Exercise 8

CRITICAL SPEEDS OF THE ROTATING SHAFT

1. Aim of the exercise

Observation and measurement of three consecutive critical speeds and corresponding modes of the actual rotating shaft. Comparison of analytically computed and measured critical speed values.

2. Theoretical introduction

The critical speed notion is associated with rotary machines and is used to describe the rotational speed at which an excessive increase in shaft deflections and resulting vibrations of the rotor housing can be observed. The critical speed occurs when the speed of rotations is equal to the natural frequency of the shaft. Two methods for determining natural frequencies of the shaft are presented in the theoretical introduction.

2.1. Rotating shaft as a continuous system

To determine natural frequencies of the uniform shaft with the mass distributed along its length, one can use the beam equation of lateral vibrations in the following form:

$$\frac{\partial^4 y}{\partial x^4} + a^2 \frac{\partial^2 y}{\partial t^2} = 0\,, \qquad (8.1)$$

where: $$a^2 = \frac{\mu}{EJ}\,, \qquad (8.2)$$

μ – mass per unit length,
EJ – flexural stiffness of the beam.

If we use the method of separation of variables, the formal solution to beam equation (8.1) takes the form:

$$y(x,t) = U(x)\,T(t)\,. \qquad (8.3)$$

The substitution of this solution in Eq. (8.1) gives:

$$U^{IV}(x)\,T(t) + a^2 U(x)\,\ddot{T}(t) = 0\,, \qquad (8.4)$$

thus:

$$\frac{U^{IV}(x)}{U(x)} = -a^2 \frac{\ddot{T}(t)}{T(t)}\,. \qquad (8.5)$$

To satisfy Eq. (8.5) for arbitrary x and t values, both sides of this equation have to be equal to a constant value denoted as k^4:

$$\frac{U^{IV}(x)}{U(x)} = -a^2 \frac{\ddot{T}(t)}{T(t)} = k^4\,. \qquad (8.6)$$

Thus, one obtains two independent differential equations:

$$U^{IV}(x) - k^4 U(x) = 0\,; \qquad (8.7)$$

$$\ddot{T}(t) + \omega_n^2 T(t) = 0\,, \qquad (8.8)$$

where: $$\omega_n^2 = \frac{k^4}{a^2}\,. \qquad (8.9)$$

The homogenous solution to Eq. (8.7) has the following form:

$$U(x) = A\sin kx + B\cos kx + C\sinh kx + D\cosh kx\,, \qquad (8.10)$$

where: *A, B, C, D* – constant quantities determined from the boundary conditions.
For simply supported ends, the boundary conditions have the following form:

$$y(0,t) = 0; \qquad y(l,t) = 0;$$
$$\left.\frac{\partial^2 y(x,t)}{\partial t^2}\right|_{x=0} = 0; \quad \left.\frac{\partial^2 y(x,t)}{\partial t^2}\right|_{x=i} = 0. \qquad (8.11)$$

Substituting solution (8.10) in boundary conditions (8.11), one receives the eigenfunction sequence in the form:

$$U_n(x) = \sin k_n x = \sin\frac{n\pi x}{l}; \quad n = 1, 2, ... \qquad (8.12)$$

The solution to Eq. (8.8) is as follows:

$$T_n(t) = K_n \cos\omega_n t + L_n \sin\omega_n t\,, \qquad (8.13)$$

where K_n and L_n are constant quantities determined from the initial conditions.

On the basis of Eqs. (8.9) and (8.12), the natural frequencies ω_n, related to successive eigenfunctions of the beam, can be determined from the following equation:

$$\omega_n = \frac{k_n^2}{a} = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EJ}{\mu}}; \quad n = 1, 2, ... \qquad (8.14)$$

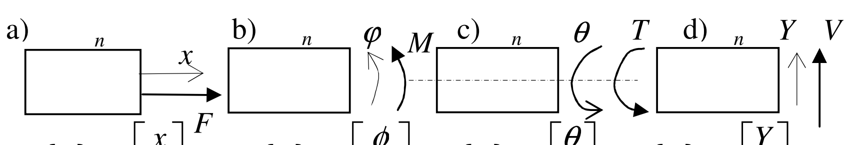
Finally, the general solution to the beam equation takes the form:

$$y(x,t) = \sum_{n=1}^{\infty}(K_n \cos\omega_n t + L_n \sin\omega_n t)\sin\frac{n\pi x}{l} \qquad (8.15)$$

2.2. Rotating shaft as a discrete system. Myklestad method

A complex or continuous system can be divided into finite elements or segments and thus it can be approximated with an equivalent discrete system. To explain this technique, one ought to introduce the concept of a state vector and a transfer matrix.

