Exercise 3

CONTROL SYSTEM REPRESENTATION

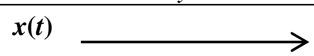
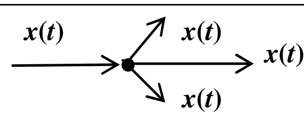
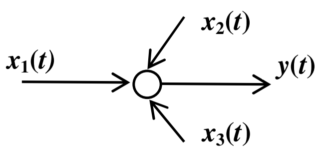
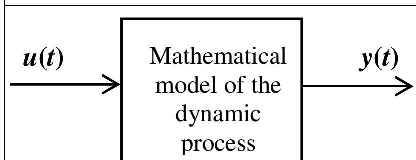
1. Aim of the exercise

Determination of transfer functions and state-space descriptions of simple control systems on the basis of their block diagrams. Control system representation with the MATLAB – SIMULINK software package.

2. Theoretical introduction

In the frequency-domain analysis of control system dynamics, the attention is usually focused on the relationship between the output signal and the input signal. The input–output relationship is characteristic of all controllable processes. The most appropriate way to visualize this relationship are **block diagrams**, at which parts of the control system are represented by geometric figures, such as rectangles, circles, triangles connected with lines. Lines with arrows represent signal flows. Table 3.1 shows elements of such diagrams.

Table 3.1. Elements of the block diagrams

Symbol	*Description*
	**Signal** $x(t)$
	**Branch point** Only one input signal – more than one output signal of the same value
	**Summing** More than one input signal – only one output signal $y(t)=\sum_{i=1}^{k}\pm x_i(t)$
	**Block** Graphical representation of the dynamic system

To build a block diagram, one should know a mathematical model of the system. Two forms of mathematical models of control systems are mostly used in the control theory: transfer functions and a state-space representation.

2.1. Transfer function description

To illustrate the transfer function description process, a one-degree-of-freedom dynamic system is taken into account. The system is shown in Fig. 3.1. The force F(t) acting on the mass *m* is understood as the input signal and the output signal is the mass displacement $y(t)$. The mass is suspended on a spring with the stiffness coefficient *k* and a dashpot representing viscous damping of the coefficient *c*.

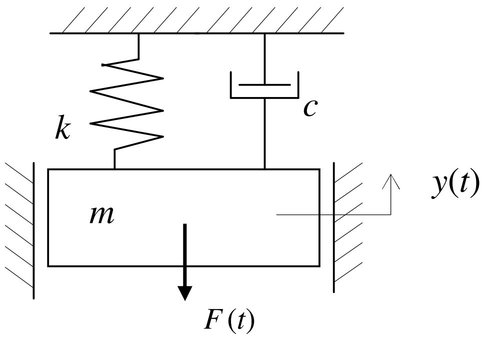


Fig. 3.1. One-degree-of-freedom dynamic system

The equation of motion of the considered model is represented in the following form:

$$m\,\ddot{y} + c\,\dot{y} + k\,y = F\,(t)\,. \tag{3.1}$$

After some transformations, equation of motion (3.1) takes the form:

$$T_1^2\,\ddot{y} + T_2\,\dot{x} + y = K\,F(t)\,, \tag{3.2}$$

where:

$$\frac{m}{k} = T_1^2\,;\;\; \frac{c}{k} = T_2\,;\;\; \frac{1}{k} = K. \tag{3.3}$$

The calculation of the Laplace transform of the left- and right-hand sides of Eq. (3.2), on assumption of zero initial conditions, gives:

$$\alpha[T_1^2\,\ddot{y} + T_2\,\dot{y} + y] = \alpha[K\,F(t)] \tag{3.4}$$

$$(T_1^2 s^2 + T_2 s + 1)\,Y(s) = K\,F(s) \tag{3.5}$$

where: $Y(s)$ – Laplace transform of the system output, and $F(s)$ – Laplace transform of the system input.

The transfer function $G(s)$ in this case has the form:

$$G(s) = \frac{Y(s)}{F(s)} = \frac{K}{T_1^2 s^2 + T_2 s + 1} \tag{3.6}$$

Generally, to determine the transfer function of the SISO (single – input – single – output) system, it is described by the following linear differential equation:

$$\begin{aligned} & a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \\ & = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \ldots + b_1 \frac{du(t)}{dt} + b_0 u(t) \end{aligned} \tag{3.7}$$

where: $a_n \ldots a_0$, $b_m \ldots b_0$ – constant coefficients, one should calculate the Laplace transform of Eq. (3.7), assuming zero initial conditions, as:

$$\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right) Y(s) = \left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0\right) U(s)\,. \tag{3.8}$$

The transfer function $G(s)$ is the ratio of the Laplace transform of the system output $Y(s)$ to the Laplace transform of the system input $U(s)$, when all initial conditions are zero:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}\,. \tag{3.9}$$

Then, the block diagram of the system has the form shown in Fig. 3.2.

