplitude Φ_1:

$$\Phi_1 = \Phi_2 - \frac{B\Phi_2\omega^2 + B\Phi_3\omega^2}{k}. \quad (10.7)$$

Similarly, using the value of Φ_1 in Eq. (10.3), we can calculate the amplitude at fixed end of the shaft Φ_0:

$$\Phi_0 = \Phi_1 - \frac{B\Phi_1\omega^2 + B\Phi_2\omega^2 + B\Phi_3\omega^2}{k_1}. \quad (10.8)$$

If ω is a natural frequency of the system (Eqs. (10.3)÷(10.5)), then the boundary condition $\Phi_0 = 0$ has to be fulfilled.

The above description shows that the Holzer's method used in the system under consideration consists in the investigation of the solutions to Eq. (10.8) in the expected range of frequency. If the currently tested value of ω is not a natural frequency of the system, then the boundary condition $\Phi_0 = 0$ is not fulfilled. Next steps are to be done with the tested value of ω, aiming at fulfilling the above condition for Φ_0 in order to estimate the natural frequency ω with the required precision.

2.1. *Example of calculations*

<u>*Problem*</u>

Determine the natural frequencies and main modes of torsional vibrations of the massless shaft with two discs of the inertia moment B_1 and B_2 (Fig.10.2) by means of the Holzer's method.

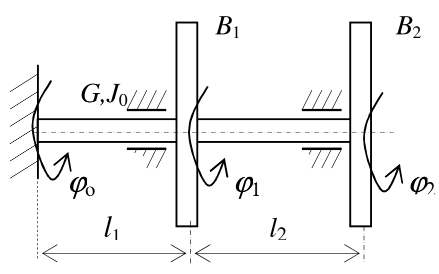


Fig. 10.2. Model of the shaft

Given data:
$B_1 = B_2 = B = 10^{-5}$ kgm^2;
$J_0 = 10^{-6}$ m^4;
$l_1 = l_2 = 0{,}1$ m;
$G = 8\times10^{10}$ Nm^{-2};

<u>*Solution*</u>

The torsional stiffness of shaft sections (of the length $l_1 = l_2 = l$) can be calculated from the relation:

$$k = \frac{GJ_0}{l} = \frac{8\times10^{10}\times10^{-6}}{0,1} = 8\times10^5 \text{ Nm rad}^{-1} \qquad \text{(a)}$$

The equations of torsional vibrations of the system under analysis (Fig. 10.2) are as follows:

$$\begin{aligned} &B\ddot{\varphi}_1 = -k\varphi_1 - k\varphi_1 + k\varphi_2 + k\varphi_0 \quad \Rightarrow \\ &\Rightarrow \; B\ddot{\varphi}_1 - k(\varphi_2 - \varphi_1) + k(\varphi_1 - \varphi_0) = 0 \end{aligned} \qquad \text{(b)}$$

$$B\ddot{\varphi}_2 = -k\varphi_2 + k\varphi_1 \quad \Rightarrow \quad B\ddot{\varphi}_2 + k(\varphi_2 - \varphi_1) = 0 \qquad \text{(c)}$$

The harmonic solutions to the differential equations (b) and (c) are expressed as:

$$\varphi_1 = \Phi_1 \sin\omega t; \qquad \varphi_2 = \Phi_2 \sin\omega t \qquad \text{(d)}$$

After the substitution of the assumed solutions (d) into the equations of motion (b) and (c), we obtain the following algebraic equations:

$$-B\Phi_1\omega^2 - k(\Phi_2 - \Phi_1) + k(\Phi_1 - \Phi_0) = 0 \qquad \text{(e)}$$

$$-B\Phi_2\omega^2 + k(\Phi_2 - \Phi_1) = 0 \qquad \text{(f)}$$

We assume the value of amplitude Φ_2 = 1 rad and take the tested value of ω = 1,5×10^5 rad/s from the expected range of frequency. Next, we determine the amplitude Φ_1 using Eq. (f):

$$\begin{aligned} &\Phi_1 = \Phi_2 - \frac{B\Phi_2\omega^2}{k} = \\ &= 1 - \frac{10^{-5}\times1\times2,25\times10^{10}}{8\times10^5} = 0.71875\text{rad,} \end{aligned} \qquad \text{(g)}$$

and the amplitude Φ_0 using Eq. (e):

$$\begin{aligned} &\Phi_0 = \Phi_1 - \frac{B\Phi_2\omega^2 + B\Phi_1\omega^2}{k} = \\ &= 0.71875 - \frac{10^{-5}\times2,25\times10^{10}(1+0,71875)}{8\times10^5} = 0.235\,\text{rad.} \end{aligned} \qquad \text{(h)}$$

The assumed value of ω = 1.5×10^5 rad/s is not the natural frequency of the system under consideration because the boundary condition $\Phi_0 = 0$ is not fulfilled. The estimation finishes with the residual value of amplitude for this frequency $\Delta\Phi = \Phi_0$ = 0.235 rad. The calculations are summarized in Table 10.1.