A **state vector** is a column of numbers, whose values represent variables at a given station in the system. The numbers describe the state of the problem. Hence, each element of the state vector is called a state variable. Typical variables and their corresponding state vectors are shown in Fig. 8.1.



$$\{Z\}_n = \begin{bmatrix} x \\ F \end{bmatrix}_n \quad \{Z\}_n = \begin{bmatrix} \phi \\ M \end{bmatrix}_n \quad \{Z\}_n = \begin{bmatrix} \theta \\ T \end{bmatrix}_n \quad \{Z\}_n = \begin{bmatrix} Y \\ V \end{bmatrix}_n$$

Fig. 8.1. State vectors, generalized forces and displacements for: a) tension, b) bending, c) torsion, d) shear.

For example, if a combination of shear and bending occurs at the station *n,* the corresponding state vector is:

$$\{Z\}_n = \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n = \begin{bmatrix} Y & \phi & M & V \end{bmatrix}_n^T. \qquad (8.16)$$

A **transfer matrix** transfers state variables from one station to another. Let us take into account a beam structure divided into segments. A typical segment of the beam, as shown in Fig. 8.2, consists of a massless span which is characterized by the bending stiffness *EJ* and the point mass m_n. The superscripts *L* and *P* denote the left- and the right-hand side of the mass m_n, respectively. The flexural properties of the segment are described by the field transfer matrix of the span and the inertial effects of the segment are described by the point transfer matrix of the mass.

To describe the field transfer matrix, consider a free-body sketch of a uniform beam of the length L_n as shown in Fig. 8.2(b). In this case, the field transfer matrix transfers the state variables of the state vector $\{Z\}^P_{n-1}$ in the left end to the state vector $\{Z\}^L_n$ in the right end of the span.

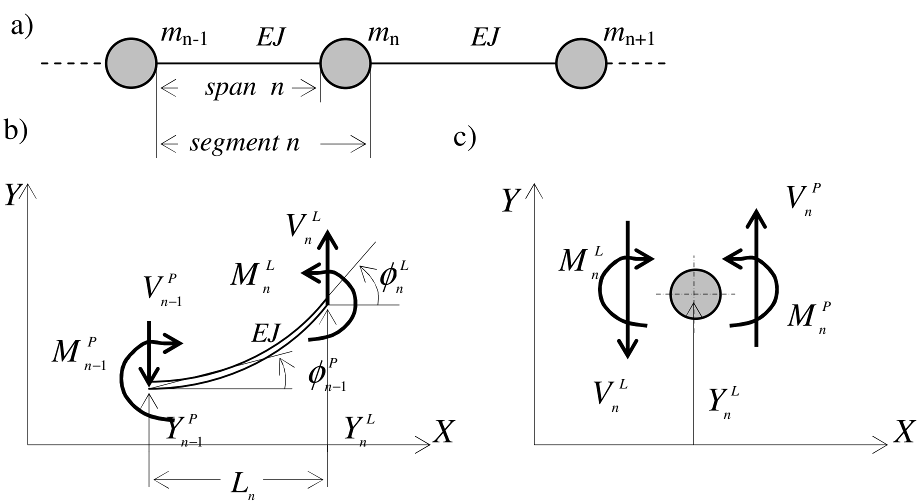


Fig. 8.2. Derivation of the transfer matrix of the beam

For the equilibrium state, we require that:

$$V_n^L = V_{n-1}^P; \quad M_n^L = M_{n-1}^P - L_n V_{n-1}^P, \qquad (8.17)$$

where: *M* - bending moment,

V - shear force.