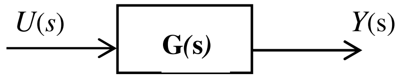


Fig. 3.2. Block diagram of the SISO dynamic system

Multidimensional linear time invariant systems are described by the **matrix transfer functions:**

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \ldots\ldots \\ Y_p(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \ldots & G_{1r}(s) \\ G_{21}(s) & G_{22}(s) & \ldots & G_{2r}(s) \\ \ldots & \ldots & \ldots & \ldots \\ G_{p1}(s) & G_{p2}(s) & \ldots & G_{pr}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \ldots\ldots \\ U_r(s) \end{bmatrix} \tag{3.10}$$

Eq. (3.10) can be written in a shorter form, with bold letters representing the matrices:

$$\mathbf{Y}(s) = \mathbf{G}(s)\,\mathbf{U}(s) \tag{3.11}$$

2.2. State-space representation

The state of a dynamic system is defined by a set of n variables, whose time evolution describes completely the internal behaviour of the system. The dimensional integer n is called the order of the state-space representation. A continuous, linear time invariant system is defined by a set of n first order differential equations which can be written in the matrix form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{3.12}$$

where: $\mathbf{x}(t)$ – n-dimensional state vector,
$\mathbf{u}(t)$ – m-dimensional input signal vector,
A - state matrix ($n \times n$),
B - input matrix ($n \times m$)

In most cases, only r functions of the state can be measured or directly observed. Thus, the r-dimensional external output vector $\mathbf{y}(t)$ can be expressed in the matrix form:

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \tag{3.13}$$

where: **C** - output matrix ($r \times n$),
D - transition matrix ($r \times m$).

State equation (3.12) and **output equation** (3.13) constitute the state-space representation of the system.

To obtain the relationship between the matrix transfer function of the system **G**(s) and its state-space description, one should assume that the initial state $\mathbf{x}(0)$ is zero. In this case, the Laplace transform of the state vector is:

$$\mathbf{X}(s) = (s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s) \tag{3.14}$$

The Laplace transform of the output is as follows:

$$\mathbf{Y}(s) = \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s) \tag{3.15}$$

Substituting Eq. (3.14) into Eq. (3.15), we obtain:

$$\mathbf{Y}(s) = \mathbf{C}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s) + \mathbf{D}\,\mathbf{U}(s) \tag{3.16}$$

On the basis of Eq. (3.16), the matrix transfer function of the system is:

$$\mathbf{G}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{3.17}$$

3. Course of the exercise

The block diagram of the feedback system, which is considered in the exercise, is shown in Fig. 3.3. In this diagram, signals are expressed in the Laplace transform forms. During the exercise, one ought to perform the following operations:

a) determine the closed loop transfer function (between $Y(s)$ and $U(s)$),
b) describe the error transfer function (between $E(s)$ and $U(s)$ - Fig. 3.3),
c) determine the unit step response on the basis of the Laplace transformation,
d) determine of the state and output matrix equations of this system,
e) describe the system definition with the MATLAB – SIMULINK software package,
f) verify the results of investigations with the MATLAB – SIMULINK procedures.

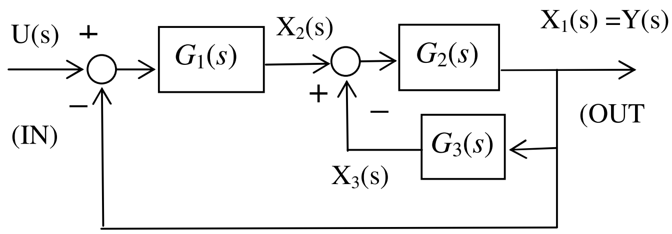


Fig. 3.3. Block diagram of the feedback system, $G_1(s) = 10/s$, $G_2(s) = 1/(s+2)$, $G_3(s) = 1/(s+3)$

4. Laboratory report should contain:

1) Aim of the exercise.
2) Block diagram of the system under consideration.
3) Results of the manual calculations.
4) Results of the computer calculations.
5) Conclusions and remarks.

References

1. Ogata K.: *Modern Control Engineering,* IV-th Edition, Prentice Hall 2002.
2. Wolovich W.A.: *Automatic Control Systems, Basic Analysis and Design,* Harcour Brace College Publishers, 1994.
3. Findeisen D.: *System Dynamics and Mechanical Vibrations,* Springer-Verlag 2000.
4. Friendland B.: *Control System Design,* McGraw-Hill Book Company, 1987.