Table 10.1 $\omega_1 = 1.5\times10^5$ rad/s $\Delta\Phi = \Phi_0 = 0.235$ rad

n	B	$B\omega^2$	Φ_n	$B\omega^2\Phi_n$	$\sum B\omega^2\Phi_n$	k	$\sum B\omega^2\Phi_n/k$
	kgm^2	Nm	rad	Nm	Nm	Nm/rad	rad
1	10^{-5}	2.25×10^5	0.7187	1.62×10^5	3.87×10^5	8×10^5	0.48375

Let us repeat now the calculations for the next value of ω with the assumed step of calculations (e.g., $\Delta\omega = 0{,}002\times10^5$ rad/s) using a computer spreadsheet, e.g., EXCEL. The results for the frequency range $\omega = 0\div1{,}748\times10^5$ rad/s are presented in Fig.10.3, where the amplitude $\Phi_0(\Delta\Phi)$ versus the frequency ω is shown. The estimated natural frequencies amount to $\omega_1 = 1{,}748\times10^5$ rad/s and $\omega_2 = 4{,}576\times10^5$ rad/s, because the boundary condition ($\Phi_0 = \Delta\Phi \approx 0$) is fulfilled for these values. The detailed calculations for these frequencies are presented in Table 10.2 and Table 10.3. We can see that the residual value $\Delta\Phi$ is close to zero.

Table 10.2 $\omega_1 = 1.748\times10^5$ rad/s $\Delta\Phi = \Phi_0 = 6.26\times10^{-5}$ rad

n	B	$B\omega^2$	Φ_n	$B\omega^2\Phi_n$	$\sum B\omega^2\Phi_n$	k	$\sum B\omega^2\Phi_n/k$
	kgm^2	Nm	rad	Nm	Nm	Nm/rad	rad
1	10^{-5}	2.05×10^5	0.618	1.89×10^5	4.93×10^5	8×10^5	0.617
2	10^{-5}	3.05×10^5	1.0	3.05×10^5	3.05×10^5	8×10^5	0.381

Table 10.3 $\omega_2 = 4.576\times10^5$ rad/s $\Delta\Phi = \Phi_0 = -1.26\times10^{-3}$ rad

n	B	$B\omega^2$	Φ_n	$B\omega^2\Phi_n$	$\sum B\omega^2\Phi_n$	k	$\sum B\omega^2\Phi_n/$k
	kgm^2	Nm	rad	Nm	Nm	Nm/rad	rad
1	10^{-5}	20.94×10^5	-1.617	-33.86×10^5	-12.92×10^5	8×10^5	-1.615
2	10^{-5}	20.94×10^5	1.0	20.94×10^5	20.94×10^5	8×10^5	2.617

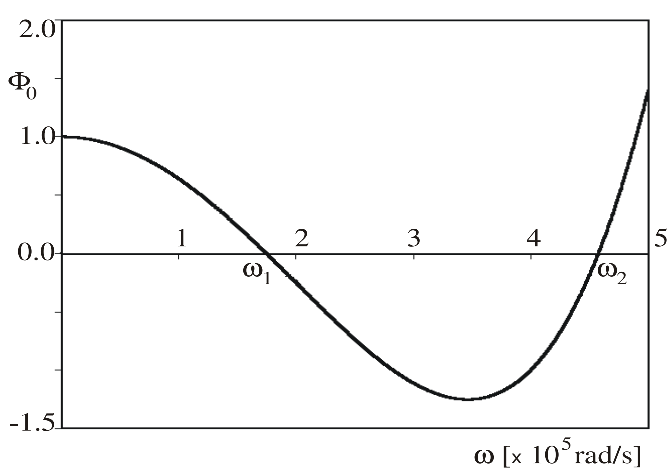


Fig. 10.3. Diagram of the amplitude $\Phi_0(\Delta\Phi)$ versus the natural frequency ω.

2.2. Principal modes of vibrations

Principal modes of torsional vibrations of the system under consideration are defined by the ratio of the vibration amplitudes corresponding to the determined natural frequencies. This ratio can be read from column 4 of Table 10.2 (for the natural frequency ω_1) or Table 10.3 (for the natural frequency ω_2). A graphic presentation of principal modes of vibration for the considered example is shown in Fig. 10.4.