According to Fig. 8.2(b), the change in the slope Φ of the span is equal to:

$$\phi_n^L - \phi_{n-1}^P = M_n^L\left(\frac{L}{EJ}\right)_n + V_n^L\left(\frac{L^2}{2EJ}\right)_n. \qquad (8.18)$$

After the substitution of Eq. (8.17) in (8.18) and rearrangement, we obtain:

$$\phi_n^L = \phi_{n-1}^P + M_{n-1}^P\left(\frac{L}{EJ}\right)_n - V_{n-1}^P\left(\frac{L^2}{2EJ}\right)_n. \qquad (8.19)$$

The change in the deflection *Y* of the span is as follows:

$$Y_n^L - Y_{n-1}^P = L_n\phi_{n-1}^P + M_n^L\left(\frac{L^2}{2EJ}\right)_n + V_n^L\left(\frac{L^3}{3EJ}\right)_n. \qquad (8.20)$$

After the substitution of V_n^L and M_n^L from Eq. (8.17) in (8.20) and rearrangement, we get:

$$Y_n^L = Y_{n-1}^P + L_n \phi_{n-1}^P + M_{n-1}^P \left(\frac{L^2}{2EJ} \right)_n + V_{n-1}^P \left(\frac{L^3}{6EJ} \right)_n . \qquad (8.21)$$

The field transfer matrix is obtained by writing Eqs. (8.17), (8.19) and (8.21) in the matrix form:

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^L = \begin{bmatrix} 1 & L & \frac{L^2}{2EJ} & -\frac{L^3}{6EJ} \\ 0 & 1 & \frac{L}{EJ} & -\frac{L^2}{2EJ} \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_{n-1}^P . \qquad (8.22)$$

To derive the point transfer matrix, consider a free-body sketch of m_n (Fig. 8.2c). The equations for the shear and the moment are as follows:

$$V_n^P = V_n^L - \omega^2 m_n Y_n^L \quad \text{and} \quad M_n^P = M_n^L - \omega^2 J_0 \phi_n^L , \qquad (8.23)$$

where J_n - mass moment of inertia of m_n about its axis normal to the (x, y) plane.

For the rigid body motion of m_n, the following relations are fulfilled:

$$\phi_n^L = \phi_n^P ; \qquad Y_n^L = Y_n^P . \qquad (8.24)$$

The point transfer matrix is obtained from Eqs. (8.23) and (8.24):

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\omega^2 J_0 & 1 & 0 \\ -\omega^2 m & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^L . \qquad (8.25)$$

Substituting $\{Z\}_n^L$ from Eq. (8.22) in (8.25), one receives the transfer matrix:

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\omega^2 J_0 & 1 & 0 \\ -\omega^2 m & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} 1 & L & \frac{L^2}{2EJ} & -\frac{L^3}{6EJ} \\ 0 & 1 & \frac{L}{EJ} & -\frac{L^2}{2EJ} \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_{n-1}^P$$

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^P = \begin{bmatrix} 1 & L & \frac{L^2}{2EJ} & -\frac{L^3}{6EJ} \\ 0 & 1 & \frac{L}{EJ} & -\frac{L^2}{2EJ} \\ 0 & -\omega^2 J_0 & 1-\omega^2 J_0 \frac{L}{EJ} & -L+\omega^2 J_0 \frac{L^2}{2EJ} \\ -\omega^2 m & -\omega^2 mL & -\omega^2 m \frac{L^2}{2EJ} & 1+\omega^2 m \frac{L^3}{6EJ} \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_{n-1}^P . \qquad (8.26)$$

Equation (8.26) can be written in the following form:

$$\{Z\}_n^P = H_n \{Z\}_{n-1}^P , \qquad (8.27)$$

where H_n is the transfer matrix of the n-th element.

Using the recurrence formula, the state vector $\{Z\}_n^P$ at a typical station n can be related to the state vector $\{Z\}_0^P$ at the boundary of the system:

$$\{Z\}_n^P = \{H_n H_{n-1} \ldots H_2 H_1\} \{Z\}_0^P . \qquad (8.28)$$

Equation (8.28) presents the recurrence formula which is used in the Myklestad method for the natural frequency calculation.

The common boundary conditions for the beam problem are presented in Table 8.1.

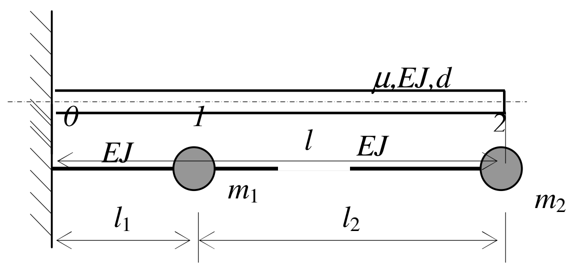
Table 8.1. Boundary condition for the beam problem

	Y	ϕ	M	V
Simple support	0	ϕ	0	V
Free end	Y	ϕ	0	0
Fixed end	0	0	M	V

For example, the deflection Y and the moment M at a simple support have to be zero, whereas the slope ϕ and the shear force V are unknown and non-zero. At the beginning point or station 0 of the beam, there are two non-zero boundary conditions, dictated by the type of support. Similarly, there are two non-zero boundary conditions at the other end of the beam.

The procedure of the Myklestad method for the natural frequency calculation consists in assuming the frequency ω and proceeding with the computation. The process repeats until the ω value that satisfies simultaneously the boundary conditions at both ends of the beam is found. This ω value is the natural frequency.

Example: Using the Myklestad method, find natural frequencies of the beam shown in Fig. 8.3.



Data:

$m = 0.222$ kg,
$l = 1$ m,
$d = 0.006$ m,
$J = \pi d^4/64 = 63.6\times10^{-12}$ m^4,
$E = 2.1 \times 10^{11}$ N/m^2,
$\mu = m/l = 0.222$ kg/m.

Fig. 8.3. Lumped-mass representation of the beam

Solution

The beam was divided into two segments of the lengths: $l_1 = 0.25$ m and $l_2 = 0.75$ m (Fig. 8.3). The mass of the first segment is concentrated at its right end. The mass of the second segment has been divided in two equal parts and located at its both ends. The lamped-mass representation of the beam:
$m_1 = 0.13875$ kg and $m_2 = 0.08325$ kg.

In this case, recurrence formula (8.28) for the natural frequency calculation has the following form:

$$\{Z\}_2^P = H_2 H_1 \{Z\}_0^P, \tag{a}$$

where: $\{Z\}_0^P = \{Y\ \phi\ M\ V\}_0^T = \{0\ 0\ M_0\ V_0\}_0^T,$ (b)

M_n and V_n are the unknown moment and shear force at the fixed end.

Applying Eq. (8.26), one receives the transfer matrices for both segments:

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_1^P = \begin{bmatrix} 1 & 0{,}25 & 2{,}34\times10^{-3} & -1{,}95\times10^{-4} \\ 0 & 1 & 1{,}87\times10^{-2} & -2{,}34\times10^{-3} \\ 0 & 0 & 1 & -0{,}25 \\ -0{,}138\omega^2 & -3{,}47\times10^{-2}\omega^2 & -3{,}245\times10^{-4}\omega^2 & 1+2{,}704\times10^{-5}\omega^2 \end{bmatrix}_n \begin{bmatrix} 0 \\ 0 \\ M_0 \\ V_0 \end{bmatrix}_0^P$$

(c)

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_2^P = \begin{bmatrix} 1 & 0{,}75 & 2{,}1\times10^{-2} & -5{,}26\times10^{-3} \\ 0 & 1 & 5{,}61\times10^{-2} & -2{,}1\times10^{-2} \\ 0 & 0 & 1 & -0{,}75 \\ -8{,}32\times10^{-2}\,\omega^2 & -6{,}24\times10^{-2}\,\omega^2 & -1{,}75\times10^{-3}\omega^2 & 1+4{,}38\times10^{-4}\omega^2 \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_1^P$$

(d)

Substituting (c) and (d) in (a) in the general description, we obtain:

$$\{Z\}_2^P = H_2 H_1 \{Z\}_0^P = [H]\{Z\}_0^P \quad \Rightarrow$$

$$\Rightarrow \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_2^P = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}_n \begin{bmatrix} 0 \\ 0 \\ M_0 \\ V_0 \end{bmatrix}_0^P . \qquad \text{(e)}$$

The moment M_2^P and the shear force V_2^P have to be zero at the free end of the beam (Table 8.1). Using the notations from (e), one can write these conditions in the following form:

$$M_2^P = H_{33}M_0 + H_{34}V_0 = 0;$$
$$V_2^P = H_{43}M_0 + H_{44}V_0 = 0. \qquad \text{(f)}$$