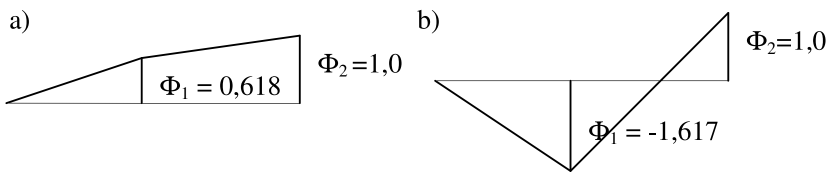
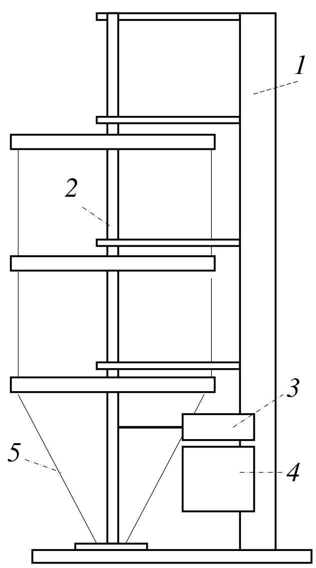


Fig. 10.4. Principal modes of vibration corresponding to: a) natural frequency ω_1, b) natural frequency ω_2

3. Experimental stand



Parameters:

B = 0.0085 kgm^2;
k = 13.14 Nm/rad;
k_1 = 6.57 Nm/rad.

Fig. 10.5. Experimental stand

A scheme of the experimental stand is shown in Fig. 10.5. It consists of vertical column *1* with the base and shaft *2* of the length 0.8 m and the diameter 0.004 m with three discs. The lower end of the shaft is fixed. The upper end can rotate in the bearing. Torsional vibrations are excited using lever mechanism *3*, which is driven by electrical engine of direct current *4*. The modes of torsional vibrations can be visualized by means of vertical threads *5*.

4. Experiment

1) Estimate torsional natural frequencies of the system using the Holzer's method. The calculations should be carried out by means of the EXCEL computer spreadsheet, according to the table of calculations presented below. The investigated range of frequency is $f = 0 \div 12$ Hz. Assume the step of frequency $\Delta f = 0.05$ Hz. Draw up a diagram of the amplitude Φ_0 versus the natural frequency ω (similar to the diagram in Fig. 10.3) on the basis of the calculations carried out. The values of natural frequencies correspond to zero places of the function $\Phi_0(\omega)$ on the diagram. The remaining amplitudes Φ_1, Φ_2 and Φ_3, which correspond to natural frequencies, determine the principal modes of vibrations.
2) Start the system exciting torsional vibrations of discs and read three consecutive resonance frequencies from the frequency measurement instrument. The system vibrates at one of natural frequencies when the local maximum of amplitudes of torsional vibrations occurs. The angles of torsional deflection of discs can be observed by means of threads. At the resonance frequency, the threads assume the shapes corresponding to the principal modes of vibrations.
3) Record the values of resonance frequencies read from the measurement instrument in the table.
4) Draw up the observed principal modes of vibrations according to the presented example (Fig. 10.4).
5) Compare the numerical calculations with the results of experiment and write the conclusions.

5. Report

The report on the experiment should contain:

1. Estimation of natural frequencies according to Table 10.4.
2. Table 10.5 with a comparison of the numerical and experimental results.
3. Diagrams presenting the principal modes of vibrations.
4. Conclusions.

Table 10.4

f	ω	Φ_3	$\Phi_2 = \Phi_3 - \frac{B\Phi_3\omega^2}{k}$	$\Phi_1 = \Phi_2 - \frac{B\Phi_2\omega^2 + B\Phi_3\omega^2}{k}$	$\Phi_0 = \Phi_1 - \frac{B\Phi_1\omega^2 + B\Phi_2\omega^2 + B\Phi_3\omega^2}{k_1}$
Hz	*rad/s*	*rad*	*rad*	*rad*	*rad*

Table 10.5

Resonance frequencies	ω_e (experiment)				ω (calculation)	$\frac{\omega_e}{\omega}$
	1	2	3	Average value		
	Hz	*Hz*	*Hz*	*rad/s*	*rad/s*	-
ω_I						
ω_{II}						
ω_{III}						

References

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2. Parszewski Z.: *Dynamika i drgania maszyn*. WNT, Warszawa 1982.
3. Osiński Z: *Teoria drgań*, PWN, Warszawa 1978.
4. Rao S.S.: *Mechnical Vibrations*, 3rd ed., Prentice Hall, NY, 1995.