For a nontrivial solution to the simultaneous homogeneous equations, the characteristic determinant of (f) has to vanish, that is:

$$\begin{vmatrix} H_{33} & H_{34} \\ H_{43} & H_{44} \end{vmatrix} = 0. \qquad \text{(g)}$$

On the basis of Eq. (g), natural frequencies of the beam can be determined. For the given data, the corresponding equation is as follows:

$$\begin{vmatrix} 1+2{,}434\times10^{-4}\omega^2 & -1-2{,}028\times10^{-5}\alpha^2 \\ -3{,}44\times10^{-3}\omega^2 - 1{,}42\times10^{-7}\omega^4 & 1+1{,}06\times10^{-3}\omega^2 + 1{,}18\times10^{-8}\omega^4 \end{vmatrix} = 0$$

(h)

or, in the developed form:

$$\omega^4 - 35{,}97\times10^3\omega^2 + 1{,}69\times10^7 = 0. \qquad \text{(i)}$$

The natural frequencies are equal to:

ω_1 = 21.8 rad/s $\Rightarrow$ f_1 = 3.47 Hz,
ω_2 = 188.4 rad/s $\Rightarrow$ f_2 = 30 Hz.

3. Measurement device

A scheme of the measurement device is shown in Fig. 8.4.

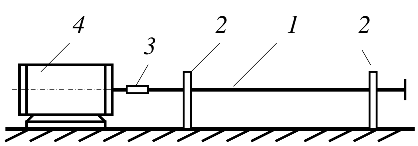


Fig. 8.4. Measurement device

Shaft (1) in the form of a steel rod is supported in self-aligning ball bearings (2). It is driven by electric motor (4) with flexible coupling (3). Both the motor and bearing housings are mounted on the common base. A potentiometer is used to vary the speed of the motor. A photoelectric transducer is used to measure the angular velocity of the shaft.

The shaft properties are as follows: mass m = 0.222 kg, diameter d = 6 mm, length: l = 1 m. In the working range of the driving motor (0÷6000 rev/min), one can observe three consecutive critical speeds of the shaft.

4. Course of the exercise

Calculate analytically three lowest values of the critical speed of the shaft treated as a continuous system using the real system data presented earlier. Record the results in Table 8.2 in the ω_n column.

Next, calculate the critical speeds of the shaft treated as a discrete system, using the Myklestad method. In the numerical calculations, use the DERIVE package. Calculate the critical speeds of the investigated shaft for different segmentations. Record the calculation results in the ω_{nM} column of Table 8.2.

In the experimental part of the exercise, measure and observe the critical speeds of the shaft. The critical speed occurs when the lateral deflections of the rotating shaft have extreme values. The measurements are to be conducted both at speeds lower than the critical one and higher than the critical one, but such at which no impacts against the limiters occur. Record the measurement results in Table 8.2 and calculate the mean values.

Table 8.2. Investigation results

	Measured frequency ω_p [rad/s]			Mean value ω_{pm}	ω_n [rad/s]	ω_{nM} [rad/s]	ω_n/ω_{pm} %	ω_{nM}/ω_{pm} %
I critical speed								
II critical speed								
III critical speed								

After completing the measurements, compare the measured and calculated values of critical speeds. Calculate the ratio ω_p/ω_n and write it in the last column of Table 8.2. Make drawings of the observed deflection shapes of the shaft.

5. **Laboratory report** should contain:

1) Aim of the exercise.
2) Experimental and calculation results in the form of Table 8.2.
3) Drawings of the observed deflection shapes of the shaft corresponding to the consecutive critical speeds.

4) Conclusions and remarks.

References

1. Tse F.S., Morse I.E., Hinkle R.T.: *Mechanical Vibrations - Theory and Applications,* Allyn and Bacon Inc., Boston, London, Sydney, Toronto, 1978.
2. Kapitaniak T.: *Wstęp do teorii drgań*. Wydawnictwo Politechniki Łódzkiej, Łódź 1992.
3. Parszewski Z.: *Drgania i dynamika maszyn*. PWN, Warszawa 1982.